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CONGRUENCE SUBMODULARITY

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Abstract

We present a countable infinite chain of conditions which are essentially weaker then congruence modularity (with exception of first two). For varieties of algebras, the third of these conditions, the so called 4-submodularity, is equivalent to congruence modularity. This is not true for single algebras in general. These conditions are characterized by Maltsev type conditions.

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A lattice L is *modular* if it satisfies the equality

$$(a \vee b) \wedge c = a \vee (b \wedge c)$$

for all $a, b, c \in L$ with $a \leq c$. Of course, the inequality

$$(a \vee b) \wedge c \geq a \vee (b \wedge c)$$

is valid trivially in every lattice whenever $a \leq c$; thus we are interested in the converse one only.

Let $A \neq \emptyset$ and L be a lattice of equivalence relations on A, i.e. L is a sublattice of the equivalence lattice $Eq\,(A)$.

It is well-known that for $\Theta, \Phi \in L$,

$$\text{(A)} \qquad \Theta \vee \Phi = (\Theta \cdot \Phi) \cup (\Theta \cdot \Phi \cdot \Theta) \cup (\Theta \cdot \Phi \cdot \Theta \cdot \Phi) \cup \cdots$$

where $\Theta \cdot \Phi$ denotes the relational product. It motivates us to introduce the following concepts:

Definition 1. A lattice L of equivalence relations on a set $A \neq \emptyset$ is called *k-submodular* ($k \geq 2$) if for all $\Theta, \Phi, \Psi \in L$ with $\Theta \subseteq \Psi$ the condition

$$\text{(B)} \qquad (\underbrace{\Theta \cdot \Phi \cdot \Theta \cdot \cdots}_{k \text{ factors}}) \cap \Psi \subseteq \Theta \vee (\Phi \vee \Psi)$$

is satisfied. An algebra $\mathcal{A}$ is *k-submodular* if $Con\,(\mathcal{A})$ is k-submodular. A variety $\mathcal{V}$ is *k-submodular* if each $\mathcal{A} \in \mathcal{V}$ has this property.

Remark 1. (a) Due to (A), an algebra $\mathcal{A}$ is congruence modular (i.e. $Con\,(\mathcal{A})$ is modular) if and only if $\mathcal{A}$ is k-submodular for each integer $k \geq 2$.

(b) Evidently, if $2 \leq m \leq k$ and $\mathcal{A}$ is congruence k-submodular then $\mathcal{A}$ is also m-submodular.

(c) The converse inclusion of (B) is valid in any lattice of equivalence relations.

(d) The product $\Theta \cdot \Phi \cdot \Theta \cdot \cdots$ (k factors) need not to be an equivalence (or congruence for $\Theta, \Phi \in Con\,(\mathcal{A})$). It is an equivalence if and only if

$$\text{(C)} \qquad \Theta \cdot \Phi \cdot \Theta \cdot \cdots = \Phi \cdot \Theta \cdot \Phi \cdot \cdots \quad \text{(with } k \text{ factors in both sides).}$$

(e) If an algebra $\mathcal{A}$ is k-permutable (i.e. (C) is valid for all $\Theta, \Phi \in Con\,(\mathcal{A})$), then $\mathcal{A}$ is congruence modular if and only if $\mathcal{A}$ is k-submodular.

Lemma 1. *Every lattice L of equivalences on a set $A \neq \emptyset$ is 3-submodular (and hence also 2-submodular).*

Proof. Let $\Theta, \Phi, \Psi \in L$ with $\Theta \subseteq \Psi$. Suppose $\langle x, y\rangle \in (\Theta \cdot \Phi \cdot \Theta) \cap \Psi$. Then $\langle x, y\rangle \in \Psi$ and there are elements $b, c \in A$ with

$$x \, \Theta \, b \, \Phi \, c \, \Theta \, y.$$

Since $\Theta \subseteq \Psi$, we have $\langle b, x\rangle \in \Psi$, $\langle y, c\rangle \in \Psi$ and, together with $\langle x, y\rangle \in \Psi$, also $\langle b, c\rangle \in \Psi$. Thus $\langle b, c\rangle \in \Phi \cap \Psi$ and hence

$$x \, \Theta \, b \, (\Phi \cap \Psi) \, c \, \Theta \, y$$

which yields $\langle x, y\rangle \in \Theta \cdot (\Phi \cap \Psi) \cdot \Theta \subseteq \Theta \vee (\Phi \cap \Psi)$. We have shown that L is 3-submodular. By (b) of Remark 1, L is also 2-submodular. ■

It is worth saying that the proof of Lemma 1 is in fact the same as the proof of the well-known result by B. Jónsson [3] that every 3-permutable algebra is congruence modular.

Theorem 1. *Let $\mathcal{V}$ be a variety of algebras and $k \geq 2$ an integer. The following conditions are equivalent:*

(1) *$\mathcal{V}$ is congruence k-submodular;*

(2) *there exist an integer $n > 0$ and $(k+1)$-ary terms $p_0, \ldots, p_n$ satisfying the following identities:*

$$p_0(x, z_1, \ldots, z_{k-1}, y) = x, \quad p_n(x, z_1, \ldots, z_{k-1}, y) = y,$$

$$p_i(x, x, z_2, z_2, z_4, z_4, \ldots) = p_{i+1}(x, x, z_2, z_2, z_4, z_4, \ldots) \text{ for } i \text{ even,}$$

$$p_i(x, z_1, z_1, z_3, z_3, \ldots, y) = p_{i+1}(x, z_1, z_1, z_3, z_3, \ldots, y) \text{ for } i \text{ odd,}$$

$$p_i(x, x, z_2, z_2, \ldots, z_{k-3}, z_{k-3}, x, x) =$$

$$= p_{i+1}(x, x, z_2, z_2, \ldots, z_{k-3}, z_{k-3}, x, x) \text{ for } i \text{ odd and } k \text{ odd,}$$

$$p_i(x, x, z_2, z_2, \ldots, z_{k-2}, z_{k-2}, x) =$$

$$= p_{i+1}(x, x, z_2, z_2, \ldots, z_{k-2}, z_{k-2}, x) \text{ for } i \text{ odd and } k \text{ even.}$$

Proof. (1)$\Rightarrow$(2): Consider the free algebra $F_v(x, y, z_1, \ldots, z_{k-1})$ of $\mathcal{V}$ generated by $k+1$ free generators $x, y, z_1, \ldots, z_{k-1}$. Further, let Θ, Φ, Ψ be the following congruences on this free algebra:

$$\Theta = \Theta(\langle x, z_1\rangle, \langle z_2, z_3\rangle, \ldots),$$

$$\Phi = \Theta(\langle z_1, z_2\rangle, \langle z_3, z_4\rangle, \ldots),$$

$$\Psi = \Theta(\langle x, y\rangle, \langle x, z_1\rangle, \langle z_2, z_3\rangle \ldots).$$

Clearly $\Theta \subseteq \Psi$ and

$$\langle x, y\rangle \in (\underbrace{\Theta \cdot \Phi \cdot \Theta \cdots}_{k \text{ factors}}) \cap \Psi.$$

Due to k-submodularity, we have also $\langle x, y\rangle \in \Theta \vee (\Phi \cap \Psi)$ and, by (C), there exist an integer $n > 0$ and elements $p_0, p_1, \ldots, p_n$ of $F_v(x, y, z_1, \cdots, z_{k-1})$ such that $p_0 = x,\ p_n = y$ and $\langle p_i, p_{i+1}\rangle \in \Theta$ for i even

(D) $$\langle p_i, p_{i+1}\rangle \in (\Phi \cap \Psi) \ \text{ for i odd.}$$

Of course, $p_i = p_i(x, z_1, \ldots, z_{k-1}, y)$ for $(k+1)$-ary terms p_i $(i = 0, \ldots, n)$. Since the factor algebras of $F_v(x, y, z_1, \cdots, z_{k-1})$ by Θ or $\Phi \cap \Psi$ are again free algebras of $\mathcal{V}$, the relations (D) give (2) immediately.

(2)$\Rightarrow$(1): Let $\mathcal{V}$ satisfy the identities of (2), let $\mathcal{A} \in \mathcal{V}$ and $\Theta, \Phi, \Psi \in$ *Con* $(\mathcal{A})$, $\Theta \subseteq \Psi$. Suppose

$$\langle a, b\rangle \in (\underbrace{\Theta \cdot \Phi \cdot \Theta \cdots}_{k \text{ factors}}) \cap \Psi.$$

Then $\langle a, b\rangle \in \Psi$ and there exist $c_1, \ldots, c_{k-1} \in A$ such that

$$a\, \Theta\, c_1\, \Phi\, c_2\, \Theta\, c_3\, \ldots b.$$

We have

$$a = p_0(a, c_1, \cdots, c_{k-1}, b),$$

$$b = p_n(a, c_1, \ldots, c_{k-1}, b).$$

Denote by $v_i = p_i(a, c_1, \ldots, c_{k-1}, b)$.

For i odd, we have

$$v_i = p_i(a, c_1, \ldots, c_{k-1}, b) \, \Psi \, p_i(a, a, c_2, c_2, \ldots, a) =$$

$$= p_{i+1}(a, a, c_2, c_2, \ldots, a) \, \Psi \, p_{i+1}(a, c_1, \ldots, c_{k-1}, b)$$

(since $\Theta \subseteq \Psi$), i.e. $\langle v_i, v_{i+1} \rangle \in \Psi$.

Further,

$$a = v_0 = p_0(a, c_1, \ldots, c_{k-1}, b) \, \Theta \, p_0(a, a, c_2, c_2 \ldots) =$$

$$= p_1(a, a, c_2, c_2, \ldots) \, \Theta \, p_1(a, c_1, \ldots, c_{k-1}, b) = v_1 \, \Phi \, p_1(a, c_1, c_1, c_3, c_3 \ldots) =$$

$$= p_2(a, c_1, c_1, c_3, c_3, \ldots) \, \Phi \, p_2(a, c_1, \ldots, c_{k-1}, b) =$$

$$= v_2 \, \Theta \, p_2(a, a, c_2, c_2, \ldots) = \ldots = b.$$

Altogether, we have $a = v_0 \, \Theta \, v_1 (\Phi \cap \Psi) \, v_2 \, \Theta \, v_3 \, (\Phi \cap \Psi) \cdots b$; thus $\langle a, b \rangle \in \Theta \vee (\Phi \cap \Psi)$ proving k-submodularity of $\mathcal{V}$. ∎

Remark 2. By Lemma 1, the identities (2) of Theorem 1 should be easily (trivially) satisfied for $k = 2$ or $k = 3$. Really, one can check that for $k = 2$, we can take $n = 3$ and

$$p_0(x, z, y) = x,$$

$$p_1(x, z, y) = z,$$

$$p_2(x, z, y) = y$$

are terms which satisfy (2) of Theorem 1.

Analogously, for $k = 3$ we can take $n = 4$ and

$$p_0(x, z_1, z_2, y) = y,$$

$$p_1(x, z_1, z_2, y) = z_1,$$

$$p_2(x, z_1, z_2, y) = z_2,$$

$$p_3(x, z_1, z_2, y) = y.$$

Congruence modular varieties were characterized by A. Day in [2]. Analysing his proof, we can find out that he properly proved the following assertion:

Proposition (A. Day). *A variety $\mathcal{V}$ is congruence modular if and only if the free algebra $F_v(x, z_1, z_2, y)$ of $\mathcal{V}$ satisfies*

$$(\Phi \cdot \Theta \cdot \Phi) \cap \Psi \subseteq \Theta \vee (\Phi \cap \Psi)$$

for each $\Theta, \Phi, \Psi \in Con\,(\mathcal{A})$ with $\Theta \subseteq \Psi$.

This result enables us to state

Theorem 2. *A variety $\mathcal{V}$ is congruence modular if and only if it is congruence 4-submodular.*

Proof. Of course, if $\mathcal{V}$ is congruence modular then, by Remark 1, $\mathcal{V}$ is also 4-submodular. Conversely, let $\mathcal{V}$ be 4-submodular and $F_v(x, z_1, z_2, y)$ be the free algebra of $\mathcal{V}$ generated by the free generators x, z_1, z_2, y. Let $\Theta, \Phi, \Psi \in Con\,(F_v(x, z_1, z_2, y))$ with $\Theta \subseteq \Psi$. Then $\Phi \cdot \Theta \cdot \Phi \subseteq \Theta \cdot \Phi \cdot \Theta \cdot \Phi$ thus also

$$(\Phi \cdot \Theta \cdot \Phi) \cap \Psi \subseteq (\Theta \cdot \Phi \cdot \Theta \cdot \Phi) \cap \Psi \subseteq \Theta \vee (\Phi \cap \Psi).$$

Applying the Proposition, $\mathcal{V}$ is congruence modular. ■

As a corollary of Theorem 1 and Theorem 2, we can derive a Maltsev condition for congruence modularity different from that of A. Day [2]:

Corollary *A variety $\mathcal{V}$ is congruence modular if and only if there exist an integer $n > 0$ and 5-ary terms $p_0, \ldots, p_n$ such that $\mathcal{V}$ satisfies the following identities:*

$$p_0(x, z_1, z_2, z_3, y) = x, \quad p_n(x, z_1, z_2, z_3, y) = y,$$

$$p_i(x, x, z, z, y) = p_{i+1}(x, x, z, z, y) \text{ for } i \text{ even,}$$

$$p_i(x, z, z, y, y) = p_{i+1}(x, z, z, y, y) \text{ for } i \text{ odd,}$$

$$p_i(x, x, z, z, x) = p_{i+1}(x, x, z, z, x) \text{ for all } i = 0, 1, \ldots, n-1.$$

One can mention that our terms occuring in the Corollary are more complex then that of A. Day [2], because they are 5-ary but Day's terms are only 4-ary. However, they can become very simple in particular cases as shown in the following:

Example 1. For a variety of groups, one can take $n = 2$ and

$$p_0(x, z_1, z_2, z_3, y) = x,$$

$$p_1(x, z_1, z_2, z_3, y) = z_1 \cdot z_2^{-1} \cdot z_3,$$

$$p_2(x, z_1, z_2, z_3, y) = y.$$

More generally, if $\mathcal{V}$ is a congruence permutable variety and $t(x, y, z)$ its Maltsev term (i.e. $t(x, z, z) = x$ and $t(x, x, z) = z$), then we can take $n = 2$ and

$$p_0(x, z_1, z_2, z_3, y) = x,$$

$$p_1(x, z_1, z_2, z_3, y) = t(x, y, z),$$

$$p_2(x, z_1, z_2, z_3, y) = y$$

which is a bit more simple than for Day's terms.

Now, we show that our Theorem 2 cannot be stated for a single algebra instead of a variety:

Example 2. Let $\mathcal{A} = (A, F)$ be a unary algebra with $A = \{a, b, c, d, e, f, g\}$ and with 3 unary operations s_1, s_2, s_3 defined as follows:

	s_1	s_2	s_3
a	c	e	d
b	d	e	c
c	e	e	b
d	e	f	a
e	e	g	a
f	e	g	b
g	d	f	c

It is an easy excercise to verify that $\mathcal{A}$ has just five congruences, i.e. the identity congruence ω, the full square A^2 and Θ, Φ, Ψ determined by their partitions as follows

Θ........$\{a, b\}, \{c, d\}, \{e, f\}, \{g\}$;

Φ........$\{b, c\}, \{d, e\}, \{f, g\}, \{a\}$;

Ψ........$\{a, b, g\}, \{c, d\}, \{e, f\}$.

Of course, $\Theta \subseteq \Psi$ and one can check easily

$$\Theta \cap \Phi = \omega = \Psi \cap \Phi, \qquad \Theta \vee \Phi = A^2 = \Psi \vee \Phi;$$

thus $Con\,(\mathcal{A}) \simeq N_5$ (the non-modular five element lattice).

Moreover, $\Theta \cdot \Phi \cdot \Theta \cdot \Phi$ is not a congruence on $\mathcal{A}$ since, e.g., $\langle a, e\rangle \in \Theta \cdot \Phi \cdot \Theta \cdot \Phi$ but $\langle e, a\rangle \notin \Theta \cdot \Phi \cdot \Theta \cdot \Phi$.

On the contrary, one can check

$$(\Theta \cdot \Phi \cdot \Theta \cdot \Phi) \cap \Psi = \Theta \subseteq \Theta \vee (\Phi \cap \Psi).$$

The checking for other combinations of congruences is trivial; thus $\mathcal{A}$ is congruence 4-submodular.

References

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