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AN INVERSE MATRIX OF AN UPPER TRIANGULAR MATRIX CAN BE LOWER TRIANGULAR

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Abstract

In this note we explain why the group of $n \times n$ upper triangular matrices is defined usually over commutative ring while the full general linear group is defined over any associative ring.

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1. Introduction

The following three results on $n \times n$ matrices with real entries can be found in standard textbooks on linear algebra.

Theorem 1. *The inverse of an invertible lower triangular matrix is lower triangular, and the inverse of an invertible upper triangular matrix is upper triangular.*

Theorem 2. *A triangular matrix is invertible if and only if all its diagonal entries are invertible.*

Theorem 3. *Two $n \times n$ matrices A and B are inverses of each other if and only if $BA = I$ or $AB = I$, where I denotes identity matrix.*

For this result, see, for example, [1], Theorem 1.7.1, p. 68 and [2]. These results can be generalized for matrices with entries from the wider classes of commutative rings (by using of determinant theory).

In handbooks on group theory for graduates, the definition of the group of invertible upper triangular matrices $T(n, R)$ is given for commutative rings with 1 only (see [6], p. 124, and [5], p. 5). This definition requires validity of theorems 1 and 2 for such rings.

It is interesting that, from the other side, a definition of a general linear group $GL(n, R)$ as the group of all invertible $n \times n$ matrices is given for matrices with entries from any associative ring R with 1 (see [6], p. 5). In commutative case $T(n, R)$ is a very natural subgroup of $GL(n, R)$, so it is an intriguing question why one cannot define $T(n, R)$ for noncommutative rings.

In this note we explain this phenomenon and prove the following

Theorem 4. *There exist a noncommutative ring R and two 2×2 matrices A, B with coefficients from R such that A is upper triangular, B is lower triangular and $AB = BA = I$.*

2. Dedekind-finite rings.

Let R be an associative ring with unit 1. We say that R is *Dedekind-finite* if $xy = 1$ implies $yx = 1$ for all x, y in R. The following rings can serve as examples of Dedekind-finite rings: finite rings, commutative rings, $End_K(K^n)$, i.e. the ring of endomorphisms of a vector space K^n (for any field K). If R is commutative, then the ring $M(n, R)$ of $n \times n$ matrices over R is Dedekind-finite too [3]. This follows from the determinant theory. In [2] one can find the various elementary proofs of this fact for matrices over a field. If R is Dedekind-finite, then the polynomials ring $R[t]$ and the ring of formal power series $R[[t]]$ are Dedekind-finite [3].

If R is not Dedekind-finite, then there exist two elements x and y such that $xy = 1$ and $yx \neq 1$ and Theorem 3 fails. The rings $R[t]$ and $R[[t]]$ are not Dedekind-finite in this case. The following example shows that $M(n, R)$ is not Dedekind-finite for $n \geq 2$ (we restrict us to $n = 2$; for the general case one can uses a similar example):

$$\begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix} \neq \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Moreover:

$$\begin{pmatrix} y & 1 \\ 0 & -x \end{pmatrix} \begin{pmatrix} x & 1 \\ 1-yx & -y \end{pmatrix} = \begin{pmatrix} x & 1 \\ 1-yx & -y \end{pmatrix} \begin{pmatrix} y & 1 \\ 0 & -x \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

This last example, taken from [7], was an inspiration for our paper. It shows that the inverse of an upper triangular matrix need not be upper triangular. It follows that Theorems 1 and 2 fail for rings which are not Dedekind-finite. Thus, the group $T(n, R)$ cannot be defined for such rings.

Infinite matrices give an example of the ring which is not Dedekind-finite. Let $\mathrm{RCFM}(R)$ be a ring of row and column finite infinite matrices [4] (indexed by positive integers) over an associative ring R with 1. Let

$$a = \begin{pmatrix} 0 & 0 & 0 & 0 & \dots \\ 1 & 0 & 0 & 0 & \dots \\ 0 & 1 & 0 & 0 & \dots \\ 0 & 0 & 1 & 0 & \dots \\ \dots & \dots & \dots & \dots & \dots \end{pmatrix} \text{ and } \quad c = \begin{pmatrix} 0 & 1 & 0 & 0 & \dots \\ 0 & 0 & 1 & 0 & \dots \\ 0 & 0 & 0 & 1 & \dots \\ 0 & 0 & 0 & 0 & \dots \\ \dots & \dots & \dots & \dots & \dots \end{pmatrix}.$$

It is clear that $ca = e$ where e denotes infinite identity matrix

$$e = \begin{pmatrix} 1 & 0 & 0 & \dots \\ 0 & 1 & 0 & \dots \\ 0 & 0 & 1 & \dots \\ \dots & \dots & \dots & \dots \end{pmatrix}$$

and ac has a first row entirely of zeros and so $ac \neq e$. Thus, Theorem 3 fails for infinite matrices.

3. Example

Examples in previous section give arise to the following question:

> *Can the inverse of an upper triangular matrix be lower triangular?*

In the case of 2×2 matrices the positive answer forces two conditions:

$$\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \begin{pmatrix} d & 0 \\ f & g \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} d & 0 \\ f & g \end{pmatrix} \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

These equations are equivalent to two systems of equations:

$$\left\{ \begin{array}{ccc} ad + bf & = & 1 \\ bg & = & 0 \\ cf & = & 0 \\ cg & = & 1 \end{array} \right. \quad \text{and} \quad \left\{ \begin{array}{ccc} da & = & 1 \\ db & = & 0 \\ fa & = & 0 \\ fb + gc & = & 1 \end{array} \right.$$

If the ring of coefficients is commutative, then such matrices must be diagonal.

Now we describe an example, announced in Theorem 4, which gives positive answer to our question.

Let $A = \begin{pmatrix} a & b \\ 0 & c \end{pmatrix}$ be a 2×2 matrix with entries in the ring $\mathrm{RCFM}(R)$ where a, c are infinite matrices defined in previous section and b is a matrix with the only nontrivial entry $b_{11} = 1$ and all other entries equal to 0:

$$b = \begin{pmatrix} 1 & 0 & 0 & \dots \\ 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & \dots \\ \dots & \dots & \dots & \dots \end{pmatrix}.$$

We put $B = \begin{pmatrix} c & 0 \\ b & a \end{pmatrix}$. Easy calculations show that $AB = BA = I$, where $I = \begin{pmatrix} e & 0 \\ 0 & e \end{pmatrix}$ is a unit matrix in $M(2, \mathrm{RCFM}(R))$, so A is invertible (and B too). Moreover, A is upper triangular, while B is lower triangular. It is clear that the diagonal entries of A and B are not invertible. ∎

We note here that the ring R of coefficients of infinite matrices in the above example can be any field, a ring of integers, or a ring of integers modulo p. An interesting open problem is the existence of such example for noncommutative Dedekind-finite ring.

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