Discussiones Mathematicae
General Algebra and Applications 22 (2002) 87–100

EQUATIONAL BASES FOR WEAK MONOUNARY VARIETIES

Grzegorz Bińczak

Institute of Mathematics, Warsaw University of Technology
pl. Politechniki 1, 00–661 Warszawa, Poland
e-mail: binczak@mini.pw.edu.pl

Abstract

It is well-known that every monounary variety of total algebras has one-element equational basis (see [5]). In my paper I prove that every monounary weak variety has at most 3-element equational basis. I give an example of monounary weak variety having 3-element equational basis, which has no 2-element equational basis.

Keywords: partial algebra, weak equation, weak variety, regular equation, regular weak equational theory, monounary algebras.

2000 AMS Mathematics Subject Classifications: 08A55, 08B05.

1. Introduction

Weak equations and varieties were studied by H. Höft [4]. An algebraic characterization of weak varieties, under a condition named "conflict free", is shown in [7]. A completeness theorem for weak equational logic was given by L. Rudak in [6]. G. Bińczak [1] characterized weak varieties as classes closed under homomorphic images and mixed products.

Basic definitions and facts about partial algebras can be found in [3] (Chapter 2) and in [2].

In this section we set up notation and terminology.

Definition 1.1. Let n be a natural number and A a set. A relation $f \subseteq A^n \times A$ is called an *n-ary partial operation in the set A* if and only if for every $a \in A^n$, $b, c \in A$, if $(a, b) \in f$ and $(a, c) \in f$, then $b = c$. If f is an n-ary partial operation in a set A, then $\mathrm{dom}(f) = \{a \in A^n \colon \exists_{b \in A}(a, b) \in f\}$.

The notation $f(a) = b$ means that $(a, b) \in f$.

Definition 1.2. A pair (F, η) is called a *type* or (*signature*) (of algebras), if F is an arbitrary set and $\eta \colon F \to \omega$. A type (F, η) is *monounary* if and only if a set F has exactly one element f and $\eta(f) = 1$. Let (F, η) be a type. Then a pair $\underline{A} = (A, (f^{\underline{A}})_{f \in F})$ is called *partial algebra* of type (F, η) if and only if $A \neq \varnothing$ and, for every $f \in F$, $f^{\underline{A}}$ is an $\eta(f)$-ary partial operation in A. We call A the *support* of $\underline{A}$. A partial algebra $\underline{A} = (A, (f^{\underline{A}})_{f \in F})$ is called *monounary* if and only if it is of some monounary type.

In the sequel we fix a monounary type (F, η) $(F = \{f\})$ and we will consider only partial algebras of this type.

Let X be a countable set of variables. The usual *monounary total term algebra* generated by a set X will be denoted by $\underline{T}(X)$. Every monounary term is of the form $x^n = \underbrace{f \ldots f}_{n \text{ times}} x$, where $x \in X$, n is a natural number and f is an operation symbol. If $n = 0$, then x^n denotes x. If $\underline{A} = (A, f^{\underline{A}})$ is a monounary partial algebra and $p = \underbrace{f \ldots f}_{n \text{ times}} x = x^n$ is a term, then the term operation $p^{\underline{A}}$ is a monounary partial operation in the set A such that $p^{\underline{A}}(a) = \underbrace{f(f(\ldots f}_{n \text{ times}}(a) \ldots))$ if $a \in \mathrm{dom}(p^{\underline{A}})$. The domain of $(x^n)^{\underline{A}}$ is defined inductively:

$\mathrm{dom}((x^0)^{\underline{A}}) = A$ and

$$\mathrm{dom}\left((x^n)^{\underline{A}}\right) = \{a \in A : a \in \mathrm{dom}\left(\left(x^{n-1}\right)^{\underline{A}}\right) \text{ and } \left(x^{n-1}\right)^{\underline{A}}(a) \in \mathrm{dom}\left(f^{\underline{A}}\right)\}.$$

If $p, r \in T(X)$, $z \in X$, $p = x^k$ and $r = y^n$ for some $x, y \in X$ and $k, n \in N$, then

$$p(z/r) = \begin{cases} y^{k+n}, & \text{if } z = x, \\ p, & \text{if } z \neq x. \end{cases}$$

A pair of terms $(p,q) \in T(X)^2$ (where $p = x^k$ and $q = y^m$ for some $x, y \in X$ and $k, m \in N$) is a *weak equation* in a partial algebra $\underline{A}$ ($\underline{A} \models p \approx q$) iff for every $a, b \in A$,

$$\text{if } x = y, \text{ then } a \in \mathrm{dom} p^{\underline{A}} \cap \mathrm{dom} q^{\underline{A}} \text{ implies } p^{\underline{A}}(a) = q^{\underline{A}}(a)$$

and

$$\text{if } x \neq y, \text{ then } a \in \mathrm{dom}\left(p^{\underline{A}}\right) \text{ and } b \in \mathrm{dom}\left(q^{\underline{A}}\right) \text{ implies } p^{\underline{A}}(a) = q^{\underline{A}}(b).$$

Instead of (p,q) we will write $p \approx q$.

Let $E \subseteq T(X)^2$, K be a class of algebras and $\underline{B} \in K$. We write:

$$K \models p \approx q \text{ iff for every } \underline{A} \in K, \qquad \underline{A} \models p \approx q,$$

$$\underline{B} \models E \text{ iff for every } p \approx q \in E, \qquad \underline{B} \models p \approx q.$$

Let $\mathrm{Eq}_w(K) = \{(p,q) \in T(X)^2 \colon K \models p \approx q\}$ and $\mathrm{Mod}_w(E) = \{\underline{A} \colon \underline{A} \models p \approx q$ for every $(p,q) \in E\}$. A class K of algebras is a *weak variety* iff $K = \mathrm{Mod}_w(\mathrm{Eq}_w(K))$. An algebraic characterization of weak varieties is shown in [1].

A set $I \subseteq T(X)$ is an *initial segment* iff for every $x^n \in I$, if $m \in N$ and $m < n$ then $x^m \in I$. A set $E \subseteq T(X)^2$ is an *equational basis* of a weak variety K iff $\mathrm{Mod}_w(E) = K$. A set $E \subseteq T(X)^2$ is a *weak equational theory* iff $E = \mathrm{Eq}_w(K)$ for some class of algebras K; equivalently (see [6]) it is closed under the following rules:

R1 $\frac{}{p \approx p}$ (reflexivity);

R2 $\frac{p \approx q}{q \approx p}$ (symmetry);

R3 $\frac{p \approx r_1, r_1 \approx r_2, \ldots, r_n \approx q}{p \approx q}$ if $r_1, \ldots, r_n \in D_E(p,q)$, where $D_E(p,q)$ is the smallest initial segment $I \subseteq T(X)$ such that $X \cup \{p,q\} \subseteq I$; and if $r \in I$, $f(s) \in I$ and $r \approx s \in E$, then $f(r) \in I$ (weak transitivity);

R4 $\frac{p \approx q}{f(p) \approx f(q)}$;

R5 $\frac{p \approx q}{p(x/r) \approx q(x/r)}$ for some $x \in X$ and $r \in T(X)$ (substitution).

Weak equational theory $E \subseteq T(X)^2$ is *nontrivial* iff there exist $p, q \in T(X)$ such that $p \neq q$ and $p \approx q \in E$. If $E_1, E_2 \subseteq T(X)^2$, then $E_1 \vdash E_2$ iff E_2 follows from E_1 by above rules, equivalently: for every weak equational theory E, if $E_1 \subseteq E$, then $E_2 \subseteq E$. If E is a weak equational theory, then a subset $E_0 \subseteq E$ is a *basis* of E iff E_0 is an equational basis of $\mathrm{Mod}_w(E)$, equivalently: $E_0 \vdash E$. Moreover, if $E_1, E_2 \subseteq T(X)^2$, $E_1 \vdash E_2$ and $E_2 \vdash E_1$, then E_1 is a basis of E iff E_2 is a basis of E.

Definition 1.3. An equation $p \approx q \in T(X)^2$ is *regular* if and only if $p = x^n$ and $q = x^m$ for some $x \in X$ and $n, m \in N$. A weak equational theory E is *regular* if and only if every equation in E is regular.

2. Regular weak equational theories

In this section we prove (Corollary 2.10) that every regular weak equational theory has a 2-element basis.

Lemma 2.1. *Let E be a weak equational theory, $n \geq 1$ and $k \geq 0$. If $x^k \approx x^{k+n} \in E$ and $m \geq k$, then $x^m \approx x^{m+rn} \in E$ for every $r \geq 0$.*

Proof. By rule R5, $x^k(x/x^{m-k}) \approx x^{k+n}(x/x^{m-k}) \in E$, so $x^m \approx x^{m+n} \in E$, which proves lemma for $r = 1$. Suppose that we prove lemma for $r \leq l$. Let $x^k \approx x^{k+n} \in E$ and $m \geq k$. Then $x^m \approx x^{m+ln} \in E$ and $x^{m+ln} \approx x^{m+ln+n} \in E$ (since $m + ln \geq k$). By rule R3, $x^m \approx x^{m+ln+n} \in E$, so $x^m \approx x^{m+(l+1)n} \in E$. ■

Lemma 2.2. *Let E be a weak equational theory, $k, n, m \geq 0$. If $x^k \approx x^{k+n} \in E$ and $x^k \approx x^{k+m} \in E$, then $x^k \approx x^{k+n+m} \in E$.*

Proof. By Lemma 2.1 $x^{k+m} \approx x^{k+m+n} \in E$ (since $x^k \approx x^{k+n} \in E$ and $k + m \geq k$). Therefore, $x^k \approx x^{k+m} \in E$ and $x^{k+m} \approx x^{k+n+m} \in E$. By rule R3, $x^k \approx x^{k+n+m} \in E$. ■

Corollary 2.3. *Let E be a weak equational theory. Let $l \geq 1$, $k, a_i, n_i \geq 0$ for $1 \leq i \leq l$. If $x^k \approx x^{k+n_1}, \ldots, x^k \approx x^{k+n_l} \in E$, then $x^k \approx x^{k+a_1n_1+\ldots a_ln_l}$ $\in E$.*

Lemma 2.4. *Let E be a weak equational theory, $n \geq 1$, $k \geq 0$ and $x \in X$. If $x^k \approx x^{k+n} \in E$, $p, q \geq 0$ and* $\max(p,q) \geq k+n$, *then for every $s \in N$, $x^s \in D_E(x^p, x^q)$.*

Proof. Suppose that there exists $s \in N$ such that $x^s \notin D_E(x^p, x^q)$. Let $r = \min\{s \in N : x^s \notin D_E(x^p, x^q)\}$. Since $x = x^0 \in D_E(x^p, x^q)$, we have $r > 0$. Moreover, $r > p$ and $r > q$, since $x^p, x^q \in D_E(x^p, x^q)$ and $D_E(x^p, x^q)$ is an initial segment. Therefore, $r - 1 \geq \max(p,q) \geq k+n$, $r-1-n \geq k$ and, by Lemma 2.1,

$$x^{r-1-n} \approx x^{r-1} \in E,$$

since $x^k \approx x^{k+n} \in E$. Moreover, $x^{r-1-n}, x^{r-1} \in D_E(x^p, x^q)$ by definition of r and $f(x^{r-1-n}) = x^{r-n} \in D_E(x^p, x^q)$, since $n \geq 1$. By definition of $D_E(x^p, x^q)$ (cf. [R3]), $f(x^{r-1}) = x^r \in D_E(x^p, x^q)$ and we have a contradiction with definition of r. ∎

Lemma 2.5. *Let E be a nontrivial weak equational theory. If $r, d \geq 1$, $s \geq r$, $k, d_0 \geq 0$, $x^k \approx x^{k+rd} \in E$ and $x^{d_0} \approx x^{d_0+d} \in E$, then $x^k \approx x^{k+sd} \in E$.*

Proof. There exists $a \geq 1$ such that $k + ard > d_0$. Then $x^{k+ard} \approx x^{k+ard+sd} \in E$, by Lemma 2.1, since $x^{d_0} \approx x^{d_0+d} \in E$. Moreover, $x^k \approx x^{k+ard}, x^{k+sd} \approx x^{k+sd+ard} \in E$, by Lemma 2.1, since $x^k \approx x^{k+rd} \in E$. Therefore,

$$x^k \approx x^{k+ard}, x^{k+ard} \approx x^{k+ard+sd}, x^{k+ard+sd} \approx x^{k+sd} \in E$$

and, by Lemma 2.4, $x^{k+ard}, x^{k+ard+sd} \in D_E(x^k, x^{k+sd})$, since $x^k \approx x^{k+rd} \in E$ and $\max(k, k+sd) = k+sd \geq k+rd$. Hence, by rule R3, $x^k \approx x^{k+sd} \in E$.

∎

Definition 2.6. Let $x \in X$ and E be a nontrivial weak equational theory. Define $R_x(E) = \{n > 0$: there exists $k \geq 0$ such that $x^k \approx x^{k+n} \in E\}$. By rule R5, $R_x(E) = R_y(E)$ for every $x, y \in X$. So, we can write $R(E)$ instead of $R_x(E)$.

Lemma 2.7. *Let E be a nontrivial weak equational theory. Then*

1. *if $n_1, n_2 \in R(E)$, then $n_1 + n_2 \in R(E)$,*
2. *if $n_1, n_2 \in R(E)$ and $n_1 - n_2 > 0$, then $n_1 - n_2 \in R(E)$,*
3. *if $n \in R(E)$ and $r \geq 0$, then $rn \in R(E)$.*

Proof.

1. If $n_1, n_2 \in R(E)$, then there exist $k_1, k_2 \geq 0$ such that $x^{k_1} \approx x^{k_1+n_1} \in E$ and $x^{k_2} \approx x^{k_2+n_2}$. Let $k = k_1 + k_2$. By Lemma 2.1, $x^k \approx x^{k+n_1} \in E$ and $x^k \approx x^{k+n_2} \in E$. Hence, $x^k \approx x^{k+n_1+n_2} \in E$ by Lemma 2.2. Therefore, $n_1 + n_2 \in R(E)$.

2. If $n_1, n_2 \in R(E)$ and $n_1 - n_2 > 0$, then there exist $k_1, k_2 \geq 0$ such that $x^{k_1} \approx x^{k_1+n_1} \in E$ and $x^{k_2} \approx x^{k_2+n_2}$. Let $k = k_1 + k_2$. By Lemma 2.1, $x^k \approx x^{k+n_1} \in E$ and $x^k \approx x^{k+n_2} \in E$. Therefore, $x^{k+n_2} \approx x^k \in E$ and $x^k \approx x^{k+n_1} \in E$. By rule R3, $x^{k+n_2} \approx x^{k+n_1} \in E$. Hence, $x^{k+n_2} \approx x^{k+n_2+(n_1-n_2)} \in E$ and $n_1 - n_2 \in R(E)$.

3. If $n \in R(E)$ and $r \geq 0$, then there exists $k \geq 0$ such that $x^k \approx x^{k+n} \in E$. By Lemma 2.1, $x^k \approx x^{k+rn} \in E$. Hence $rn \in R(E)$. ■

Corollary 2.8. *Let E be a nontrivial weak equational theory. If $a_1, \ldots, a_n \in R(E)$, then $\gcd(a_1, \ldots, a_n) \in R(E)$.*

Proof. Let $d = \gcd(a_1, \ldots, a_n)$. Then $d = b_1a_1 + \ldots + b_ia_i + c_{i+1}a_{i+1} + \ldots + c_na_n$, where $b_j \geq 0$ and $c_j < 0$. By Lemma 2.7, $d_1 = b_1a_1 + \ldots + b_ia_i \in R(E)$ and $d_2 = -(c_{i+1}a_{i+1} + \ldots + c_na_n) \in R(E)$. Hence, $d = d_1 - d_2 \in R(E)$ by Lemma 2.7. ■

If E is a nontrivial weak equational theory, then a set $R(E)$ is infinite. Suppose that $R(E) = \{a_1, \ldots, a_n, \ldots\}$. Let $d_n = \gcd(a_1, \ldots, a_n)$ for $n \geq 1$. Then we have a sequence $d_1 \geq d_2 \geq \ldots > 0$. Therefore, there exists $n \geq 1$ such that $d_n = d_{n+1} = \ldots$. By Corollary 2.8, $d = d_n \in R(E)$ and $d = gcd(R(E))$ (i.e. $d|k$ for every $k \in R(E)$ and if there exists $d_0 \geq 1$ such that $d_0|k$ for every $k \in R(E)$, then $d|d_0$). Moreover, $R(E) = \{kd \in N{:}\, k > 0 \text{ and } k \in N\}$.

Lemma 2.9. *Let E be a nontrivial weak equational theory. Let $d = \gcd(R(E))$, $x \in X$ and $d_0 = \min\{k \geq 0\colon x^k \approx x^{k+d} \in E\}$. Let $k_0 = \min\{k \geq 0\colon$ there exists $n > k$ such that $x^k \approx x^n \in E\}$. Let further $l_0 = \min\{k \geq 0\colon x^{k_0} \approx x^{k_0+k} \in E\}$ and*

$$E_0 = \{x^{k_0} \approx x^{k_0+l_0}, x^{d_0} \approx x^{d_0+d}\}.$$

Then $E_0 \subseteq E$ and for every weak equational theory E' such that $E_0 \subseteq E'$ and for every $y^k \approx y^m \in E$, we have $y^k \approx y^m \in E'$.

Proof. Let E' be a weak equational theory such that $E_0 \subseteq E'$ and $y^k \approx y^m \in E$. We show that $y^k \approx y^m \in E'$. By rule R2 (symmetry), we can assume that $m \geq k$. Let $m = k + n$ for some $n \geq 0$. We show that $y^k \approx y^{k+n} \in E'$.

If $n = 0$, then $y^k \approx y^{k+n} \in E'$ by rule R1.

By rule R5, $E_0' = \{y^{k_0} \approx y^{k_0+l_0}, y^{d_0} \approx y^{d_0+d}\} \subseteq E' \cap E$.

Suppose that $n > 0$. Then $n \in R(E)$ and $d|n$. Hence, there exists $r \geq 1$ such that $n = rd$.

If $d_0 \leq k$, then $y^k \approx y^{k+rd} \in E'$ by Lemma 2.1, since $y^{d_0} \approx y^{d_0+d} \in E'$.

Suppose that $d_0 > k$. Since $k_0 \leq k$ by definition of k_0, we have $k_0 \leq k < d_0$.

We know that $l_0 \in R(E)$. Hence, there exists $r_0 \geq 1$ such that $l_0 = r_0 d$. Suppose that $r \geq r_0$. Then $y^k \approx y^{k+l_0} \in E'$ by Lemma 2.1, since $y^{k_0} \approx y^{k_0+l_0} \in E'$ and $k_0 \leq k$. Therefore, $y^k \approx y^{k+rd} \in E'$ by Lemma 2.5, since $y^k \approx y^{k+r_0 d} \in E'$, $y^{d_0} \approx y^{d_0+d} \in E'$ and $r \geq r_0$. Hence, $y^k \approx y^{k+n} \in E'$, since $n = rd$.

Suppose that $r_0 > r$. We show that $k + n \geq d_0 + d$.

Suppose that $k + n < d_0 + d$. Then $d_0 - 1 + d \geq k + n$ and $y^{d_0-1+nd} \in D_E(y^{d_0-1}, y^{d_0-1+d})$ by Lemma 2.4, since $d_0-1+d \geq k+n$. By Lemma 2.1, we have $y^{d_0-1} \approx y^{d_0-1+nd} \in E$ and $y^{d_0-1+nd} \approx y^{d_0-1+d} \in E$, since $d_0 - 1 \geq k$, $d_0 - 1 + d \geq d_0$, $n - 1 \geq 0$, $y^k \approx y^{k+n} \in E$ and $y^{d_0} \approx y^{d_0+d} \in E$. We have

$$y^{d_0-1} \approx y^{d_0-1+nd}, y^{d_0-1+nd} \approx y^{d_0-1+d} \in E$$

and $y^{d_0-1+nd} \in D_E(y^{d_0-1}, y^{d_0-1+d})$. By rule R3, we obtain $y^{d_0-1} \approx y^{d_0-1+d} \in E$ and we have a contradiction with definition of d_0.

We know that $y^k \approx y^{k+r_0d} \in E'$, since $k \geq k_0$ and $y^{k_0} \approx y^{k_0+r_0d} = y^{k_0} \approx y^{k_0+l_0} \in E'_0 \subseteq E'$. Moreover, $k + rd = k + n \geq d_0 + d > d_0$ and $k+r_0d-(k+rd) = (r_0-r)d \geq 0$ implies $y^{k+rd} \approx y^{k+r_0d} \in E'$ by Lemma 2.1. We have $y^{k+r_0d} \in D_{E'}(y^k, y^{k+n})$, by Lemma 2.4, since $k + n \geq d_0 + d$ and $y^{d_0} \approx y^{d_0+d} \in E'_0 \subseteq E'$. Therefore,

$$y^k \approx y^{k+r_0d}, y^{k+r_0d} \approx y^{k+rd} \in E', y^{k+r_0d} \in D_{E'}(y^k, y^{k+rd})$$

and, by rule R3, $y^k \approx y^{k+rd} = y^k \approx y^{k+n} \in E'$. ■

Corollary 2.10. *Every regular weak equational theory E has a 2-element basis.*

Proof. If E is not nontrivial weak equational theory, then $E = \{(p,p) \in T(X)^2 \colon p \in T(X)\}$ and every 2-element subset of E is a basis of E.

Suppose that E is nontrivial regular weak equational theory. Let $d = \gcd(R(E)) \in R(E)$. Fix $x \in X$. Then the set $\{k \geq 0 \colon x^k \approx x^{k+d} \in E\}$ is not empty. Let $d_0 = \min\{k \geq 0 \colon x^k \approx x^{k+d} \in E\}$. Let $k_0 = \min\{k \geq 0$: there exists $n > k$ such that $x^k \approx x^n \in E\}$. Let $l_0 = \min\{k \geq 0$: $x^{k_0} \approx x^{k_0+k} \in E\}$. We show that

$$E_0 = \{x^{k_0} \approx x^{k_0+l_0}, x^{d_0} \approx x^{d_0+d}\}$$

is a basis of E.

We know that $E_0 \subseteq E$ by definitions of d_0, l_0, k_0.
Let E' be a weak equational theory such that $E_0 \subseteq E'$. We show that $E \subseteq E'$. Let $z^k \approx y^m \in E$. Then $z = y$, since E is regular and, by Lemma 2.9, we have $y^k \approx y^m \in E'$. Therefore, $E \subseteq E'$ and E_0 is a basis of E. ■

3. Main theorem

Lemma 3.1. *Let E be a weak equational theory, $p, q \in N$, $x, y \in X$ and $x \neq y$. If $x^p \approx y^q \in E$, $p' \geq p$ and $q' \geq q$, then $x^{p'} \approx y^{q'} \in E$.*

Proof. If $x^p \approx y^q \in E$, then $x^p(x/x^{p'-p}) \approx y^q(x/x^{p'-p}) = x^{p'} \approx y^q \in E$ by rule R5. Hence, $x^{p'}(y/y^{q'-q}) \approx y^q(y/y^{q'-q}) = x^{p'} \approx y^{q'} \in E$ by rule R5. ∎

Lemma 3.2. *Let E be a weak equational theory, $n \geq 1$, $x, y \in X$ and $x \neq y$. Let $p, q, k \in N$. If $x^k \approx x^{k+n}, x^p \approx y^q \in E$, then $x^p \approx y^{k+n} \in E$.*

Proof. There exists $a \geq 1$ such that $k + an > p$. By Lemma 3.1, $x^{k+an} \approx y^q \in E$. Hence $y^{k+an} \approx y^q \in E$ by rule R5 and $y^q \approx y^{k+an} \in E$ by rule R2.

By rule R5, $y^k \approx y^{k+n} \in E$ (since $x^k \approx x^{k+n} \in E$). Therefore, $y^{k+n} \approx y^{k+an} \in E$ by Lemma 2.1, since $k + n \geq k$, $k + an = k + n + (a-1)n$ and $a - 1 \geq 0$. By rule R2, we have $y^{k+an} \approx y^{k+n} \in E$.
We have

$$x^p \approx y^q, y^q \approx y^{k+an}, y^{k+an} \approx y^{k+n} \in E$$

and $y^q, y^{k+an} \in D_E(x^p, y^{k+n})$ by Lemma 2.4, since $y^k \approx y^{k+n} \in E$. Hence, $x^p \approx y^{k+n} \in E$ by rule R3. ∎

Lemma 3.3. *Let E be a weak equational theory, $x, y \in X$ and $x \neq y$. If $m < l$, $x^n \approx y^k \in E$ and $x^m \approx y^l \in E$, then $x^n \approx y^l \in E$.*

Proof. By rule R5, $x^m \approx x^l \in E$. Hence, $x^n \approx y^l$ by Lemma 3.2, since $l = m + (l - m)$ and $l - m \geq 1$. ∎

Theorem 3.4. *Every monounary weak variety of partial algebras has an at most 3-element equational basis.*

Proof. Let V be a weak monounary variety and $E = \mathrm{Eq}_w(V) \subseteq T_F(X)^2$. We show that E has at most 3-element basis.

If E is a trivial weak equational theory, then $E = \{(p,p) \in T(X)^2\colon p \in T(X)\}$ and every 3-element subset of E is a basis of E.

If E is a regular weak equational theory, then E has a 2-element basis by Corollary 2.10.

Suppose that E is not regular. Then there exist $x, y \in X$, $p, q \in N$ such that $x^p \approx y^q \in E$ and $x \neq y$. By Lemma 3.1, $x^{\max(p,q)} \approx y^{\max(p,q)} \in E$ and the set $\{n \geq 0\colon\ x^n \approx y^n \in E\}$ is not empty.

Let $m = \min\{n \geq 0\colon x^n \approx y^n \in E\}$ and $d = \gcd(R(E)) \in R(E)$.

Observe that the set $\{k \geq 0\colon\ x^k \approx x^{k+d} \in E\}$ is not empty. Let $d_0 = \min\{k \geq 0\colon\ x^k \approx x^{k+d} \in E\}$ and let $k_0 = \min\{k \geq 0\colon \text{there exists } n > k \text{ such that } x^k \approx x^n \in E\}$. Let further $l_0 = \min\{k \geq 0\colon\ x^{k_0} \approx x^{k_0+k} \in E\}$, $k_1 = \min\{k \geq 0\colon \text{ there exists } n \geq k \text{ such that } x^k \approx y^n \in E\}$ and let $l_1 = \min\{k > 0\colon\ x^{k_1} \approx y^{k_1+k} \in E\}$.

We show that

$$E_1 = \{x^{k_0} \approx x^{k_0+l_0}, x^{d_0} \approx x^{d_0+d}, x^m \approx y^m, x^{k_1} \approx y^{k_1+l_1}\}$$

is a basis of E.

Obviously, $E_1 \subseteq E$ by definitions of l_1, d_0, l_0.

Let E' be a weak equational theory such that $E_1 \subseteq E'$. Let $z^p \approx t^q \in E$ for some $z, t \in X$ and $p, q \geq 0$. If $z = t$, then $z^p \approx t^q \in E'$ by Lemma 2.9. Suppose that $z \neq t$. By rule R5, $x^p \approx y^q \in E$. By rule R2 (symmetry), we can assume that $q \geq p$.

1. If $q = p$, then $m \leq q$ and $x^m \approx y^m \in E_1 \subseteq E'$. Hence, $x^p \approx y^q \in E'$ by Lemma 3.1 and $z^p \approx t^q \in E'$ by rule R5.

2. If $q > p$, then $x^{k_1} \approx y^q \in E$ by Lemma 3.3, since $x^{k_1} \approx y^{k_1+l_1} \in E_1 \subseteq E$. Hence, $k_1 + l_1 \leq q$ by definition of l_1. Moreover, $k_1 \leq p$ by definition of k_1. Therefore, $x^p \approx y^q \in E'$ by Lemma 3.1, since $x^{k_1} \approx y^{k_1+l_1} \in E_1 \subseteq E'$, $k_1 \leq p$ and $k_1 + l_1 \leq q$. Hence, $z^p \approx t^q \in E'$ by rule R5.

We proved that $E \subseteq E'$ and E_1 is a 4-element basis of E.

Now we show some connections between exponents of equations in E_1.

By Lemma 3.1, $x^m \approx y^{m+1} \in E$, since $x^m \approx y^m \in E_1 \subseteq E$. Hence, $x^m \approx x^{m+1} \in E$ by rule R5. Therefore, $1 \in R(E)$ and $d = 1$. Moreover, $k_1 \leq m$ by definition of k_1, since $x^m \approx y^{m+1} \in E$.

We show that $m \leq k_1 + l_1 \leq m + 1$ and $m \leq d_0 + 1 \leq m + 1$.

a) By Lemma 3.1, $x^{k_1+l_1} \approx y^{k_1+l_1} \in E$ (since $x^{k_1} \approx y^{k_1+l_1} \in E$). Hence, $m \leq k_1 + l_1$ by definition of m.

b) By Lemma 3.1, $x^m \approx y^{m+1} \in E$, since $x^m \approx y^m \in E$. Hence, $x^m \approx x^{m+1} \in E$ by rule R5. Therefore, $x^{k_1} \approx y^{m+1} \in E$ by Lemma 3.2, since $x^{k_1} \approx y^{k_1+l_1} \in E$. Hence, $k_1 + l_1 \leq m + 1$ by definition of l_1.

c) We know that $x^m \approx x^{m+1} \in E$. Hence, $d_0 \leq m$ by definition of d_0, since $d = 1$.

d) By Lemma 3.2, $x^m \approx y^{d_0+1} \in E$, since $x^m \approx y^m \in E$ and $x^{d_0} \approx x^{d_0+1} \in E$. By rule R5, $y^m \approx x^{d_0+1} \in E$ and $x^{d_0+1} \approx y^m \in E$ by rule R2. Therefore, $x^{d_0+1} \approx y^{d_0+1}$ by Lemma 3.2, since $x^{d_0+1} \approx y^m \in E$ and $x^{d_0} \approx x^{d_0+1} \in E$. Hence, $m \leq d_0 + 1$ by definition of m.

Consider the following cases:

1. $m = k_1 + l_1$. Then $E_1^1 = E_1 \setminus \{x^m \approx y^m\}$ is a 3-element basis for E, because $\{x^{k_1} \approx y^{k_1+l_1}\} \vdash \{x^m \approx y^m\}$ by Lemma 3.1, $x^{k_1} \approx y^{k_1+l_1} \in E_1^1$ and $E_1^1 \vdash E_1$.

2. $m + 1 = k_1 + l_1$ and $k_1 = m$. Then $E_1^2 = E_1 \setminus \{x^{k_1} \approx y^{k_1+l_1}\}$ is a 3-element basis for E, because $\{x^m \approx y^m\} \vdash x^{k_1} \approx y^{k_1+l_1}$ by Lemma 3.1, $x^m \approx y^m \in E_1^2$ and $E_1^2 \vdash E_1$.

3. $m + 1 = k_1 + l_1$, $k_1 < m$ and $m = d_0$. Then $E_1^3 = E_1 \setminus \{x^{d_0} \approx x^{d_0+1}\}$ is a 3-element basis for E, because $\{x^m \approx y^m\} \vdash \{x^m \approx y^{m+1}\} \vdash \{x^{d_0} \approx x^{d_0+1}\}$ by Lemma 3.1 and rule R5 ($d_0 = m$), $x^m \approx y^m \in E_1^3$ and $E_1^3 \vdash E_1$.

4. $m + 1 = k_1 + l_1$, $k_1 < m$ and $m = d_0 + 1$. We show that $E_1^4 = \{x^{k_0} \approx x^{k_0+l_0}, x^{d_0} \approx x^{d_0+1}, x^{k_1} \approx y^m\}$ is a 3-element basis of E. By Lemma 3.1, $\{x^{k_1} \approx y^m\} \vdash \{x^m \approx y^m, x^{k_1} \approx y^{k_1+l_1}\}$, since $k_1 \leq m$ and $m \leq m + 1 = k_1 + l_1$. Hence, $E_1^4 \vdash E_1$. By Lemma 3.2, $\{x^{d_0} \approx x^{d_0+1}, x^{k_1} \approx y^{k_1+l_1}\} \vdash \{x^{k_1} \approx y^m\}$, since $m = d_0 + 1$. Hence, $E_1 \vdash E_1^4$ and E_1^4 is a 3-element basis of E. ∎

Example 3.5. The weak monounary variety $V = \mathrm{Mod}_w(\{x^2 \approx y^2, x^1 \approx y^3, x^0 \approx x^3\})$ has no 2-element basis.

Proof. Define the following monounary algebras (digits denote elements of the support and arrows show how the unique partial 1-ary operation acts):

$$\underline{A}_1 :\ \begin{tikzcd} 0 & 1 \arrow[loop right] \end{tikzcd}, \qquad \underline{A}_2 :\ \begin{tikzcd} 0 \arrow[r] & 1 \arrow[loop right] \end{tikzcd},$$

$$\underline{A}_3 :\ \begin{tikzcd} 0 \arrow[loop right] & 1 \arrow[loop right] \end{tikzcd}, \qquad \underline{A}_4 :\ \begin{tikzcd} 0 \arrow[r] & 1 \arrow[r] & 2 \end{tikzcd},$$

$$\underline{A}_5 :\ \begin{tikzcd} & 2 \\ 0 \arrow[loop right] & 1 \arrow[u] \end{tikzcd}, \qquad \underline{A}_6 :\ \begin{tikzcd} 0 \arrow[r] & 1 \arrow[r] & 2 \\ 3 \arrow[r] & 4 \arrow[r] & 5 \end{tikzcd}.$$

It is easy to see that $\underline{A}_1 \in V$ and $\underline{A}_4 \in V$. Let $E = \mathrm{Eq}_w(V)$. Observe that

$$(*) \qquad x^0 \approx y^n \notin E \qquad \text{for every } n \in N,$$

because $\underline{A}_1 \not\models x^0 \approx y^n$ for every $n \in N$ (since $x^0(0) = 0 \neq 1 = y^n(1)$) and $\underline{A}_1 \in V$. Hence, any basis of E cannot contain an equation $x^0 \approx y^n$ for some $n \in N$.

Suppose that E_0 is a 2-element basis of E. Consider the following three cases:

1. E_0 has two not regular equation. Then we can assume that $E_0 = \{x^n \approx y^m, x^k \approx y^l\}$ for some $x, y \in X$, $x \neq y$ and $n, m, k, l \in N$. Moreover $n, m, k, l \geq 1$ by $(*)$. Therefore, $\underline{A}_2 \models E_0$, since $\underline{A}_2 \models x^p \approx y^q$ for every $p, q \geq 1$. Hence, $\underline{A}_2 \in V$, since E_0 is a basis of E. But $\underline{A}_2 \notin V$, since $\underline{A}_2 \not\models x^0 \approx x^3$ $((x^0)^{\underline{A}_2}(0) = 0 \neq 1 = (x^3)^{\underline{A}_2}(0))$, a contradiction.

2. E_0 has exactly one regular equation. Then we can assume that $E_0 = \{x^n \approx x^{n+k}, x^m \approx y^{m+l}\}$ for some $x, y \in X$, $x \neq y$ and $n, k, m, l \in N$.

If $k = 0$, then $E'_0 = \{x^m \approx y^{m+l}\}$ is a basis of E, which is impossible by the previous case. Thus $k \geq 1$. Moreover, $m \geq 1$ by $(*)$. Then $\underline{A}_2 \vDash x^m \approx y^{m+l}$ and $\underline{A}_2 \notin V$. Therefore, $\underline{A}_2 \nvDash x^n \approx x^{n+k}$, since E_0 is a basis of E. But $\underline{A}_2 \vDash x^p \approx x^q$ for $p, q \geq 1$. Hence $n = 0$.

Observe that $\underline{A}_4 \in V$, $\underline{A}_4 \nvDash x^0 \approx x^1$ $((x^0)^{\underline{A}_4}(0) = 0 \neq 1 = (x^1)^{\underline{A}_4}(0))$ and $\underline{A}_4 \nvDash x^0 \approx x^2$ $((x^0)^{\underline{A}_4}(0) = 0 \neq 2 = (x^2)^{\underline{A}_4}(0))$. Hence $k \geq 3$. Then $\underline{A}_5 \vDash x^0 \approx x^k$ and $\underline{A}_5 \notin V$, since $\underline{A}_5 \nvDash x^1 \approx y^3$ $((x^1)^{\underline{A}_5}(1) = 2 \neq 0 = (y^3)^{\underline{A}_5}(0))$. Thus $\underline{A}_5 \nvDash x^m \approx y^{m+l}$, since E_0 is a basis of E. But $\underline{A}_5 \vDash x^p \approx y^q$ for $p, q \geq 2$. Hence $m = 1$.

Moreover, $\underline{A}_4 \in V$, $\underline{A}_4 \nvDash x^1 \approx y^1$ $((x^1)^{\underline{A}_4}(0) = 1 \neq 2 = (y^1)^{\underline{A}_4}(1))$ and $\underline{A}_4 \nvDash x^1 \approx y^2$ $((x^1)^{\underline{A}_4}(0) = 1 \neq 2 = (y^2)^{\underline{A}_4}(0))$. Hence $l \geq 2$. Therefore, $E_0 = \{x^0 \approx x^k, x^1 \approx y^{1+l}\}$, $k \geq 3$ and $l \geq 2$. Then $\underline{A}_6 \vDash E_0$ and $\underline{A}_6 \in V$, since E_0 is a basis of E. But $\underline{A}_6 \notin V$, since $\underline{A}_6 \nvDash x^2 \approx y^2$ $((x^2)^{\underline{A}_6}(0) = 2 \neq 5 = (y^2)^{\underline{A}_6}(3))$, a contradiction.

3. E_0 has two regular equation. Then we can assume that $E_0 = \{x^n \approx x^m, x^k \approx x^l\}$ for some $x \in X$ and $n, m, k, l \in N$. Therefore, $\underline{A}_3 \vDash E_0$ and $\underline{A}_3 \in V$, since E_0 is a basis of E. But $\underline{A}_3 \notin V$, since $\underline{A}_3 \nvDash x^2 \approx y^2$ $((x^2)^{\underline{A}_3}(0) = 0 \neq 1 = (y^2)^{\underline{A}_3}(1))$, a contradiction. ■

From this example we know that there exists a weak monounary variety with 3-element basis, which has no 2-element basis.

REFERENCES

[1] G. Bińczak, *A characterization theorem for weak varieties*, Algebra Universalis **45** (2001), 53–62.

[2] P. Burmeister, *A Model – Theoretic Oriented Approach to Partial Algebras*, Akademie-Verlag, Berlin 1986.

[3] G. Grätzer, *Universal Algebra*, (the second edition), Springer-Verlag, New York 1979.

[4] H. Höft, *Weak and strong equations in partial algebras*, Algebra Universalis **3** (1973), 203–215.

[5] E. Jacobs and R. Schwabauer, *The lattice of equational classes of algebras with one unary operation*, Amer. Math. Monthly **71** (1964), 151–155.

[6] L. Rudak, *A completness theorem for weak equational logic*, Algebra Universalis **16** (1983), 331–337.

[7] L. Rudak, *Algebraic characterization of conflict-free varieties of partial algebras*, Algebra Universalis **30** (1993), 89–100.

Received 19 April 2002
Revised 2 July 2002