# ON GENERALIZED Hom-FUNCTORS OF CERTAIN SYMMETRIC MONOIDAL CATEGORIES 

Hans-Jürgen Vogel<br>University of Potsdam, Institute of Mathematics PF 6015 53, D-14415 Potsdam, Germany<br>e-mail: vogel@rz.uni-potsdam.de<br>or hans-juergen.vogel@freenet.de<br>In memory of<br>Prof. Dr. habil. Herbert Lugowski<br>(17. 06. 1925-10. 05. 2001)


#### Abstract

It is well-known that for each object $A$ of any category $\mathcal{C}$ there is the covariant functor $H^{A}: \mathcal{C} \rightarrow$ Set, where $H^{A}(X)$ is the set $\mathcal{C}[A, X]$ of all morphisms out of $A$ into $X$ in $\mathcal{C}$ for an arbitrary object $X \in|\mathcal{C}|$ and $H^{A}(\varphi), \varphi \in \mathcal{C}[X, Y]$, is the total function from $\mathcal{C}[A, X]$ into $\mathcal{C}[A, Y]$ defined by $\mathcal{C}[A, X] \ni u \mapsto u \varphi \in \mathcal{C}[A, Y]$.

If $\underline{\mathcal{C}}$ is a $d t s$-category, then $H^{A}$ is in a natural manner a $d$-monoidal functor with respect to $$
\begin{array}{r} \widetilde{H^{A}}=\left(\widetilde{H^{A}}\langle X, Y\rangle: \mathcal{C}[A, X] \times \mathcal{C}[A, Y] \rightarrow \mathcal{C}[A, X \otimes Y],\right. \\ \left.\left(\left(u_{1}, u_{2}\right) \mapsto d_{A}\left(u_{1} \otimes u_{2}\right)\right)|X, Y \in| \mathcal{C} \mid\right) \end{array}
$$ and $$
i_{H^{A}}:\{\emptyset\} \rightarrow \mathcal{C}[A, I],\left(\emptyset \mapsto t_{A}\right) .
$$

This construction can be generalized to functors $H^{e}$ from any $d h t h \nabla s$-category $\underline{K}$ into the category $\underline{P a r}$ related to arbitrary subidentities $e$ of $\underline{K}$ (cf. Schreckenberger [3]). Each such generalized Homfunctor $H^{e}$ related to any subidentity $e \leq 1_{A}, o_{A, A} \neq e$, turns out to be a monoidal $d h t h \nabla s$-functor from $\underline{K}$ into $\underline{P a r}$.


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## 1. Introduction

The development of a functorial semantic of partial algebras requires the knowledge about functors between certain symmetric monoidal categories which preserve the special monoidal structure except for isomorphisms. In [8] was shown that each functor between diagonal-halfterminal-halfdiagonal-inversional-symmetric monoidal categories ( $d h t h \nabla s$-categories), which respects the monoidal structure and the diagonal morphisms with regard to a morphism family $\widetilde{F}$ of the image category, also preserves the canonical partial order relation, the totality and the injectivity of morphisms, and the terminal morphisms as well as the diagonal inversion morphisms with respect to the same family of isomorphisms $\widetilde{F}$.

The morphism class of a category $K$ will be denoted by $K$ too, the object class of $K$ by $|K|$, and the set of all morphisms in $K$ between objects $A$ and $B$ by $K[A, B]$.

Definition 1.1. Let $K^{\bullet}$ be a symmetric monoidal category in the sense of Eilenberg-Kelly [1].

A sequence $\left(K^{\bullet} ; d\right)$ is called diagonal-symmetric monoidal category (shortly ds-category; see [6]), if $d=\left(d_{A} \in K[A, A \otimes A]|A \in| K \mid\right)$ is a family of morphisms of $K$ such that
(D1) $\forall A, A^{\prime} \in|K| \forall \varphi \in K\left[A, A^{\prime}\right]\left(\varphi d_{A^{\prime}}=d_{A}(\varphi \otimes \varphi)\right)$,
(D2) $\forall A \in|K|\left(d_{A}\left(d_{A} \otimes 1_{A}\right)=d_{A}\left(1_{A} \otimes d_{A}\right) a_{A, A, A}\right)$,
(D3) $\forall A \in|K|\left(d_{A} s_{A, A}=d_{A}\right)$,
(D4) $\quad \forall A, B \in|K|\left(\left(d_{A} \otimes d_{B}\right) b_{A, A, B, B}=d_{A \otimes B}\right)$
are fulfilled.
$\left(K^{\bullet}, d, t\right)$ is called diagonal-terminal-symmetric monoidal category $(d t s$-category; see [6] $)$, if $\left(K^{\bullet}, d\right)$ is a $d s$-category with a family $t=\left(t_{A} \mid A \in\right.$ $|K|)$ of terminal morphisms $t_{A} \in K[A, I]$ such that the conditions

$$
\begin{equation*}
\forall A, A^{\prime} \in|K| \forall \varphi \in K\left[A, A^{\prime}\right]\left(\varphi t_{A^{\prime}}=t_{A}\right) \text { and } \tag{T1}
\end{equation*}
$$

(DTR) $\forall A \in|K|\left(d_{A}\left(1_{A} \otimes t_{A}\right) r_{A}=1_{A}\right)$
are right.
$\left(K^{\bullet} ; d, t, o\right)$ will be called diagonal-halfterminal-symmetric monoidal category (shortly dhts-category; see [2], [4], [6]), if $d$ is a morphism family as above, $t=\left(t_{A} \in K[A, I]|A \in| K \mid\right)$ is a family of morphisms in $K$, and $o: I \rightarrow O$ is a distinguished morphism in $K$ related to a distiguished object $O \in|K|, O \neq I$, such that

$$
\begin{equation*}
\forall A, A^{\prime} \in|K| \forall \varphi \in K\left[A, A^{\prime}\right]\left(d_{A}(\varphi \otimes \varphi)=\varphi d_{A^{\prime}}\right) \tag{D1}
\end{equation*}
$$

$(\mathrm{DTR}) \quad \forall A \in|K|\left(d_{A}\left(1_{A} \otimes t_{A}\right) r_{A}=1_{A}\right)$,
$(\mathrm{DTL}) \quad \forall A \in|K|\left(d_{A}\left(t_{A} \otimes 1_{A}\right) l_{A}=1_{A}\right)$,
(DTRL) $\left.\forall A_{1}, A_{2} \in|K|\left(d_{A_{1} \otimes A_{2}}\left(\left(1_{A_{1}} \otimes t_{A_{2}}\right) r_{A_{1}} \otimes\left(t_{A_{1}} \otimes 1_{A_{2}}\right) l_{A_{2}}\right)=1_{A_{1} \otimes A_{2}}\right)\right)$,

$$
\begin{equation*}
\forall A, B \in|K|\left(t_{A \otimes B}=\left(t_{A} \otimes t_{B}\right) t_{I \otimes I}\right) \tag{TT}
\end{equation*}
$$

$$
\begin{equation*}
\forall A \in|K|(A \otimes O=O \otimes A=O) \tag{O1}
\end{equation*}
$$

(o1) $\quad \forall A \in|K| \forall \varphi \in K[A, O]\left(t_{A} O=\varphi\right)$, and
(o2) $\quad \forall A \in|K| \forall \psi \in K[O, A]\left(\left(1_{A} \otimes t_{O}\right) r_{A}=\psi\right)$
are fulfilled.
$\left(K^{\bullet} ; d, t, \nabla, o\right)$ is called a diagonal-halfterminal-halfdiagonal-inversionalsymmetric monoidal category (for short $d h t h \nabla s$-category; in [6] dht $\nabla$-symmetric category), if ( $K^{\bullet} ; d, t, o$ ) is a dhts-category endowed with a morphism family

$$
\nabla=\left(\nabla_{A} \in K[A \otimes A, A]|A \in| K \mid\right) \text { fulfilling }
$$

$\left(\mathrm{D}_{1}^{*}\right) \quad \forall A \in|K|\left(d_{A} \nabla_{A}=1_{A}\right)$,
$\left(\mathrm{D}_{2}^{*}\right) \quad \forall A \in|K|\left(\nabla_{A} d_{A} d_{A \otimes A}=d_{A \otimes A}\left(\nabla_{A} d_{A} \otimes 1_{A \otimes A}\right)\right)$.
The zero morphisms $o_{A, B}$ absorb all other morphisms at composition and $\otimes$-operation in any dhts-category. Because of (o1) and (o2), the unit morphism $1_{O}$ is identical with the zero morphism $o_{O, O}$.

The category $\underline{P a r}$ of all partial functions between arbitrary sets is an example for a $d h t h \nabla s$-category.

In view of the properties of the category $\underline{\text { Par }}$ we only will consider $d h t h \nabla s$-categories fulfilling the conditions

$$
\begin{equation*}
\forall A, B \in|K|(A \otimes B=O \Rightarrow(A=O \vee B=O)) \tag{O3}
\end{equation*}
$$

(o3)

$$
\begin{aligned}
& \forall A, B, C, D \in|K| \forall \varphi \in K[A, B] \forall \psi \in K[C, D] \\
& \quad\left(\varphi \otimes \psi=o_{A \otimes C, B \otimes D} \Rightarrow\left(\varphi=o_{A, B} \vee \psi=o_{C, D}\right)\right)
\end{aligned}
$$

Remark that $\left(K^{\bullet} ; d\right)$ is a $d s$-category for each $d h t s$ - category $\left(K^{\bullet} ; d, t, o\right)$ and $\nabla$ is the only family in a $d h t h \nabla s$-category with the properties $\left(\mathrm{D}_{1}^{*}\right)$ and $\left(\mathrm{D}_{2}^{*}\right)$, cf. [3].

The class $T_{K}:=\left\{\varphi \in K \mid \varphi t_{\text {codom } \varphi}=t_{\text {dome }}\right\}$ forms a dts-subcategory $\underline{T}_{K}$ of a dhts-category $\underline{K}:=\left(K^{\bullet} ; d, t, o\right)$ and $\left(A \otimes B, p_{1}^{A, B}:=\left(1_{A} \otimes t_{B}\right) r_{A}\right.$, $\left.p_{2}^{A, B}:=\left(t_{A} \otimes 1_{B}\right) l_{B}\right)$ is a categorical product in $\underline{T}_{K}$, but not in the whole category $\underline{K}$. The morphisms $p_{1}^{A, B}$ and $p_{2}^{A, B}$ are called the canonical projections concerning $A$ and $B$ ([2]).

The relation $\leq$ defined by

$$
\varphi \leq \psi: \Leftrightarrow \exists A, A^{\prime} \in|K| \quad\left(\varphi, \psi \in K\left[A, A^{\prime}\right] \wedge \varphi=d_{A}(\varphi \otimes \psi) p_{2}^{A^{\prime}, A^{\prime}}\right)
$$

is a partial order relation and it is compatible with composition and $\otimes$-operation of morphisms (see [3]).

The following conditions are equivalent in any dhts-category (see [4]):

$$
\begin{aligned}
& \varphi=d_{A}(\varphi \otimes \psi) p_{2}^{A^{\prime}, A^{\prime}} \\
& \varphi=d_{A}(\psi \otimes \varphi) p_{1}^{A^{\prime}, A^{\prime}} \\
& \varphi d_{A^{\prime}}=d_{A}(\varphi \otimes \psi) \\
& \varphi d_{A^{\prime}}=d_{A}(\psi \otimes \varphi)
\end{aligned}
$$

Moreover, each $d h t h \nabla s$-category has the properties

$$
\begin{aligned}
& \left(\mathrm{h} \nabla_{1}\right) \quad \forall A, A^{\prime} \in|K| \forall \varphi \in K\left[A, A^{\prime}\right]\left(\nabla_{A} \varphi d_{A^{\prime}}=d_{A \otimes A}\left(\nabla_{A} \varphi \otimes(\varphi \otimes \varphi) \nabla_{A^{\prime}}\right)\right), \\
& \left(\mathrm{hT}_{1}\right) \quad \forall A, A^{\prime} \in|K| \forall \varphi \in K\left[A, A^{\prime}\right]\left(\varphi t_{A^{\prime}} d_{I}=d_{A}\left(\varphi t_{A^{\prime}} \otimes t_{A}\right)\right),
\end{aligned}
$$

therefore $\nabla_{A} \varphi \leq(\varphi \otimes \varphi) \nabla_{A^{\prime}}$ and $\varphi t_{A^{\prime}} \leq t_{A}$ for all morphisms $\varphi \in K\left[A, A^{\prime}\right]$ and all objects $A, A^{\prime} \in|K|$.

Each morphism set $K[A, B]$ of a dhth $\nabla s$-category $\underline{K}$ forms a meetsemilattice with respect to $\varphi \wedge \psi=d_{A}(\varphi \otimes \psi) \nabla_{B}$. This semilattice has the minimum $o_{A, B}$, maximal elements are the total functions. Especially, the morphism sets $K[A, I]$ possess a maximum, namely $t_{A}$.

The basic morphisms related to the distingushed object $I$ in any symmetric monoidal category, any dhts-category, or even any $d h t h \nabla s$-category have some interesting properties as follows:

Lemma 1.2. Let $K^{\bullet}$ be a symmetric monidal category. Then the following equalities hold:

$$
r_{I}=l_{I}([3]), a_{I, I, I}=r_{I}^{-1} \otimes r_{I}, b_{A, I, I, B}=1_{A \otimes I} \otimes 1_{I \otimes B}([6]), s_{I, I}=1_{I \otimes I} .
$$

Moreover, every dhts-category $\underline{K}$ has in addition the properties

$$
\begin{aligned}
& d_{I}=r_{I}^{-1}, \quad r_{I} d_{I}=1_{I \otimes I}, \quad t_{I}=1_{I}([3]), \quad t_{I \otimes I}=r_{I}, \\
& i \in \text { iso }_{K}[I, I] \Rightarrow i=t_{I}, \quad \forall X \in|K| \forall x \in K[I, X]\left(x \in \text { iso } K \Rightarrow x^{-1}=t_{X}\right) .
\end{aligned}
$$

Finally, if $\underline{K}$ is a dhth $\nabla$ s-category, then the additional property

$$
\nabla_{I}=r_{I}
$$

is true.

Proof. The identity $a_{A, I, B}\left(r_{A} \otimes 1_{B}\right)=1_{A} \otimes l_{B}$ is one of the defining properties of monoidal-symmetric categories, hence $a_{I, I, I}\left(r_{I} \otimes 1_{I}\right)=1_{I} \otimes r_{I}$ by $r_{I}=l_{I}$ and $a_{I, I, I}=\left(r_{I}^{-1} \otimes r_{I}\right)$, since all right-identity morphisms are isomorphisms.
$s_{A, I} l_{A}=r_{A}$ is a further defining identity, hence $s_{I, I} l_{I}=r_{I}=l_{I}$ and therefore $s_{I, I}=1_{I \otimes I}$, because $l_{I}$ is an isomorphism in $K$.

In any $d h t s$-category one has the defining identity $d_{A}\left(1_{A} \otimes t_{A}\right) r_{A}=1_{A}$, hence $1_{I}=d_{I}\left(1_{I} \otimes t_{I}\right) r_{I}=d_{I}\left(1_{I} \otimes 1_{I}\right) r_{I}=d_{I} r_{I}$ since $t_{I}=1_{I}$, consequently $d_{I}=r_{I}^{-1}$ and $r_{I} d_{I}=1_{I \otimes I}$.

Each coretraction $\varphi \in K[A, B]$ of a $d h t s$-category has the property $\varphi t_{B}=t_{A}$. Because $d_{I}$ is even an isomorphism, one observes $d_{I} t_{I \otimes I}=t_{I}=$ $1_{I}$, therefore $t_{I \otimes I}=1_{I \otimes I} t_{I \otimes I}=r_{I} d_{I} t_{I \otimes I}=r_{I} 1_{I}=r_{I}$.

One of the characterizing conditions of the diagonal inversions in a $d h t h \nabla s$-category is $d_{A} \nabla_{A}=1_{A}$. Therefore, $\nabla_{I}=1_{I \otimes I} \nabla_{I}=r_{I} d_{I} \nabla_{I}=r_{I}$ as above.

Now let $i \in K[I, I]$ be an isomorphism of a dhts-category $\underline{K}$. Then $i=i 1_{I}=i t_{I}=t_{I}$, because of $1_{i}=t_{I}$.

Let $x \in K[I, X]$ be an isomorphism in a dhts-category $\underline{K}$. Then one obtains by the same manner as above $1_{I}=t_{I}=x t_{X}$, hence the assertion.
J. Schreckenberger introduced in [3] the important concept of a subidentity in any dhts-category $\underline{K}$ in the following way: A morphism $e \in K$ is called subidentity of an object $A \in|K|$, if $e \leq 1_{A}$ with respect to the canonical order relation in $\underline{K}$. The set of all subidentities of an object $A \in|K|$ will be denoted by $E_{K}(A)$, i.e.

$$
E_{K}(A):=\left\{e \in K[A, A] \mid e \leq 1_{A}\right\} .
$$

Each morphism $\varphi \in K\left[A, A^{\prime}\right]$ determines in a natural manner a subidentity

$$
\alpha(\varphi):=d_{A}\left(1_{A} \otimes \varphi\right) p_{1}^{A, A^{\prime}}, \text { the subidentity of } \varphi .
$$

Subidenties possess a lot of important properties as follows ([3], [9]):
Theorem 1.3. Let $\underline{K}=\left(K^{\bullet}, d, t, \nabla, o\right)$ be a dhth $\nabla s$-category. Then the following claims hold:
(E1) $\forall A \in|K| \forall e \in E_{K}(A)(e e=e)$,
(E2) $\forall A \in|K| \forall e_{1}, e_{2} \in E_{K}(A)\left(e_{1} e_{2}=e_{2} e_{1}\right)$,
(E3) $\forall A \in|K| \forall e_{1}, e_{2} \in E_{K}(A)\left(e_{1} e_{2}=\inf \left\{e_{1}, e_{2}\right\}\right)$, and
(E4) $\forall A \in|K| \forall e_{1}, e_{2} \in E_{K}(A)\left(e_{1} \leq e_{2} \Leftrightarrow e_{1} e_{2}=e_{1}\right)$, i.e.
the set $E_{K}(A)$ forms together with the morphism composition a meetsemilattice with maximal element $1_{A}$ and minimal element $o_{A, A}$ related to the canonical partial order relation for each $A \in|K|$.

Subidenties related to arbitrary morphisms of $K$ have the following properties:

$$
\text { ( } \alpha \text { ( } \begin{aligned}
& \forall A, A^{\prime} \in|K| \forall \varphi \in K[A, A]\left(\alpha(\varphi):=d_{A}\left(1_{A} \otimes \varphi\right) p_{1}^{A, A^{\prime}}\right. \\
& \left.=d_{A}\left(\varphi \otimes 1_{A}\right) p_{2}^{A^{\prime}, A} \leq 1_{A}\right),
\end{aligned}
$$

$$
\text { ( } \alpha 2 \text { ) } \forall A, A^{\prime} \in|K| \forall \varphi \in K[A, A](\alpha(\varphi) \varphi=\varphi) \text {, }
$$

$$
\text { ( } \alpha 3 \text { ) } \forall A, A^{\prime} \in|K| \forall \varphi \in K[A, A]\left(\alpha(\varphi) t_{A}=\varphi t_{A^{\prime}}\right) \text {, }
$$

$$
\text { ( } \alpha 4 \text { ) } \forall A, A^{\prime} \in|K| \forall \varphi \in K[A, A]\left(\alpha(\varphi)=1_{A} \Leftrightarrow \varphi t_{A^{\prime}}=t_{A}\right) \text {, }
$$

$$
\text { ( } \alpha 5 \text { ) } \forall A, A^{\prime} \in|K| \forall \varphi \in K[A, A]\left(\alpha(\varphi)=o_{A, A} \Leftrightarrow \varphi=o_{A, A^{\prime}}\right) \text {, }
$$

$$
\text { ( } \alpha 6 \text { ) } \forall A \in|K| \forall e \in E_{K}(A)(\alpha(e)=e) \text {, }
$$

$$
\text { ( } \alpha 7 \text { ) } \forall A, A^{\prime} \in|K| \forall \varphi \in K\left[A, A^{\prime}\right] \forall e \in E_{K}(A)(e \alpha(\varphi)=\alpha(e \varphi) \leq e) \text {, }
$$

$$
\text { ( } \alpha 8 \text { ) } \forall A, A^{\prime} \in|K| \forall \varphi \in K\left[A, A^{\prime}\right] \forall e \in E_{K}(A)(e \varphi=\varphi \Leftrightarrow \alpha(\varphi) \leq e) \text {, }
$$

$$
\text { ( } \alpha 9 \text { ) } \forall A, A^{\prime} \in|K| \forall \varphi \in K\left[A, A^{\prime}\right] \forall e \in E_{K}(A)(e \leq \alpha(\varphi) \Rightarrow \alpha(e \varphi)=e) \text {, }
$$

$$
\text { ( } \alpha 10 \text { ) } \forall A, A^{\prime} \in|K| \forall \varphi, \psi \in K\left[A, A^{\prime}\right](\varphi \leq \psi \Leftrightarrow \alpha(\varphi) \psi=\varphi) \text {, }
$$

$$
\text { ( } \alpha 11 \text { ) } \forall A, A^{\prime} \in|K| \forall \varphi, \psi \in K\left[A, A^{\prime}\right](\varphi \leq \psi \Rightarrow \alpha(\varphi) \leq \alpha(\psi)) \text {, }
$$

$$
(\alpha 12) \forall A, A^{\prime} \in|K| \forall \varphi, \psi, \xi \in K\left[A, A^{\prime}\right]
$$

$$
(\varphi \leq \xi \wedge \psi \leq \xi \wedge \alpha(\varphi) \leq \alpha(\psi) \Rightarrow \varphi \leq \psi)
$$

$$
\text { ( } \alpha 13 \text { ) } \forall A, A^{\prime} \in|K| \forall \varphi, \psi \in K\left[A, A^{\prime}\right](\alpha(\varphi)=\alpha(\psi) \wedge
$$

$$
\exists \xi(\varphi \leq \xi \wedge \psi \leq \xi) \Rightarrow \varphi=\psi)
$$

( $\alpha 14) \forall A, A^{\prime} \in|K| \forall \varphi, \psi \in K\left[A, A^{\prime}\right](\varphi \leq \psi \wedge \alpha(\varphi)=\alpha(\psi) \Rightarrow \varphi=\psi)$,
( $\alpha 15$ ) $\forall A, A^{\prime}, B \in|K| \forall \varphi \in K\left[A, A^{\prime}\right] \forall \psi \in K\left[A^{\prime}, B\right](\alpha(\varphi \psi) \leq \alpha(\varphi))$,
( $\alpha 16$ ) $\forall A, A^{\prime}, B \in|K| \forall \varphi \in K\left[A, A^{\prime}\right] \forall \psi \in K\left[A^{\prime}, B\right](\varphi \alpha(\psi)=\alpha(\varphi \psi) \varphi)$,
( $\alpha 17$ ) $\forall A, A^{\prime}, B \in|K| \forall \varphi \in K\left[A, A^{\prime}\right] \forall \psi \in K\left[A^{\prime}, B\right](\alpha(\varphi \psi)=\alpha(\varphi \alpha(\psi)))$,

$$
\begin{aligned}
&(\alpha 18) \forall A, A^{\prime}, B, B^{\prime} \in|K| \forall \varphi \in K\left[A, A^{\prime}\right] \forall \varphi^{\prime} \in K\left[B, B^{\prime}\right] \\
&\left(\alpha\left(\varphi \otimes \varphi^{\prime}\right)=\alpha(\varphi) \otimes \alpha\left(\varphi^{\prime}\right)\right), \\
&(\alpha 19) \forall A, A_{1}, A_{2} \in|K| \forall \varphi_{i} \in K\left[A, A_{i}\right](i=1,2) \\
&\left(\alpha\left(d_{A}\left(\varphi_{1} \otimes \varphi_{2}\right)\right)=\alpha\left(\varphi_{1}\right) \alpha\left(\varphi_{2}\right)\right), \\
&(\alpha 20) \forall A, A_{1}, A_{2} \in|K| \forall \varphi \in K\left[A, A_{1}\right] \forall \psi \in T_{K}\left[A_{1}, A_{2}\right] \forall \chi \in T_{K}\left[A, A_{2}\right] \\
&\left(\alpha(\varphi)=\alpha\left(d_{A}(\varphi \otimes \chi)\right)=\alpha\left(d_{A}(\varphi \psi \otimes \chi)\right)\right), \\
&(\alpha 21) \forall A, A^{\prime} \in|K| \forall \varphi_{1}, \varphi_{2} \in K\left[A, A^{\prime}\right] \\
&\left(\alpha\left(\varphi_{1}\right) \varphi_{2}=\varphi_{2} \Leftrightarrow \alpha\left(\varphi_{2}\right) \leq \alpha\left(\varphi_{1}\right)\right), \\
&(\alpha 22) \quad \forall A \in|K|\left(\alpha\left(\nabla_{A}\right)=\alpha\left(\nabla_{A} d_{A}\right)=\nabla_{A} d_{A}\right), \\
&(\alpha 23) \forall A, A^{\prime} \in|K| \forall \varphi_{1}, \varphi_{2} \in K\left[A, A^{\prime}\right] \\
&\left(\alpha\left(\left(\varphi_{1} \otimes 1_{A^{\prime}}\right) \nabla_{A^{\prime}}\right)=\alpha\left(\left(\varphi_{2} \otimes 1_{A^{\prime}}\right) \nabla_{A^{\prime}}\right) \Leftrightarrow \varphi_{1}=\varphi_{2}\right), \\
&(\alpha 24) \forall A, A^{\prime} \in|K| \forall \varphi_{1}, \varphi_{2} \in K\left[A, A^{\prime}\right] \\
&\left(\alpha\left(\varphi_{2}\right) \varphi_{1}=\alpha\left(\varphi_{1}\right) \varphi_{2} \Rightarrow d_{A}\left(\varphi_{1} \otimes \varphi_{2}\right) \nabla_{A^{\prime}}=d_{A}\left(\varphi_{1} \otimes \varphi_{2}\right) p_{i}^{A^{\prime}, A^{\prime}},(i=1,2)\right), \\
&(\alpha 25) \forall A, A^{\prime} \in|K| \forall \varphi_{1}, \varphi_{2} \in K\left[A, A^{\prime}\right] \\
&\left(\alpha\left(\varphi_{1}\right) \varphi_{2}=d_{A}\left(\varphi_{1} \otimes \varphi_{2}\right) \nabla_{A^{\prime}} \Leftrightarrow \alpha\left(\varphi_{2}\right) \varphi_{1}=d_{A}\left(\varphi_{1} \otimes \varphi_{2}\right) \nabla_{A}\right), \\
&(\alpha 26) \forall A, A^{\prime} \in|K| \forall \varphi \in K\left[A, A^{\prime}\right]\left(\alpha\left(\nabla_{A} \varphi\right)=\nabla_{A} \varphi d_{A}\right), \\
&(\alpha 27) \forall A, A^{\prime} \in|K| \forall \varphi \in K\left[A, A^{\prime}\right] \forall \varphi^{*} \in K\left[A^{\prime}, A\right] \\
&\left(\alpha(\varphi)=\varphi \varphi^{*} \Leftrightarrow\left(\varphi \varphi^{*} \varphi=\varphi \wedge \varphi \varphi^{*} \leq 1_{A}\right)\right) .
\end{aligned}
$$

## 2. Monoidal functors

In applications to theories of algebraic structures, functors $F: \underline{K} \rightarrow \underline{K^{\prime}}$ between $d h t h \nabla s$-categories are of interest which preserve in addition to the functor properties the $d h t h \nabla s$-structure with respect to a family $\widetilde{F}=$ $(\widetilde{F}\langle X, Y\rangle|X, Y \in| K \mid)$ of isomorphisms $\widetilde{F}\langle X, Y\rangle: X F \otimes Y F \rightarrow(X \otimes Y) F$
in $\underline{K}^{\prime}$ and an isomorphism $i_{F}$ between $I^{\prime}$ and $I F$, where $I$ and $I^{\prime}$ are the distinguished objects in $\underline{K}$ and $\underline{K}^{\prime}$, respectively, ([2], [4], [8]).

Definition 2.1 ([8]). A functor $F: K^{\bullet} \rightarrow K^{\bullet \bullet}$ between symmetric monoidal categories $K^{\bullet}$ and $K^{\bullet \bullet}$ is called monoidal with respect to a family of morphisms

$$
\widetilde{F}=(\widetilde{F}\langle X, Y\rangle: X F \otimes Y F \rightarrow(X \otimes Y) F|X, Y \in| K \mid) \text { of } K^{\prime}
$$

and to a morphism

$$
i_{F}: I^{\prime} \rightarrow I F,
$$

for short $\left(F, \widetilde{F}, i_{F}\right): K^{\bullet} \rightarrow K^{\bullet}$, iff the following conditions are fulfilled:

$$
(F \sim) \forall X, Y \in|K|\left(\widetilde{F}\langle X, Y\rangle \in \operatorname{iso}\left(K^{\prime}\right)\right),
$$

(FI) $\quad i_{F} \in \operatorname{iso}\left(K^{\prime}\right)$,
(FA) $\quad \forall X, Y, Z \in|K|\left(\left(1_{X F}^{\prime} \otimes \widetilde{F}\langle Y, Z\rangle\right) \widetilde{F}\langle X, Y \otimes Z\rangle\left(a_{X, Y, Z} F\right)=\right.$ $\left.=a_{X F, Y F, Z F}^{\prime}\left(\widetilde{F}\langle X, Y\rangle \otimes 1_{Z F}^{\prime}\right) \widetilde{F}\langle X \otimes Y, Z\rangle\right)$,
(FR) $\forall X \in|K|\left(\widetilde{F}\langle X, I\rangle\left(r_{X} F\right)=\left(1_{X F} \otimes i_{F}^{-1}\right) r_{X F}^{\prime}\right)$,
(FS) $\quad \forall X, Y \in|K|\left(\widetilde{F}\langle X, Y\rangle\left(s_{X, Y} F\right)=s_{X F, Y F}^{\prime} \widetilde{F}\langle Y, X\rangle\right)$,
(FM) $\forall \varphi: X \rightarrow Y, \forall \psi: U \rightarrow V \in K$

$$
((\varphi F \otimes \psi F) \widetilde{F}\langle Y, V\rangle=\widetilde{F}\langle X, U\rangle(\varphi \otimes \psi) F)
$$

Corollary 2.2. Let $\left(F, \widetilde{F}, i_{F}\right): K^{\bullet} \rightarrow K^{\bullet \bullet}$ be a monoidal functor between symmetric monoidal categories. Then

$$
\text { (FL) } \forall X \in|K|\left(\widetilde{F}\langle I, X\rangle\left(l_{X} F\right)=\left(i_{F}^{-1} \otimes 1_{X F}\right) l_{X F}^{\prime}\right) \text {. }
$$

Proof. The validity of (FL) is a consequence of (FR) and (FS) by the properties of symmetric monoidal categories in the following way:

$$
\begin{array}{rlr}
\widetilde{F}\langle I, X\rangle\left(l_{X} F\right) & =s_{I F, X F}^{\prime} \widetilde{F}\langle X, I\rangle\left(s_{I, X} F\right)^{-1}\left(l_{X} F\right) & ((\mathrm{FS})) \\
& =s_{I F, X F}^{\prime} \widetilde{F}\langle X, I\rangle\left(\left(s_{X, I} l_{X}\right) F\right) & \left(s_{A, B}^{-1}=s_{B, A}\right) \\
& =s_{I F, X F}^{\prime} \widetilde{F}\langle X, I\rangle\left(r_{X} F\right) & \left(s_{I X} l_{X}=r_{X}\right) \\
& =s_{I F, X F}^{\prime}\left(1_{X F}^{\prime} \otimes i_{F}^{-1}\right) r_{X F}^{\prime} & ((\mathrm{FR})) \\
& =\left(i_{F}^{-1} \otimes 1_{X F}^{\prime}\right) s_{I^{\prime}, X F}^{\prime} r_{X F}^{\prime} &  \tag{FR}\\
& =\left(i_{F}^{-1} \otimes 1_{X F}^{\prime}\right) l_{X F}^{\prime} . &
\end{array}
$$

Definition 2.3 ([8]). A monoidal functor $\left(F, \widetilde{F}, i_{F}\right): \underline{K} \rightarrow \underline{K^{\prime}}$ between $d s$-categories $\underline{K}$ and $\underline{K}^{\prime}$ is called $d$-monoidal, if in addition the condition
(FD) $\quad \forall A \in|K|\left(d_{A} F=d_{A F}^{\prime} \widetilde{F}\langle A, A\rangle\right)$
holds.
Obviously, the identical functor of $K^{\bullet}$ forms a monoidal functor

$$
\begin{aligned}
& \left(1_{K},\left(\tilde{1}_{K}\langle X, Y\rangle=1_{X F \otimes Y F}|X, Y \in| K \mid\right), i_{1_{K}}=1_{I}\right): K^{\bullet} \rightarrow K^{\bullet} \\
& (X \mapsto X, \varphi \mapsto \varphi)
\end{aligned}
$$

and the constant functor from $K^{\bullet}$ into $K^{\bullet \bullet}$ too, $\left(E,\left(\widetilde{E}\langle X, Y\rangle=1_{I^{\prime}}^{\prime}|X, Y \in| K \mid\right), i_{E}=1_{I^{\prime}}^{\prime}\right): K^{\bullet} \rightarrow K^{\bullet}\left(X \mapsto I^{\prime}, \varphi \mapsto 1_{I^{\prime}}^{\prime}\right)$, where $K^{\bullet}$ and $K^{\bullet \bullet}$ are arbitrary symmetric monoidal categories.

Moreover:
Proposition 2.4. Let $\underline{K}$ and $\underline{K}^{\prime}$ be dhts-categories such that there are the distinguished zero-objects $O \in|K|, O^{\prime} \in\left|K^{\prime}\right|$. Then

$$
\begin{aligned}
& E_{0}: \underline{K} \rightarrow \underline{K^{\prime}} \text {, defined by } X \mapsto\left\{\begin{array}{ll}
I^{\prime} & \text { if } X \neq O, \\
O^{\prime} & \text { if } X=O,
\end{array}\right. \text { and } \\
& (\varphi: X \rightarrow Y) \mapsto \begin{cases}1_{I^{\prime}}^{\prime} & \text { if } X \neq O \wedge Y \neq O \wedge \varphi \neq o_{X, Y}, \\
o_{I^{\prime}, I^{\prime}}^{\prime} & \text { if } X \neq O \wedge Y \neq O \wedge \varphi=o_{X, Y}, \\
t_{O^{\prime}}^{\prime} & \text { if } X=O \wedge Y \neq O, \\
o^{\prime} & \text { if } X \neq O \wedge Y=O, \\
1_{O^{\prime}}^{\prime} & \text { if } X=Y=O,\end{cases}
\end{aligned}
$$

is d-monoidal with respect to

$$
\widetilde{E_{0}}\langle X, Y\rangle=\left\{\begin{array}{ll}
r_{I^{\prime}}^{\prime} & \text { if } X \neq O \neq Y, \\
1_{O^{\prime}}^{\prime} & \text { otherwise },
\end{array} \text { and } i_{E_{0}}=1_{I^{\prime}}^{\prime}\right.
$$

Proof. The functor properties are easy to verify by consideration of the separate cases. By definition, all morphisms $E_{0}\langle X, Y\rangle$ are isomorphisms and $i_{E_{0}}$ is an isomorphism too.

Ad (FA): If $X, Y$, and $Z$ all are different from $O$, then

$$
\begin{gathered}
\left(1_{X E_{0}}^{\prime} \otimes \widetilde{E_{0}}\langle Y, Z\rangle\right) \widetilde{E_{0}}\langle X, Y \otimes Z\rangle\left(a_{X, Y, Z} E_{0}\right)=\left(1_{I^{\prime}}^{\prime} \otimes r_{I^{\prime}}^{\prime}\right) r_{I^{\prime}}^{\prime} 1_{I^{\prime}}^{\prime}=\left(1_{I^{\prime}}^{\prime} \otimes r_{I^{\prime}}^{\prime}\right) l_{I^{\prime}}^{\prime} \\
=l_{I^{\prime} \otimes I^{\prime}}^{\prime} r_{I^{\prime}}^{\prime}=a_{I^{\prime}, I^{\prime}, I^{\prime}}^{\prime}\left(l_{I^{\prime}}^{\prime} \otimes 1_{I^{\prime}}^{\prime}\right) r_{I^{\prime}}^{\prime}=a_{I^{\prime}, I^{\prime}, I^{\prime}}^{\prime}\left(r_{I^{\prime}}^{\prime} \otimes 1_{I^{\prime}}^{\prime}\right) r_{I^{\prime}}^{\prime} \\
=a_{X E_{0}, Y E_{0}, Z E_{0}}^{\prime}\left(\widetilde{E_{0}}\langle X, Y\rangle \otimes 1_{Z E_{0}}^{\prime}\right) \widetilde{E_{0}}\langle X \otimes Y, Z\rangle .
\end{gathered}
$$

On the other hand:

$$
\begin{aligned}
X= & O \vee Y=O \vee Z=O \\
& \Rightarrow X \otimes(X \otimes Z)=O=(X \otimes Y) \otimes Z
\end{aligned}
$$

$$
\begin{aligned}
& \Rightarrow a_{X, Y, Z} E_{0}=o_{O, O} E_{0}=o_{O^{\prime}, O^{\prime}}^{\prime} \wedge a_{X E_{0}, Y E_{0}, Z E_{0}}^{\prime}=o_{O^{\prime}, O^{\prime}}^{\prime} \\
& \Rightarrow\left(1_{X E_{0}}^{\prime} \otimes \widetilde{E_{0}}\langle Y, Z\rangle\right) \widetilde{E_{0}}\langle X, Y \otimes Z\rangle\left(a_{X, Y, Z} E_{0}\right)=o_{O^{\prime}, O^{\prime}}^{\prime} \wedge \\
& a_{X E_{0}, Y E_{0}, Z E_{0}}^{\prime}\left(\widetilde{E_{0}}\langle X, Y\rangle \otimes 1_{Z E_{0}}^{\prime}\right) \widetilde{E_{0}}\langle X \otimes Y, Z\rangle=o_{O^{\prime}, O^{\prime}}^{\prime} .
\end{aligned}
$$

Ad (FR): Because $i_{E_{0}} \in K^{\prime}\left[I^{\prime}, I E_{0}\right]$ is an isomorphism in $K^{\prime}$ and $I E_{0}=I^{\prime}$, one has $i_{E_{0}}=1_{I^{\prime}}^{\prime}$ by Lemma 1.2. Let $X \neq O$. Then
$\widetilde{E}_{0}\langle X, I\rangle\left(r_{X} E_{0}\right)=r_{I^{\prime}}^{\prime} 1_{I^{\prime}}^{\prime}=\left(1_{I^{\prime}}^{\prime} \otimes 1_{I^{\prime}}^{\prime}\right) r_{I^{\prime}}^{\prime}=\left(1_{I^{\prime}}^{\prime} \otimes i_{E_{0}}^{-1}\right) r_{X E_{0}}^{\prime}$.
Otherwise, $X=O$ implies $X E_{0}=O^{\prime} \wedge X \otimes I=O \otimes I=O$, hence $\widetilde{E_{0}}\langle X, I\rangle=\widetilde{E}_{0}\langle O, I\rangle=1_{O^{\prime}}^{\prime}=o_{O^{\prime}, O^{\prime}}^{\prime} \wedge r_{X} E_{0}=1_{O} E_{0}=1_{O^{\prime}}^{\prime}=o_{O^{\prime}, O^{\prime}}^{\prime}=$ $r_{O^{\prime}}^{\prime}=r_{X E_{0}}^{\prime}$. Therefore

$$
\begin{aligned}
\widetilde{E}_{0}\langle X, I\rangle\left(r_{X} E_{0}\right)=1_{O^{\prime}}^{\prime} 1_{O^{\prime}}^{\prime} & =1_{O^{\prime}}^{\prime}=o_{O^{\prime}, O^{\prime}}^{\prime}=\left(o_{O^{\prime}, O^{\prime}}^{\prime} \otimes i_{E_{0}}^{-1}\right) o_{O^{\prime}, O^{\prime}}^{\prime} \\
& =\left(1_{X E_{0}}^{\prime} \otimes i_{E_{0}}^{-1}\right) r_{X E_{0}}^{\prime} .
\end{aligned}
$$

Ad (FS): Since $s_{I^{\prime}, I^{\prime}}^{\prime}=1_{I^{\prime} \otimes I^{\prime}}^{\prime}$, we have

$$
\widetilde{E_{0}}\langle X, Y\rangle\left(s_{X, Y} E_{0}\right)=r_{I^{\prime}}^{\prime} 1_{I^{\prime}}^{\prime}=s_{I^{\prime}, I^{\prime}}^{\prime} r_{I^{\prime}}^{\prime}=s_{X E_{0}, Y E_{0}}^{\prime} \widetilde{E_{0}}\langle Y, X\rangle
$$

for $X \neq O \neq Y$.
Assuming $X=O$ or $Y=O$ one obtains

$$
X \otimes Y=O=Y \otimes X \text { and } \widetilde{E_{0}}\langle X, Y\rangle=1_{O^{\prime}}^{\prime}=o_{O^{\prime}, O^{\prime}}^{\prime}=\widetilde{E_{0}}\langle Y, X\rangle
$$

hence $\widetilde{E_{0}}\langle X, Y\rangle\left(s_{X, Y} E_{0}\right)=o_{O^{\prime}, O^{\prime}}^{\prime}=s_{X E_{0}, Y E_{0}}^{\prime} \widetilde{E_{0}}\langle Y, X\rangle$.

Ad (FM): For $O \notin\{X, Y, U, V\} \wedge \varphi \neq o_{X, Y} \wedge \psi \neq o_{U, V}$ one has $\varphi \otimes \psi \neq o_{X \otimes U, Y \otimes V}$, therefore

$$
\left(\varphi E_{0} \otimes \psi E_{0}\right) \widetilde{E_{0}}\langle Y, V\rangle=\left(1_{I^{\prime}}^{\prime} \otimes 1_{I^{\prime}}^{\prime}\right) r_{I^{\prime}}^{\prime}=r_{I^{\prime}}^{\prime} 1_{I^{\prime}}^{\prime}=\widetilde{E_{0}}\langle X, U\rangle\left((\varphi \otimes \psi) E_{0}\right) .
$$

In the case $O \in\{X, Y, U, V\}$ one obtains

$$
\begin{aligned}
& \varphi \otimes \psi=o_{X \otimes U, Y \otimes V} \wedge(\varphi \otimes \psi) E_{0}=o_{(X \otimes U) E_{0},(Y \otimes V) E_{0}}^{\prime} \wedge \\
& \varphi E_{0} \otimes \psi E_{0}=o_{X E_{0} \otimes U E_{0}, Y E_{0} \otimes V E_{0}}^{\prime}, \text { hence } \\
& \left(\varphi E_{0} \otimes \psi E_{0}\right) \widetilde{E_{0}}\langle Y, V\rangle=o_{X E_{0} \otimes U E_{0}, Y E_{0} \otimes V E_{0}} \widetilde{E}_{0}\langle Y, V\rangle=o_{X E_{0} \otimes U E_{0},(Y \otimes V) E_{0}}^{\prime} \\
& \quad=\widetilde{E}_{0}\langle X, U\rangle o_{(X \otimes U) E_{0},(Y \otimes V) E_{0}}^{\prime}=\widetilde{E}_{0}\langle X, U\rangle\left((\varphi \otimes \psi) E_{0}\right) .
\end{aligned}
$$

Ad (FD): The assumption $X \neq O$ yields directly

$$
d_{X} E_{0}=1_{I^{\prime}}^{\prime}=d_{I^{\prime}}^{\prime}, r_{I^{\prime}}^{\prime}=d_{X E_{0}}^{\prime} \widetilde{E}_{0}\langle X, X\rangle .
$$

For $X=O$ one has:

$$
d_{X} E_{0}=o_{O, O} E_{0}=o_{O^{\prime}, O^{\prime}}^{\prime}=d_{X E_{0}}^{\prime} \widetilde{E_{0}}\langle X, X\rangle
$$

Each $d$-monoidal functor ( $F, \widetilde{F}, i_{F}$ ) between $d h t s$-categories possesses the following properties (see [3], [8]):
( $\mathrm{FI}^{*}$ ) $\quad t_{I F}^{\prime}=i_{F}^{-1}$,
(Fmon) $\forall \varphi, \psi \in K(\varphi \leq \psi \Rightarrow \varphi F \leq \psi F)$,
(FT) $\quad \forall X \in|K|\left(t_{X} F t_{I F}^{\prime}=t_{X F}^{\prime}\right)$,
(FP) $\quad \forall X, Y \in|K|\left(p_{j}^{X, Y} F=(\widetilde{F}\langle X, Y\rangle)^{-1} p_{j}^{\prime X F, Y F} ; \quad j=1,2\right)$,
(FE) $\quad \forall A \in|K|\left(e \leq 1_{A} \Rightarrow e F \leq 1_{A F}\right)$,
$(\mathrm{FE} \alpha) \quad \forall X, Y \in|K| \forall \varphi \in K[X, Y]((\alpha(\varphi)) F=\alpha(\varphi F))$.

Let $\underline{K}, \underline{K^{\prime}}$ be $d h t h \nabla s$-categories and let $\left(F, \widetilde{F}, i_{F}\right): \underline{K} \rightarrow \underline{K^{\prime}}$ be a $d$-monoidal functor. Then in addition the following properties hold ([8]):

$$
\begin{array}{ll}
\text { (Finf) } & \forall X, Y \in|K| \forall \varphi, \psi \in K[X, Y] \\
& \left(\left(d_{X}(\varphi \otimes \psi) \nabla_{Y}\right) F=d_{X F}^{\prime}(\varphi F \otimes \psi F) \nabla_{Y F}^{\prime}\right), \\
\text { (Finj) } & \forall X, Y \in|K| \forall \varphi \in K[X, Y]
\end{array}
$$

$$
\begin{array}{ll} 
& \left((\varphi \otimes \varphi) \nabla_{Y}=\nabla_{X} \varphi \Rightarrow(\varphi F \otimes \varphi F) \nabla_{Y F}^{\prime}=\nabla_{X F}^{\prime}(\varphi F)\right), \\
(F \nabla) \quad & \forall X \in|K|\left(\nabla_{X F}=\widetilde{F}\langle X, X\rangle \nabla_{X}^{\prime} F\right), \\
\left(F \nabla_{1}\right) \quad & \forall X, Y, U \in|K| \forall \varphi \in K[X, U] \forall \psi \in K[Y, U] \\
& \left(\left((\varphi \otimes \psi) \nabla_{U}\right) F=\widetilde{F}\langle X, Y\rangle((\varphi \otimes \psi) F) \nabla_{U F}^{\prime}\right) \\
\left(F \nabla_{2}\right) \quad & \forall X, Y \in|K| \forall \varphi \cdot \psi \in K[X, Y] \\
& \left((\varphi \otimes \psi) \nabla_{Y}=\nabla_{X} \varphi \Rightarrow(\varphi F \otimes \psi F) \nabla_{Y F}^{\prime}=\nabla_{X F}^{\prime}(\varphi F)\right) .
\end{array}
$$

Obviously, property (Finj) is a special case of $\left(\mathrm{F} \nabla_{2}\right)$ and this property expresses once more the monotony of the functor $F$, namely $\varphi \leq \psi \Rightarrow \varphi F \leq \psi F$.
The so-called zero functor $Z: \underline{K} \rightarrow \underline{K^{\prime}}$ is defined by $X Z=O^{\prime}$ for all objects $X \in|K|$ and $\varphi Z=1_{O^{\prime}}^{\prime}$ for all morphisms $\varphi \in K$. Trivially, this functor is a $d$-monoidal one.

Definition 2.5 ([8]). A $d$-monoidal functor $\left(F, \widetilde{F}, i_{F}\right)$ between $d h t s$-categories will be called dht-monoidal functor, iff either $F=Z$ or, besides the conditions of a $d$-monoidal functor, the condition

$$
\begin{equation*}
O F=O^{\prime} \wedge \forall X \in|K|\left(X F=O^{\prime} \Rightarrow X=O\right) \tag{FZ}
\end{equation*}
$$

is fulfilled.
Proposition $2.6([8])$. Let $\left(F, \widetilde{F}, i_{F}\right): \underline{K} \rightarrow \underline{K}^{\prime}$ be a dht-monoidal functor such that $F \neq Z$. Then one obtains:

$$
\begin{aligned}
& \forall X \in|K|\left(\widetilde{F}\langle X, O\rangle=\widetilde{F}\langle O, X\rangle=1_{O^{\prime}}^{\prime}\right) \\
& \forall X, Y \in|K|\left(o o_{X, Y} F=o_{X F, Y F}^{\prime}\right) \\
& o F=t_{I F}^{\prime} o^{\prime} \quad\left(\Leftrightarrow o^{\prime}=i(o F)\right)
\end{aligned}
$$

By the structure of $d h t s$-categories $\underline{K}$ and $\underline{K^{\prime}}$, each functor $F: \underline{K} \rightarrow \underline{K^{\prime}}$ determines with respect to arbitrary objects $X, Y \in|K|$ the morphisms

$$
F^{*}\langle X, Y\rangle:=d_{(X \otimes Y) F}^{\prime}\left(p_{1}^{X, Y} F \otimes p_{2}^{X, Y} F\right) \in K^{\prime}[(X \otimes Y) F, X F \otimes Y F]
$$

in the category $K^{\prime}$.

In the case that $\left(F, \widetilde{F}, i_{F}\right): \underline{K} \rightarrow \underline{K^{\prime}}$ is a $d$-monoidal functor, the morphisms $\widetilde{F}\langle X, Y\rangle$ are uniquely determined by

$$
(\widetilde{F}\langle X, Y\rangle)^{-1}=d_{(X \otimes Y) F}^{\prime}\left(p_{1}^{X, Y} F \otimes p_{2}^{X, Y} F\right)=F^{*}\langle X, Y\rangle
$$

(see [2]).
Moreover:
Theorem 2.7 (see [8]). Assume that $F: \underline{K} \rightarrow \underline{K}^{\prime}$ is any functor from a dht-symmetric category $\underline{K}$ into a dht-symmetric category $\underline{K}^{\prime}$ satisfying the following conditions:
(F*) $\quad \forall X, Y \in|K|\left(F^{*}\langle X, Y\rangle \in\right.$ iso $\left.\left(K^{\prime}\right)\right)$,
$\left(\mathrm{FI}^{*}\right) \quad t^{\prime}{ }_{I F} \in \operatorname{iso}\left(K^{\prime}\right)$,
$\left(\mathrm{FM}^{*}\right) \quad \forall \varphi, \psi \in|K|\left((\varphi \otimes \psi) F F^{*}\left\langle X^{\prime}, Y^{\prime}\right\rangle=F^{*}\langle X, Y\rangle(\varphi F \otimes \psi F)\right)$.
Then $\left(F, \widetilde{F}, i_{F}\right): \underline{K} \rightarrow \underline{K}^{\prime}$ is d-monoidal with $\widetilde{F}\langle X, Y\rangle:=\left(F^{*}\langle X, Y\rangle\right)^{-1}$, $i_{F}:=t^{\prime-1}{ }_{I F}$.

## 3. Properties of the Hom-functors

Any $d t s$-category contains not necessarily an initial object $O$. For $d t s$-categories $\underline{K}$ without initial objects one has the following fact.

Theorem 3.1. Let $\underline{K}$ be a dts-category and let $H^{A}$ be the usual Homfunctor from the underlying category $K$ into the category Set with reference to any object $A \in|K|$. Then $H^{A}$ is a d-monoidal functor related to $\widetilde{H^{A}}$ and $i_{H^{A}}$, where

$$
\widetilde{H^{A}}=\left(\widetilde{H^{A}}\langle X, Y\rangle: K[A, X] \times K[A, Y] \rightarrow K[A, X \otimes Y]|X, Y \in| K \mid\right),
$$

defined by

$$
\left.\left(u_{1}, u_{2}\right) \mapsto\left(\widetilde{H^{A}}\langle X, Y\rangle\right)\left(\left(u_{1}, u_{2}\right)\right):=d_{A}\left(u_{1} \otimes u_{2}\right)\right),
$$

and

$$
i_{H^{A}}:\{\emptyset\} \rightarrow K[A, I] \text {, defined by }\left(\emptyset \mapsto t_{A}\right) .
$$

Proof. Let $A$ be any object of the category $K$. Then the functor properties of $H^{A}$ are well-known.
$\operatorname{Ad}(\mathrm{F} \sim)$ : Each ordered pair $\left(u_{1}, u_{2}\right) \in K[A, X] \times K[A, Y], X, Y \in|K|$, determines uniquely the morphism $d_{A}\left(u_{1} \otimes u_{2}\right) \in K[A, X \otimes Y]$. Conversely, each morphism $u \in K[A, X \otimes Y]$ determines the both morphisms $u p_{1}^{X, Y} \in$ $K[A, X], u p_{2}^{X, Y} \in K[A, Y]$ and $d_{A}\left(u_{1} \otimes u_{2}\right) p_{i}^{X, Y}=u_{i}$ for $i=1,2$ shows that $\widetilde{H^{A}}\langle X, Y\rangle$ is an isomorphism in Set.
Ad (FI): In any dts-category, the set $K[A, I]$ consists of one element only, therefore the only total function $i_{H^{A}}$ from $I^{\text {Set }}=\{\emptyset\}$ onto $I H^{A}=K[A, I]$ is an isomorphism in Set and $i_{H^{A}}^{-1}=t_{I H^{A}}^{\text {Set }}$.
Ad (FA): Let $u \in X H^{A}, v \in Y H^{A}, w \in Z H^{A}$ be arbitrary morphisms, $X, Y, Z \in|K|$. Then one has

$$
\begin{aligned}
& \left(a_{X H^{A}, Y H^{A}, Z H^{A}}^{S S e}\left(\widetilde{H^{A}}\langle X, Y\rangle \times 1_{Z H^{A}}^{S e t}\right) \widetilde{H^{A}}\langle X \otimes Y, Z\rangle\right)((u,(v, w))) \\
& \quad=\left(\left(\widetilde{H^{A}}\langle X, Y\rangle \times 1_{Z H^{A}}^{S e t}\right) \widetilde{H^{A}}\langle X \otimes Y, Z\rangle\right)(((u, v), w)) \\
& \quad=\widetilde{H^{A}}\langle X \otimes Y, Z\rangle\left(\left(d_{A}(u \otimes v), w\right)\right)=d_{A}\left(d_{A}(u \otimes v) \otimes w\right)
\end{aligned}
$$

and, on the other hand,

$$
\begin{aligned}
& \left(\left(1_{X H^{A}}^{S e t} \times \widetilde{H^{A}}\langle Y, Z\rangle\right) \widetilde{H^{A}}\langle X, Y \otimes Z\rangle\left(a_{X, Y, Z} H^{A}\right)\right)((u,(v, w))) \\
& \quad=\left(\widetilde{H^{A}}\langle X, Y \otimes Z\rangle\left(a_{X, Y, Z} H^{A}\right)\right)((u, d(v \otimes w))) \\
& \quad=\left(a_{X, Y, Z} H^{A}\right)\left(d_{A}\left(u \otimes d_{A}(v \otimes w)\right)\right) \\
& \quad=d_{A}\left(u \otimes d_{A}(v \otimes w)\right) a_{X, Y, Z}=d_{A}\left(1_{A} \otimes d_{A}\right) a_{A, A, A}((u \otimes v) \otimes w) \\
& \quad=d_{A}\left(d_{A}(u \otimes v) \otimes w\right),
\end{aligned}
$$

hence

$$
\begin{aligned}
& a_{X H^{A}, Y H^{A}, Z H^{A}}^{S e t}\left(\widetilde{H^{A}}\langle X, Y\rangle \times 1_{Z H^{A}}^{S e t}\right. \\
&\left.\widetilde{H^{A}}\langle X \otimes Y, Z\rangle\right) \\
&=\left(1_{X H^{A}}^{S e t} \times \widetilde{H^{A}}\langle Y, Z\rangle\right) \widetilde{H^{A}}\langle X, Y \otimes Z\rangle\left(a_{X, Y, Z} H^{A}\right) .
\end{aligned}
$$

Ad (FR): The morphism set $K[A, I]$ consists in each $d t s$-category of one element $t_{A}$ only. So one obtains for each morphism $u \in K[A, X]=X H^{A}$, $X \in|K|$,

$$
\begin{aligned}
\left(\widetilde{H^{A}}\langle X, I\rangle\left(r_{X} H^{A}\right)\right)\left(\left(u, t_{A}\right)\right) & =\left(r_{X} H^{A}\right)\left(d_{A}\left(u \otimes t_{A}\right)\right) \\
& =d_{A}\left(u \otimes t_{A}\right) r_{X}=d_{A}\left(1_{A} \otimes t_{A}\right) r_{A} u=u
\end{aligned}
$$

and

$$
\left(\left(1_{X H^{A}}^{\text {Set }} \times t_{I H^{A}}^{\text {Set }}\right) r_{X H^{A}}^{\text {Set }}\right)\left(\left(u, t_{A}\right)\right)=r_{X H^{A}}^{\text {Set }}((u, \emptyset))=u,
$$

thus the validity of (FR).

Ad (FS): Since for all $u \in X H^{A}, v \in Y H^{A}, X, Y \in|K|$, the morphisms

$$
\begin{gathered}
\left(\widetilde{H^{A}}\langle X, Y\rangle\left(s_{X, Y} H^{A}\right)((u, v))=\left(s_{X, Y} H^{A}\right)\left(d_{A}(u \otimes v)\right)=d_{A}(u \otimes v) s_{X, Y}\right. \\
=d_{A} s_{A, A}(v \otimes u)=d_{A}(v \otimes u) \text { and } \\
\left(s_{X H^{A}, Y H^{A}}^{S e t} \widetilde{H^{A}}\langle Y, X\rangle\right)((u, v))=\left(\widetilde{H^{A}}\langle Y, X\rangle\right)((v, u))=d_{A}(v \otimes u)
\end{gathered}
$$

coincide, the condition is fulfilled.

Ad (FM): The equation

$$
\begin{aligned}
& \left(\left(\varphi H^{A} \times \psi H^{A}\right) \widetilde{H^{A}}\langle Y, V\rangle\right)((u, v))=\left(\widetilde{H^{A}}\langle Y, V\rangle\right)((u \varphi, v \psi))=d_{A}(u \varphi \otimes v \psi) \\
& \quad=d_{A}(u \otimes v)(\varphi \otimes \psi)=\left((\varphi \otimes \psi) H^{A}\right)\left(d_{A}(u \otimes v)\right) \\
& \quad=\left(\widetilde{H^{A}}\langle X, U\rangle\left((\varphi \otimes \psi) H^{A}\right)\right)((u, v))
\end{aligned}
$$

is valid for all objects $X, Y, U, V \in|K|$ and all morphisms $\varphi \in K[X, Y], \psi \in$ $K[U, V], u \in K[A, X]=X H^{A}, v \in K[A, U]=U H^{A}$, therefore (FM) is an identity.

Ad (FD): For each $X \in|K|$ and all $u \in X H^{A}$ one has

$$
\left(d_{X} H^{A}\right)(u)=u d_{X}=d_{A}(u \otimes u)=\widetilde{H^{A}}\langle X, X\rangle((u, u))=\left(d_{X H^{A}}^{S e t} \widetilde{H^{A}}\langle X, X\rangle\right)(u) .
$$

The argumentation above shows that the proof of some properties only need the $d$-monoidal-symmetric structure of the category $K$, so one obtains:

Corollary 3.2. Let $K^{\bullet}$ be a ds-category. Then there is to each pair $(X, Y)$ of objects of $K^{\bullet}$ and each Hom-functor $H^{A}: K^{\bullet} \rightarrow$ Set a function $\widetilde{H^{A}}\langle X, Y\rangle$ : $K[A, X] \times K[A, Y] \rightarrow K[A, X \otimes Y]$ such that the conditions (FA), (FS), (FM), (FD) are valid.

Every $d t s$-category contains the distinguished terminal object $I$, i.e. that the set $K[X, I]$ consists of exatly one element $t_{X}$. Conversely, in general there are no information about the sets $K[I, X]$. There are possibly objects $X$ in $\underline{K}$ such that $K[I, X]=\emptyset$. The dts-category $\underline{\text { Set }}$ has the property that $S e t[I, X]$ contains at least one element for each $X \neq \emptyset$.

Let $\left(K^{\bullet}, d\right)$ be a $d s$-category containing an initial object $O$ with the property
(O1) $\quad \forall X \in|K|(O \otimes X=O=X \otimes O)$,
then every morphism set $K[O, X]$ constists of exactly one element, say $z_{X}$. In this case, $\left(K^{\bullet}, d\right)$ has the property

$$
\begin{aligned}
& \mathrm{(zz}) \quad \forall A, B, X \in|K|, \forall \varphi \in K[A, B] \\
& \quad\left(z_{A} \varphi=z_{B} \wedge \varphi \otimes z_{X}=z_{B \otimes X} \wedge z_{X} \otimes \varphi=z_{X \otimes B}\right)
\end{aligned}
$$

Proposition 3.3. Let $\underline{K}$ be a ds-category containing an initial object $O$ with the property (O1). Then this object $O$ induces a special d-monoidal functor $H^{O}: \underline{K} \rightarrow \underline{\text { Set }}$ as follows:

$$
\begin{aligned}
& \forall X \in|K|\left(X H^{O}=\left\{z_{X}\right\}\right), \\
& \forall U, V \in|K| \forall \varphi \in K[U, V]\left(\varphi H^{O}: K[O, U] \rightarrow K[O, V]\right. \\
& \left.\qquad\left(z_{U} \mapsto z_{U} \varphi=z_{V}\right)\right) \\
& \forall X, Y \in|K|\left(\widetilde{H^{O}}\langle X, Y\rangle: K[O, X] \times K[O, Y] \rightarrow K[O, X \otimes Y]\right. \\
& \left.\qquad\left(\left(z_{X}, z_{Y}\right) \mapsto z_{X \otimes Y}\right)\right) \\
& i_{H^{O}}:\{\emptyset\} \rightarrow K[O, I]\left(\emptyset \mapsto z_{I}\right)
\end{aligned}
$$

Proof. Since every morphism set $K[O, X]$ consists of exactly one element $z_{X}$ for each object $X \in|K|$, all sets of the form $K[O, X] \times K[O, Y]$ consist of one element too. Therefore, all functions $\widetilde{H^{O}}\langle X, Y\rangle$ are isomorphisms, the function $i_{H^{\circ}}$ is an isomorphism and all conditions (FA), (FR), (FS), (FM), and (FD) are fulfilled.
Remark that generally the $H o m$-functor $H^{A}$ is not a $d$-monoidal functor for arbitrary dhts-categories. To each element $\left(u_{1}, u_{2}\right) \in K\left[A, X_{1}\right] \times K\left[A, X_{2}\right]$ there is in fact allways the morphism $d_{A}\left(u_{1} \otimes u_{2}\right) \in K\left[A, X_{1} \otimes X_{2}\right]$, but $d\left(u_{1} \otimes u_{2}\right) p_{1}^{X_{1}, X_{2}}$ has not to be equal to $u_{i}(i=1,2)$ in general, therefore, $\widetilde{H^{A}}\left\langle X_{1}, X_{2}\right\rangle$ must not be an isomorphism in Par. The concept of a subidenty in $d h t s$-categories introduced by J. Schreckenberger ([3]) allows a modification of the concept of a Hom-functor such that one obtains a $d$-monoidal functor.

## 4. Functors Defined By Subidentities

Theorem 4.1 ([3]). Let $\underline{K}$ be a dhts-category. Each subidentity $e \leq 1_{A}$ in $\underline{K}$ determines a d-monoidal functor $\left(H^{e}, \tilde{H}^{e}, i_{H^{e}}\right): \underline{K} \rightarrow \underline{\text { Par }}$ by
(1) $X H^{e}:=\{u \in K[A, X] \mid \alpha(u)=e\}, X \in|K|$,
(2) $\quad(\varphi: X \rightarrow Y) H^{e}:=\varphi H^{e}: X H^{e} \rightarrow Y H^{e}(\in$ Par $)$, defined by

$$
\left(\varphi H^{e}\right)(u):=u \varphi \text { for } u \in D\left(\varphi H^{e}\right):=\left\{u \in X H^{e} \mid u \alpha(\varphi)=u\right\}
$$

(3) $\widetilde{H^{e}}\langle X, Y\rangle: X H^{e} \times Y H^{e} \rightarrow(X \otimes Y) H^{e}$, defined by

$$
\left(u_{1}, u_{2}\right) \mapsto \tilde{H}^{e}\langle X, Y\rangle\left(\left(u_{1}, u_{2}\right)\right):=d_{A}\left(u_{1} \otimes u_{2}\right)
$$

(4) $i_{H^{e}}:\{\emptyset\}=I^{\text {Par }} \rightarrow I H^{e}=\{u \in K[A, I] \mid \alpha(u)=e\}$.

Proof. Obviosly, $u \varphi \in K[A, Y]$ and $\alpha(u \varphi)=\alpha(u \alpha(\varphi))=\alpha(u)=e($ with respect to ( $\alpha 17$ ) and (2)) shows $u \varphi \in X^{\prime} H^{e}$.
$1_{X} H^{e}$ is the identical function $i d_{X H^{e}}$ of $X H^{e}$ since $D\left(1_{X} H^{e}\right)=\{u \in$ $\left.X H^{e} \mid u 1_{X}=u\right\}=X H^{e}$ and $\left(1_{X} H^{e}\right)(u)=u 1_{X}=u$ for all $u \in X H^{e}$, i.e. $1_{X} H^{e}=i d_{X H^{e}}$.

Let $\varphi \in K[X, Y], \psi \in K[Y, Z]$ be arbitrary morphisms in $K$. Then, by the properties of the subidenties and the defining conditions above, $(\varphi \psi) H^{e}=$ $\left(\varphi H^{e}\right)\left(\psi H^{e}\right)$ because of

$$
\begin{aligned}
& u \in D\left((\varphi \psi) H^{e}\right) \Rightarrow \alpha(u)=e \wedge u \alpha(\varphi \psi)=u \\
& \Rightarrow u \alpha(\varphi)=\alpha(u \varphi) u=\alpha(u \alpha(\varphi \psi) \varphi) u=\alpha(u \alpha(\varphi \psi) \alpha(\varphi)) u \\
& =\alpha(u \alpha(\varphi \psi)) u=\alpha(u) u=u \\
& \wedge u \varphi \alpha(\psi)=u \alpha(\varphi \psi) \varphi=u \varphi \wedge \alpha(u \varphi)=\alpha(u \alpha(\varphi))=\alpha(u)=e \\
& \Rightarrow u \in D\left(\varphi H^{e}\right) \wedge u \varphi \in D\left(\psi H^{e}\right) \Rightarrow u \in D\left(\left(\varphi H^{e}\right)\left(\psi H^{e}\right)\right) \\
& u \in D\left(\left(\varphi H^{e}\right)\left(\psi H^{e}\right)\right) \Rightarrow u \in D\left(\varphi H^{e}\right) \wedge u \varphi \in D\left(\psi H^{e}\right) \\
& \Rightarrow \alpha(u)=e \wedge u \alpha(\varphi)=u \wedge \alpha(u \varphi)=e \wedge u \varphi \alpha(\psi)=u \varphi \\
& \Rightarrow u \alpha(\varphi \psi)=\alpha(u \varphi \psi) u=\alpha(u \varphi \alpha(\psi)) u=\alpha(u \varphi) u=u \alpha(\varphi)=u \\
& \Rightarrow u \in D\left((\varphi \psi) H^{e}\right) ; \\
& u \in D\left((\varphi \psi) H^{e}\right)=D\left(\left(\varphi H^{e}\right)\left(\psi H^{e}\right)\right) \\
& \Rightarrow\left((\varphi \psi) H^{e}\right)(u)=u(\varphi \psi)=(u \varphi) \psi=\left(\left(\varphi H^{e}\right)(u)\right) \psi=\left(\left(\varphi H^{e}\right)\left(\psi H^{e}\right)\right)(u)
\end{aligned}
$$

Therefore, $H^{e}$ is a functor from $\underline{K}$ into $\underline{P a r}$.

To apply Theorem 2.7, one has to prove the conditions (F*), (FI*), and ( $\mathrm{FM}^{*}$ ) as follows:

Ad (F*): The function $\left(H^{e}\right)^{*}\langle X, Y\rangle=d_{(X \otimes Y) H^{e}}^{P a r}\left(p_{1}^{X, Y} H^{e} \times p_{2}^{X, Y} H^{e}\right)$ fulfils

$$
\left(\left(H^{e}\right)^{*}\langle X, Y\rangle\right)(u)=\left(u p_{1}^{X, Y}, u p_{2}^{X, Y}\right) \in X H^{e} \times Y H^{e}
$$

for each $u \in(X \otimes Y) H^{e}$, because of $\alpha\left(p_{j}^{X, Y}\right)=1_{X \otimes Y}, j=1,2$, i.e. $\left(H^{e}\right)^{*}\langle X, Y\rangle$ is always a total function from $(X \otimes Y) H^{e}$ into $X H^{e} \times Y H^{e}$.
Each ordered pair $\left(u_{1}, u_{2}\right) \in X H^{e} \times Y H^{e}$ determines uniquely the morphism $d_{A}\left(u_{1} \otimes u_{2}\right) \in K[A, X \otimes Y]$. Moreover, this morphism belongs to $(X \otimes Y) H^{e}$, since $\alpha\left(d_{A}\left(u_{1} \otimes u_{2}\right)\right)=\alpha\left(u_{1}\right) \alpha\left(u_{2}\right)=e e=e((\alpha 19))$, and $d_{A}\left(u_{1} \otimes u_{2}\right) \in D\left(d_{(X \otimes Y) H^{e}}^{P a r}\left(p_{1}^{X, Y} H^{e} \times p_{2}^{X, Y} H^{e}\right)\right)$ due to $\alpha\left(p_{j}^{X, Y}\right)=1_{X \otimes Y}$ and $d_{A}\left(u_{1} \otimes u_{2}\right) p_{1}^{X, Y}=\alpha\left(u_{2}\right) u_{1}=\alpha\left(u_{1}\right) u_{1}=u_{1}, d_{A}\left(u_{1} \otimes u_{2}\right) p_{2}^{X, Y}=\alpha\left(u_{1}\right) u_{2}=$ $\alpha\left(u_{2}\right) u_{2}=u_{2}$, hence $\left(H^{e}\right)^{*}\langle X, Y\rangle\left(d_{A}\left(u_{1} \otimes u_{2}\right)\right)=\left(u_{1}, u_{2}\right)$, i.e. $\left(H^{e}\right)^{*}\langle X, Y\rangle$ is a surjective function.
The property $d_{A}\left(u p_{1}^{X, Y} \otimes u p_{2}^{X, Y}\right)=u d_{X \otimes Y}\left(p_{1}^{X, Y} \otimes p_{2}^{X, Y}\right)=u$ shows in conclusion that $\left(H^{e}\right)^{*}\langle X, Y\rangle$ is an isomorphism in Par for all objects $X, Y \in$ $|K|$.
Ad $\left(\mathrm{FI}^{*}\right)$ : The set $I H^{e}=\{u \in K[A, I] \mid \alpha(u)=e\}$ is the one element set $\left\{e t_{A}\right\}$, since

$$
\alpha\left(e t_{A}\right)=\alpha\left(e \alpha\left(t_{A}\right)\right)=\alpha(e)=e
$$

and $\alpha(u)=e$ implies

$$
\begin{aligned}
e t_{A} & =\alpha(u) t_{A}=d_{A}\left(1_{A} \otimes u\right) p_{1}^{A, I} t_{A}=d_{A}\left(t_{A} \otimes u\right)\left(1_{I} \otimes t_{I}\right) r_{I} \\
& =d_{A}\left(t_{A} \otimes u\right)\left(t_{I} \otimes 1_{I}\right) l_{I}=d_{A}\left(t_{A} \otimes 1_{A}\right) l_{A} u=u .
\end{aligned}
$$

Therefore, $i_{H^{e}}: I^{\text {Par }}=\{\emptyset\} \rightarrow\left\{e t_{A}\right\}=I H^{e}$, defined by $\emptyset \mapsto e t_{A}$, is an isomorphism in Par and $i_{H^{e}}=\left(t_{I H^{e}}^{\text {Par }}\right)^{-1}$.

Ad $\left(\mathrm{FM}^{*}\right)$ : For every morphism $u \in(X \otimes Y) H^{e}$ and all morphisms $\varphi \in$ $K[X, U], \psi \in K[Y, V]$ one obtains

$$
\begin{gathered}
\left.\left(\left(H^{e}\right)^{*}\langle X, Y\rangle\right)\left(\varphi H^{e} \times \psi H^{e}\right)\right)(u)=\left(\varphi H^{e} \times \psi H^{e}\right)\left(\left(p_{1}^{X, Y} H^{e}\right)(u),\left(p_{2}^{X, Y} H^{e}\right)(u)\right) \\
=\left(\varphi H^{e} \times \psi H^{e}\right)\left(u p_{1}^{X, Y}, u p_{2}^{X, Y}\right)=\left(u p_{1}^{X, Y} \varphi, u p_{2}^{X, Y} \psi\right)
\end{gathered}
$$

and

$$
\begin{aligned}
((\varphi \otimes \psi) & \left.\left.H^{e}\left(H^{e}\right)^{*}\langle U, V\rangle\right)(u)=\left(H^{e}\right)^{*}\langle U, V\rangle\right)(u(\varphi \otimes \psi)) \\
& =\left(u(\varphi \otimes \psi) p_{1}^{U, V}, u(\varphi \otimes \psi) p_{2}^{U, V}\right)=\left(u\left(\varphi \otimes \psi t_{V}\right) r_{U}, u\left(\varphi t_{U} \otimes \psi\right) l_{V}\right) \\
& =\left(u\left(\alpha(\varphi) \varphi \otimes \alpha(\psi) t_{Y}\right) r_{U}, u\left(\alpha(\varphi) t_{X} \otimes \alpha(\psi) \psi\right) l_{V}\right) \\
& =\left(u(\alpha(\varphi) \otimes \alpha(\psi))\left(1_{X} \otimes t_{Y}\right) r_{X} \varphi, u(\alpha(\varphi) \otimes \alpha(\psi))\left(t_{X} \otimes 1_{Y}\right) l_{Y} \psi\right) \\
& \left.=(u \alpha(\varphi \otimes \psi)) p_{1}^{X, Y} \varphi, u \alpha(\varphi \otimes \psi) p_{2}^{X, Y} \psi\right) \\
& =\left(u p_{1}^{X, Y} \varphi, u p_{2}^{X, Y} \psi\right)
\end{aligned}
$$

respectively, hence

$$
\left.\left(H^{e}\right)^{*}\langle X, Y\rangle\right)\left(\varphi H^{e} \times \psi H^{e}\right)=(\varphi \otimes \psi) H^{e}\left(H^{e}\right)^{*}\langle U, V\rangle
$$

for all morphisms $\varphi, \psi \in K$.
The application of Theorem 2.7 shows that $\left(H^{e},\left(\left(H^{e}\right)^{*}\right)^{-1},\left(t_{I H^{e}}^{P a r}\right)^{-1}\right)$ is a $d$-monoidal functor.

Proposition 4.2 (see [3]). The d-monoidal functor $\left(H^{e}, \widetilde{H}^{e}, i_{H^{e}}\right): \underline{K} \rightarrow$ Par is even dht-monoidal if the conditions
$e \neq o_{A, A}$ and $\forall X \in|K| \backslash\{O\}\left(T_{K}[I, X]=\left\{\varphi \in K[I, X] \mid \varphi t_{X}=t_{I}\right\} \neq \emptyset\right)$
are fulfilled.

Proof. It remains to show the condition (FZ).
Since $e \neq o_{A, A}$, we have $A \neq O . O H^{e}$ is a subset of $K[A, O]=\left\{o_{A, O}\right\}$. Because of $\alpha\left(o_{A, O}\right)=o_{A, A} \neq e$, the set $O H^{e}$ is the empty set in Par.

Now let $X \neq O$ any object of $K$. Then

$$
T_{K}[A, X]=\left\{\varphi \in K[A, X] \mid \varphi t_{X}=t_{A}\right\} \neq \emptyset,
$$

since $\exists \psi \in T_{K}[I, X]\left(\alpha\left(t_{A} \psi\right)=\alpha\left(t_{A} \alpha(\psi)\right)=\alpha\left(t_{A} 1_{I}\right)=1_{A}\right)$, hence $t_{A} \psi \in$ $T_{K}[A, X]$. Then et $_{A} \psi$ belongs by $\alpha\left(e t_{A} \psi\right)=e$ to $X H^{e}$, hence $X H^{e} \neq \emptyset$.

Remark that in every $d h t s$-category $\underline{K}$ having the property

$$
\forall X \in|K| \backslash\{O\}\left(T_{K}[I, X]=\left\{\varphi \in K[I, X] \mid \varphi t_{X}=t_{I}\right\} \neq \emptyset\right)
$$

one obtains by the same reasons as in the proof above

$$
\forall X, Y \in|K| \backslash\{O\}\left(T_{K}[X, Y] \neq \emptyset\right) .
$$

Since the set $E_{K}(A)$ of all subidentities of an object $A$ in a dhts-category $\underline{K}$ forms a semilattice with maximal element $1_{A}$ and minimal element $o_{A, A}$, there are two particular cases for functors $H^{e}$, namely $e=o_{A, A}$ and $e=1_{A}$, respectively.

Corollary 4.3. Let $\underline{K}$ be any dhts-category. Then each object $A \in|K|$ determines the trivial ds-functor ( $H^{o_{A, A}}, \widehat{H^{o_{A, A}},}, i_{H^{o_{A, A}}}$ ) : $K \rightarrow$ Par defined by

$$
\begin{aligned}
& \forall X \in|K|\left(X H^{o_{A, A}}=\left\{o_{A, X}\right\}\right), \\
& \forall X, Y \in|K| \forall \varphi \in K[X, Y]\left(\varphi H^{o_{A, A}}:\left\{o_{A, X}\right\} \rightarrow\left\{o_{A, Y}\right\},\right. \\
& \left.\quad o_{A, X} \mapsto o_{A, X}\right), \\
& \forall X, Y \in|K| \widetilde{H^{o_{A}, A}}\langle X, Y\rangle: X H^{o_{A, A}} \times Y H^{o_{A, A}} \rightarrow(X \otimes Y) H^{o_{A, A}}, \\
& \\
& \left.\quad\left(o_{A, X}, o_{A, Y}\right) \mapsto o_{A, X \otimes Y}\right),
\end{aligned}
$$

$$
i_{H^{o} A, A}:\{\emptyset\} \rightarrow I H^{o_{A, A}}, \emptyset \mapsto o_{A, I} .
$$

Proof. If $e=o_{A, A}$, then $X H^{e}=\left\{u \in K[A, X] \mid \alpha(u)=e=o_{A, A}\right\}=$ $\left\{o_{A, X}\right\}$ for all $X \in|K|$ since $\alpha(u)=o_{A, A} \Rightarrow u=o_{A, X}$ by ( $\alpha 5$ ). Therefore, all sets $X H^{o_{A, A}}$ are one element sets and all functions being in consideration are isomorphisms between one element sets.

Corollary 4.4. Let $\underline{K}$ be a dhts-category such that

$$
\forall X \in|K| \backslash\{O\}\left(T_{K}[I, X]=\left\{\varphi \in K[I, X] \mid \varphi t_{X}=t_{I}\right\} \neq \emptyset\right)
$$

Then $\left(H^{1_{A}}, \widetilde{H^{1_{A}}}, e_{1_{A}}\right): \underline{K} \rightarrow \underline{\text { Par }}$ is a dht-monoidal functor for each object $A \in|K| \backslash\{O\}$ which maps every object $X \in|K| \backslash\{O\}$ to the set $T_{K}[A, X]$ of all total morphisms $u \in K[A, X]$, i.e. the usual Hom-functor $H^{A}: T_{K} \rightarrow$ Set is a restriction of $H^{1_{A}}: K \rightarrow$ Par.

Proof. It remains to show that $\varphi H^{1_{A}}: T_{K}[A, X] \rightarrow T_{K}[A, Y]$ is a total function for all $\varphi \in T_{K}[X, Y], X, Y \in|K| \backslash\{O\}$. Because of $(\alpha 4)$, one has $\alpha(\varphi)=1_{X}$ for $\varphi \in T_{K}[X, Y]$, hence $D\left(\varphi H^{1_{A}}\right)=\left\{u \in T_{K}[A, X] \mid u \alpha(\varphi)=\right.$ $\left.u 1_{X}=u\right\}=T_{K}[X, Y]$, thus $\varphi H^{1_{A}}$ id a total function.
The functors $H^{e}$ related to subidentities $e$ in $d h t s$-categories represent an important tool for the construction of full, faithful, and representative functors from a dhts-category $\underline{K}$ into $\underline{P a r}$, see the papers by J. Schreckenberger [3] and [4].

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