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ON p-SEMIRINGS

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Abstract

A class of semirings, so called p-semirings, characterized by a natural number p is introduced and basic properties are investigated. It is proved that every p-semiring is a union of skew rings. It is proved that for some p-semirings with non-commutative operations, this union contains rings which are commutative and possess an identity.

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1. Introduction

Due to their application in theoretical computer science, semirings have been widely investigated in the last decade. For an extensive list of papers, see the monographs [7] and [8].

The aim of the present paper is to introduce a class of semirings, based on semigroups with some particular properties, as follows.

In the paper [5], a notion of an anti-inverse semigroup was introduced and its properties are described. As a generalization, a p-semigroup, $p \in \mathbb{N}$

is defined and investigated in [2] and [3], so that, for $p = 1$, anti-inverse semigroups are obtained.

In the present paper, we define a p-semiring, $p \in \mathbb{N}$, whose additive semigroup is a p-semigroup. The class of p-semirings does not coincide with any other known class of semirings. A subclass of this class is a variety, as proved in [4].

Among other properties of p-semirings, we prove that they are regular, and that each element of a p-semiring possesses his own additive zero (neutral element). As the main result of the paper, we prove that each p-semiring is covered by skew rings (i.e., by algebras which differ from rings by the single fact that their additive group does not have to be commutative). We also investigate particular p-semirings, generally with non-commutative operations, which are union of rings (commutative, with unit). Finally, we present some examples, and an algorithm for the construction of p-semirings.

2. Preliminaries

We recall some notions and properties of p-semigroups. For more details, see [3].

Let $(S;+)$ be a semigroup and $p \in \mathbb{N}$. For $x \in S$ denote by px the sum $x + x + \ldots + x$ (p-times). Introduce the relation τ_p on $(S;+)$ by:

$$x \tau_p y \text{ if and only if } x + py + x = y \text{ and } py + x + py = x.$$

If $x \tau_p y$ for $x, y \in S$, then py is called a *p-element* of x.

A semigroup $(S,+)$ is called a *p-semigroup* if each element has a p-element.

The following propositions are proved in [3].

Lemma 1. *Let S be a semigroup and $p \in \mathbb{N}$. Then S is a p-semigroup if and only if for each $x \in S$ there is $y \in S$ such that*

$$2x = (p+1)y, \;\; py + x = (2p+1)x + p^2 y, \;\; (4p+1)x = x. \qquad \blacksquare$$

Lemma 2. *For each element of a p-semigroup S, $x + 4px = 4px + x = x$.* $\blacksquare$

By the preceding lemma, in a p-semigroup S every element x possesses *its own zero* $0_x := 4px$.

Lemma 3. *If $x\tau_p y$ in a p-semigroup S, then the following holds:*

(i) *if p is even, then* (a) $p^2y = 0_x$*;* (b) $x + y = y + x$;

(ii) *if p is odd, then $p^2y = py$.* ■

Lemma 4. *Let S be a p-semigroup, where p is an odd number. Then $x\tau_p y$ for each pair $x, y \in S$ if and only if S is a group, each element of which is its own inverse.* ■

Next we recall some definitions concerning semirings.

A *semiring* is a structure $(S; +, \cdot)$ with two binary operations on a nonempty set S, so that both operations are associative, and the second is distributive with respect to the first; in other words, for all $x, y, z \in S$ the following identities hold:

$$x + (y + z) = (x + y) + z, \quad x \cdot (y \cdot z) = (x \cdot y) \cdot z,$$

and

$$x \cdot (y + z) = (x \cdot y) + (x \cdot z), \quad (x + y) \cdot z = (x \cdot z) + (y \cdot z).$$

In the sequel, we sometimes omit the sign and parentheses for the second operation, i.e., in some cases we write xy instead of $(x \cdot y)$ and so on. In addition, as in the case of semigroups, we denote $x + x + \ldots + x$ (n times) by nx (for any $n \in \mathbb{N}$).

By some authors (see [7] and [8]) the first operation is assumed to be commutative, and also a neutral element with respect to the first operation (or both) is supposed to exist. We use the most general definition as above, without these additional requirements.

If A is a nonempty subset of a semiring S, then, as usual, we denote by $\langle A \rangle$ the subsemiring generated by A; in particular, if $A = \{a\}$, we denote the corresponding subsemiring by $\langle a \rangle$.

Recall that a semiring $(S; +, \cdot)$ is *additively regular* if $(S; +)$ is a regular semigroup, i.e., if for each $x \in S$ there is $y \in S$ such that $x = x + y + x$.

We say that a semiring $(S; +, \cdot)$ is a *skew ring* if its additive semigroup $(S; +)$ is a group. Obviously, if $(S; +)$ is an Abelian group, then a semiring (skew ring) $(S; +, \cdot)$ is a ring.

3. Results

Let $(S;+,\cdot)$ be a semiring, and $p\in\mathbb{N}$. Let us define the relation θ_p on S, as follows.

$$x\theta_p y \text{ if and only if the following three equalities hold :}$$

$$x+py+x=y;\ \ py+x+py=x;\ \ 4px^2=4px.$$

Obviously, $x\theta_p y$ if and only if $x\tau_p y$ in the semigroup $(S;+)$ and $4px^2=4px$.

If $x\theta_p y$ in a semiring $(S;+,\cdot)$, then we say that py is a *p-element* of x. A semiring $(S;+,\cdot)$ is a *p-semiring* for fixed $p\in\mathbb{N}$ if each element in S possesses a p-element.

Theorem 5. *Let S be a semiring and $p\in\mathbb{N}$. Then S is a p-semiring if and only if for each $x\in S$ there is $y\in S$ so that the following four equalities hold:*

$$2x=(p+1)y,\ \ py+x=(2p+1)x+p^2y,\ \ (4p+1)x=x,\ \ 4px^2=4px.$$

Proof. Obvious, by Lemma 1 (since $(S;+)$ is a p-semigroup) and by the definition of a p-semiring. ■

Recall that a *zero of an element* x in S under $+$ is an element 0_x, such that $0_x+x=x+0_x=x$.

Corollary 6. *Let $(S;+,\cdot)$ be a p-semiring, for some $p\in\mathbb{N}$. Then:*

(i) *S is an additively regular semiring;*

(ii) *each element of S possesses its own zero 0_x, where $0_x=4px$;*

(iii) *if $x\theta_p y$, then $0_x=0_y$;*

(iv) *if x is an element in S such that $2px=0_x$ and if $x\theta_p y$, then $py+x=x+p^2y$.* ■

Lemma 7. *Let $p\in\mathbf{N}$ and let S be a p-semiring. If $a,b\in S$, $a\theta_p b$, and $m\in\mathbf{N}$, then the following holds:*

(i) *$a\cdot 0_a=0_a\cdot a=0_a$;*

(ii) *$0_a^2=0_a$;*

(iii) $0_a \cdot pb = pb \cdot 0_a = 0_a$;

(iv) $4pa^m = 0_a$;

(v) $a^m + 0_a = 0_a + a^m = a^m$;

(vi) $4p(pb)^m = 0_a$;

(vii) $(pb)^m + 0_a = 0_a + (pb)^m = (pb)^m$.

Proof. Let $a\theta_p b$. Then:

(i) $a \cdot 0_a = a \cdot 4pa = 4pa^2 = 4pa = 0_a$.

Similarly, $0_a \cdot a = 0_a$.

(ii) $0_a^2 = 4pa \cdot 0_a = 4p(a \cdot 0_a) = 4p0_a = 0_a$.

(iii) $0_a \cdot pb = 4\ 0_a \cdot pb = 0_a \cdot 4pb = 0_a \cdot 0_b = 0_a$.

Similarly, $pb \cdot 0_a = 0_a$.

For $m = 1$, equalities (iv) – (vii) are trivial. Let $m \geqslant 2$. Then we have:

(iv) $4pa^m = a^m + a^m + \ldots + a^m = a^{m-1} \cdot (a + a + \ldots + a) = a^{m-1}(4pa) = a^{m-1} \cdot 0_a = 0_a$.

(v) $a^m + 0_a = a^m + 4pa^m = (4p + 1)a^m = a^m$.

Similarly, $0_a + a^m = a^m$.

(vi) $4p(pb)^m = (pb)^m + (pb)^m + \ldots + (pb)^m = (pb)^{m-1} \cdot (pb + pb + \ldots + pb) = (pb)^{m-1} \cdot 4p(pb) = (pb)^{m-1} \cdot p(4pb) = (pb)^{m-1} \cdot 0_a = 0_a$.

(vii) $(pb)^m + 0_a = (pb)^m + 4p(pb)^m = (4p + 1)(pb)^m = (pb)^m$. ■

Proposition 8. *Let S be a p-semiring, where p is an odd number. Then $x\theta_p y$ for each pair x, y of elements in S if and only if S is a ring each element of which is its own additive inverse.*

Proof. Let $x\theta_a y$ for all $x, y \in S$. Then by Lemma 4, we have that $x+x=0$ for all $x \in S$, and the proof of one implication is complete.

Conversely, if $x+x=0$ for all $x \in S$, then again by Lemma 4, $x+py+x=y$ and $py+x+py=x$ for all $x, y \in S$. Now, since $0=2z$ for all $z \in S$, we have that $2(2px^2)=0$, i.e., $4px^2=0=4px$. Therefore, $x\theta_p y$ for arbitrary $x, y \in S$. ∎

Example 1. Let $S=\{e,a,b,c\}$, and let operations $+$ and $\cdot$ be defined by the following tables:

$+$	e	a	b	c
e	e	a	b	c
a	a	e	c	b
b	b	c	e	a
c	c	b	a	e

$\cdot$	e	a	b	c
e	e	e	e	e
a	e	a	e	a
b	e	e	b	b
c	e	a	b	c

It is straightforward to show that it satisfies conditions of Proposition 8. Of course, each Boolean ring is p-semiring for any p.

Next we investigate particular substructures of p-semirings.

Lemma 9. *Let $a\theta_a b_i, i=1,\ldots,m$, in a p-semiring S and let $x=a_1a_2\ldots a_n$, where $a_i \in \{a, pb_1, \ldots, pb_n\}$. Then x possesses an additive zero 0_x; moreover, $0_x=0_a$.*

Proof. Let $x=a_1a_2\ldots a_n$. Then $0_x=4px=a_1a_2\ldots a_n+a_1a_2\ldots a_n+\ldots+a_1a_2\ldots a_n=a_1a_2\ldots a_{n-1}(a_n+a_n+\ldots+a_n)=a_1a_2\ldots a_{n-1}\cdot 4pa_n=a_1a_2\ldots a_{n-1}\cdot 0_a=\ldots=0_a$, after repeating the procedure n times.

Further, $x+0_a=x+0_x=x$, and similarly, $0_a+x=x$. ∎

Let a be an arbitrary element of a p-semiring S. Denote by B_a the set of all p-elements of a:

$$B_a := \{px \in S \mid a\theta_p x\}.$$

In addition, for any subset I_a of B_a, denote by GI_a the subsemiring of S, generated by $\{a\} \cup I_a$:

$$GI_a := \langle \{a\} \cup I_a \rangle.$$

Theorem 10. *Let S be a p-semiring for an arbitrary $p \in \mathbb{N}$, and let $a \in S$. Then for each subset I_a of B_a, the subsemiring $GI_a = \langle \{a\} \cup I_a \rangle$ is a skew ring.*

Proof. Let S be a p-semiring, $a \in S$ and $x \in GI_a$. Then $x = x_1 + x_2 + \ldots + x_k$, where $x_i = a_1^i a_2^i \ldots a_{m_i}^i$, and $a_t^i = a$, or $a_t^i = pb \in I_a$. By Lemma 9, $0_{x_i} = 0_a$ and $x_i + 0_a = 0_a + x_i = x_i$, $i = 1, 2, \ldots, k$. Therefore, $x + 0_a = 0_a + x = x$, and 0_a is the additive zero in GI_a.

Further, let

$$x' = (4p-1)x_k + (4p-1)x_{k-1} + \ldots + (4p-1)x_1.$$

Then,

$$\begin{aligned} x + x' &= x_1 + x_2 + \ldots + x_k + (4p-1)x_k + (4p-1)x_{k-1} + \ldots + (4p-1)x_1 \\ &= x_1 + x_2 + \ldots + x_{k-1} + 0_a + (4p-1)x_{k-1} + \ldots + (4p-1)x_1 \\ &= x_1 + x_2 + \ldots + x_{k-1} + (4p-1)x_{k-1} + \ldots + (4p-1)x_1 \\ &= \ldots = 0_a. \end{aligned}$$

Similarly, $x' + x = 0_a$ and every element has the additive inverse. Hence, GI_a is a skew ring. ∎

Corollary 11. *If $p \in \mathbb{N}$ and S is a p-semiring, then*

$$S = \bigcup_{a \in S} \langle a \rangle.$$

In other words, every p-semiring is a union of skew rings. ∎

From the above, it is clear that a p-semiring whose additive semigroup is commutative, is a union of rings. The converse is not true in general, i.e., the fact that a p-semiring is a union of rings does not imply additive commutativity. This is shown by the class of p-semirings constructed in the following example.

Example 2. We describe a construction of a disjoint union of rings which is a p-semiring, but not a ring. Let $(S_i, +_i, \cdot_i)$, $i = 1, 2$ be two rings, which are also p-semirings. Let 0_1 and 0_2 be additive zeros in S_1 and S_2, respectively.

On the set $S = S_1 \cup S_2$ define operations $+$ and $\cdot$ as follows: for $x_i \in S_i, y_j \in S_j$, $i, j = 1, 2$

$$x_i + y_j := \begin{cases} x_i +_i y_j, & \text{if} \quad i = j \\ y_j, & \text{if} \quad i < j \\ x_i, & \text{if} \quad j < i. \end{cases}$$

$$x_i \cdot y_j := \begin{cases} x_i \cdot_i y_j, & \text{if} \quad i = j \\ 0_i, & \text{if} \quad i \neq j. \end{cases}$$

It is easy to check that $(S, +, \cdot)$ is a p-semiring, but not a ring.

Let S be a semiring with the property that for every $x \in S$ there is $n \in \mathbb{N}$ such that $x^{n+1} = x$. Obviously, in terms of semigroups, elements of such semiring are periodic under multiplication, with index 1. Therefore we say that S is a *multiplicatively periodic semiring.* In the following we investigate multiplicatively periodic p-semirings.

Lemma 12. *Every multiplicatively periodic skew ring is a commutative ring.*

Proof. Let S be an multiplicatively periodic skew ring. We have to prove that both operations are commutative.

Let $x, y \in S$. Then, there are $m, n \in \mathbb{N}$, such that $x^{n+1} = x$, and $y^{m+1} = y$. Further,

$$\begin{aligned} (x + y)(y^m + x^n) &= x \cdot (y^m + x^n) + y \cdot (y^m + x^n) \\ &= x \cdot y^m + x^{n+1} + y^{m+1} + y \cdot x^n \\ &= x \cdot y^m + x + y + y \cdot x^n. \end{aligned}$$

On the other hand,

$$\begin{aligned} (x + y)(y^m + x^n) &= (x + y) \cdot y^m + (x + y) \cdot x^n \\ &= x \cdot y^m + y^{m+1} + x^{n+1} + y \cdot x^n \\ &= x \cdot y^m + y + x + y \cdot x^n. \end{aligned}$$

Hence, since S is a group under addition, we get $x + y = y + x$, and S is a ring.

The second part is a well known Theorem of Jacobson, (see, e.g., [9]): A ring in which every element x satisfies the equality $x^{n+1} = x$ for some $n \in \mathbb{N}$, is commutative. ■

Corollary 13. *Let S be a multiplicatively periodic p-semiring. Then, for each $a \in S$, GI_a is a commutative ring.*

Proof. By Theorem 10 and Lemma 12. ■

Due to Corollary 11, it is obvious that every multiplicatively periodic p-semiring is a union of commutative rings.

Next we prove more, namely that each multiplicatively periodic p-semiring is a union of commutative rings with identity.

We use the following lemma.

Lemma 14. *Let S be a multiplicatively periodic p-semiring. Then $2px = 0_x$ for each $x \in S$.*

Proof. Let S be a multiplicatively periodic p-semiring for some $p \in \mathbb{N}$. Since $x^{n+1} = x$ for some $n \in \mathbb{N}$, we have

$$2x = (x + x)^{n+1} = x^{n+1} + x^{n+1} + \ldots + x^{n+1} = x + x + \ldots + x = 2^{n+1}x.$$

So we have $2x = 2^{n+1}x$. Now since $4px = 0_x$, it follows that $4k(px) = 0_x$ for each $k \in \mathbb{N}$, hence also for $k = 2^{n-1}$. Therefore, $4p(2^{n-1}x) = 0_x$, i.e., $p(2^{n+1}x) = 0_x$, and hence $p(2x) = 0_x$, finally $2px = 0_x$. ■

Proposition 15. *Let S be a multiplicatively periodic p-semiring for some $p \in \mathbb{N}$. If $a\theta_p b$ in S and $I_a = \{pb\}$, then GI_a is a commutative ring with identity.*

Proof. GI_a is a commutative ring by Corollary 13 and we prove that it has an identity.

By Lemma 14, we have $2pb = 0_b$, and by Corollary 6 (iii), $2pb = 0_a$, since $a\theta_p b$. Therefore, for any $r, s \in \mathbb{N}$, we have $2a^r(pb)^s = a^r(pb)^{s-1}(2pb) = a^r(pb)^{s-1} \cdot 0_a = 0_a$, where $a^r(pb)^0 = a^r$. In addition, for $s > 1$ we have $2(pb)^s = (pb)^{s-1}(2pb) = (pb)^{s-1} \cdot 0_a = 0_a$.

Since S is multiplicatively periodic, we have that $a^{m+1} = a$ and $(pb)^{n+1} = pb$, for some $m, n \in \mathbb{N}$. We prove that the identity in the ring GI_a is $a^m + (pb)^n + a^m(pb)^n$.

Observe that each element of the ring GI_a can be represented by

$$x = j_1^{(0)}a + j_2^{(0)}a^2 + \ldots + j_m^{(0)}a^m + j_1 a^{u_1}(pb)^{v_1} + j_2 a^{u_2}(pb)^{v_2} + \ldots + j_t a^{u_t}(pb)^{v_t} =$$

$$= \sum_{i=1}^{m} j_i^{(0)} a^i + \sum_{i=1}^{t} j_i a^{u_i}(pb)^{v_i},$$

where $j_i^{(0)} \in \{0, 1, \ldots, k-1\}$, $i = 1, \ldots, m$ (k is the smallest positive integer such that $ka = 0_a$), $j_i \in \{0, 1\}$, $u_i \in \{0, 1, \ldots, n\}$, $v_i \in \{0, 1, \ldots, m\}$, $i = 1, 2, \ldots, t$, $t \in \mathbb{N}$. We also define $0a^i = 0_a$, $0a^{u_i}(pb)^{v_i} = 0_a$, $j_i a^0 (pb)^{v_i} = j_i(pb)^{v_i}$, $j_i a^{u_i}(pb)^0 = j_i a^{u_i}$, $j_i a^0(pb)^0 = 0_a$ ($i \in \mathbb{N}$). Further,

$$\begin{aligned}
x \cdot (a^m + (pb)^n + a^m(pb)^n) &= x \cdot a^m + x \cdot (pb)^n + x \cdot a^m(pb)^n \\
&= \sum_{i=1}^{m} j_i^{(0)} a^{m+i} + \sum_{i=1}^{t} j_i a^{m+u_i}(pb)^{v_i} + \sum_{i=1}^{m} j_i^{(0)} a^i (pb)^n \\
&\quad + \sum_{i=1}^{t} j_i a^{u_i}(pb)^{n+v_i} + \sum_{i=1}^{m} j_i^{(0)} a^{m+i}(pb)^n + \sum_{i=1}^{t} j_i a^{m+u_i}(pb)^{n+v_i} \\
&= \sum_{i=1}^{m} j_i^{(0)} a^i + \sum_{i=1}^{t} j_i a^{u_i}(pb)^{v_i} + \sum_{i=1}^{m} j_i^{(0)} a^i (pb)^n \\
&\quad + \sum_{i=1}^{t} j_i a^{u_i}(pb)^{v_i} + \sum_{i=1}^{m} j_i^{(0)} a^i (pb)^n + \sum_{i=1}^{t} j_i a^{u_i}(pb)^{v_i} \\
&= \sum_{i=1}^{m} j_i^{(0)} a^i + 2\sum_{i=1}^{t} j_i a^{u_i}(pb)^{v_i} + 2\sum_{i=1}^{m} j_i^{(0)} a^i (pb)^n + \sum_{i=1}^{t} j_i a^{u_i}(pb)^{v_i} \\
&= \sum_{i=1}^{m} j_i^{(0)} a^i + \sum_{i=1}^{t} j_i (2a^{u_i}(pb)^{v_i}) + \sum_{i=1}^{m} j_i^{(0)} (2a^i (pb)^n) + \sum_{i=1}^{t} j_i a^{u_i}(pb)^{v_i} \\
&= \sum_{i=1}^{m} j_i^{(0)} a^i + 0_a + 0_a + \sum_{i=1}^{t} j_i a^{u_i}(pb)^{v_i} = x.
\end{aligned}$$

Thus, GI_a is a commutative ring with identity $a^m + (pb)^n + a^m(pb)^n$. ■

Corollary 16. *Every multiplicatively periodic p-semiring is a union of commutative rings with identity.* ■

References

[1] B. Budimirović, *On a class of p-semirings*, M.Sc. Thesis, Faculty of Sciences, University of Novi Sad, 2001.

[2] V. Budimirović, *A Contribution to the Theory of Semirings*, Ph.D. Thesis, Fac. of Sci., University of Novi Sad, Novi Sad, 2001.

[3] V. Budimirović, *On p-semigroups*, Math. Moravica **4** (2000), 5–20.

[4] V. Budimirović and B. Šešelja, *Operators H, S and P in the classes of p-semigroups and p-semirings*, Novi Sad J. Math. **32** (2002), 127–132.

[5] S. Bogdanović, S. Milić and V. Pavlović, *Anti-inverse semigroups*, Publ. Inst. Math. (Beograd) (N.S.) **24** (**38**) (1978), 19–28.

[6] K. Głazek, *A guide to the Literature on Semirings and their Applications in Mathematics and Information Sciences*, Kluwer Acad. Publ. Dordrecht 2002.

[7] J.S. Golan, *The theory of semirings with applications in mathematics and theoretical computer sciences*, Longman Scientific & Technical, Harlow 1992.

[8] U. Hebisch and H.J. Weinert, *Semirings, Algebraic theory and applications in mathematics and computer sciences*, World Scientific, Singapore 1999.

[9] I.N. Herstein, *Wedderburn's Theorem and a Theorem of Jacobson*, Amer. Math. Monthly **68** (1961), 249–251.

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