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POWER-ORDERED SETS

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Abstract

We define a natural ordering on the power set $\mathfrak{P}(Q)$ of any finite partial order Q, and we characterize those partial orders Q for which $\mathfrak{P}(Q)$ is a distributive lattice under that ordering.

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1. Introduction

For an unstructured set X, the power set $\mathfrak{P}(X)$, equipped with the partial order of inclusion, is a Boolean algebra. When we consider a partially ordered (finite) set $(Q, \leq)$, there is another (perhaps more natural) ordering on $\mathfrak{P}(Q)$:

For $A, B \subseteq Q$, let $A \leq B$ iff there is a 1-1 map $\pi : A \to B$ with $a \leq \pi(a)$ for all $a \in A$.

(For infinite sets this relation $\leq$ is in general not antisymmetric.)

We call the structure $(\mathfrak{P}(Q), \leq)$ a "power-ordered set". We will show that $(\mathfrak{P}(Q), \leq)$ is a distributive lattice iff Q is a chain or a horizontal sum (see Definition 3.1) of chains. We also remark that the complement operation on $\mathfrak{P}(X)$ is an involutory anti-automorphism of $(\mathfrak{P}(Q), \leq)$.

2. Powers of chains

Let L be a linear order. We will show that $\mathfrak{P}(L)$ is a distributive lattice. Our proof also gives an explicit description of the lattice operations of the power-ordered set $\mathfrak{P}(L)$ by representing $\mathfrak{P}(L)$ as a sublattice of a product of chains.

Setup 2.1. Let L be a linear order, $n \in \{1, 2, \ldots\}$. Let $-\infty \notin L$, and let $\bar{L} := \{-\infty\} \cup L$, with the obvious order.

Let $L^{\langle n \rangle}$ be the set of all n-tuples $(x_1, \ldots, x_n) \in \bar{L}^n$ which satisfy:

- $x_1 \geq x_2 \geq \cdots \geq x_n$;
- for all $\ell \in \{1, \ldots, n-1\}$: if $x_\ell \neq -\infty$, then $x_\ell > x_{\ell+1}$.

That is, we consider all strictly decreasing k-tuples from L, for $0 \leq k \leq n$, but we make them into n-tuples by appending the necessary number of copies of $-\infty$.

Fact 2.2. *Let L, $\bar{L}$, $L^{\langle n \rangle}$ be as above. Then*

- *$\bar{L}^n$, as a product of distributive lattices, is again a distributive lattice*
- *$L^{\langle n \rangle}$ is a sublattice of $\bar{L}^n$.*

Lemma 2.3. *Let L be a finite linear order.*

1. *Let $D, E \subseteq L$ be nonempty sets of the same cardinality. Then we can inductively analyse the relation $D \leq E$ in the power-ordered set $\mathfrak{P}(L)$ as follows:*

$$D \leq E \iff (D \setminus \{\max D\}) \leq (E \setminus \{\max E\}) \text{ and } \max D \leq \max E;$$

2. *If D and E are enumerated in decreasing order by $d_1 > \cdots > d_k$ and $e_1 > \cdots > e_k$, respectively, then*

$$D \leq E \iff d_1 \leq e_1 \;\&\; \cdots \;\&\; d_k \leq e_k.$$

Proof.
Proof of (1): $\Leftarrow$ is clear. Conversely, assume that π witnesses $D \leq E$.

Define a function $\hat{\pi} : D \to E$ as follows: if $\pi(\max D) = \max E$, then $\hat{\pi} = \pi$. Otherwise, let $\pi(x_0) = \max E$, for some (unique) $x_0 \in D \setminus \{\max D\}$ and let $y_0 = \pi(\max D)$. Define $\hat{\pi}(x_0) = y_0$, $\hat{\pi}(\max D) = \max E = \pi(x_0)$, and $\hat{\pi}(x) = \pi(x)$ otherwise.

Then also $\hat{\pi}$ witnesses $D \leq E$. [Why? We have to check $x_0 \leq \hat{\pi}(x_0)$. This follows from $x_0 \leq \max D \leq \pi(\max D) = \hat{\pi}(x_0)$.] Moreover, we have $\hat{\pi}(\max D) = \max E$. Now let $\pi_0 : D \setminus \{\max D\} \to E \setminus \{\max E\}$ be the restriction of π. Then π_0 witnesses $(D \setminus \{\max D\}) \leq (E \setminus \{\max E\})$.

Proof of (2) : This follows from (1) by induction. ∎

Fact 2.4. *If $E \subseteq L$, and E is enumerated in decreasing order by $e_1 > \cdots > e_k$, then:*

1. *for any $\ell \leq k$, every ℓ-element subset of E is $\leq \{e_1, \ldots, e_\ell\}$;*

2. *for any $\ell \leq k$, and any ℓ-element set $D \subseteq L$, we have $D \leq E$ iff $D \leq \{e_1, \ldots, e_\ell\}$.*

This fact allows us to reduce the question "$A \leq B$" to a question "$A \leq B'$", where B' has the same number of elements as A. Lemma 3.3 can then be used to compare A and B':

Conclusion 2.5. *Let L be a finite linear order with n elements, and let $L^{\langle n \rangle}$ be defined as above. Then $\mathfrak{P}(L)$ is (as a partial order, hence also as a lattice) isomorphic to $L^{\langle n \rangle}$.*

So $\mathfrak{P}(L)$ is a distributive lattice.

We can compute meet and join in $\mathfrak{P}(L)$ as follows: If $D = \{d_1, \ldots, d_\ell\} \subseteq L$ and $E = \{e_1, \ldots, e_k\} \subseteq L$, both in decreasing order, and $\ell \leq k$, then

- $D \wedge E = \{d_1 \wedge e_1, \ldots, d_\ell \wedge e_\ell\}$;
- $D \vee E = \{d_1 \vee e_1, \ldots, d_\ell \vee e_\ell, e_{\ell+1}, \ldots, e_k\}$.

Proof. The map $h : (x_1, \ldots, x_n) \mapsto \{x_1, \ldots, x_n\} \setminus \{-\infty\}$ is a bijection from $L^{\langle n \rangle}$ onto $\mathfrak{P}(L)$. We have to check that h and h^{-1} preserve order:

Let $(d_1, \ldots, d_n), (e_1, \ldots, e_n) \in L^{\langle n \rangle}$, and let $D := h(d_1, \ldots, d_n)$, $E := h(e_1, \ldots, e_n)$. If $(d_1, \ldots, d_n) \leq (e_1, \ldots, e_n)$ in the product partial order, then the map $\pi : D \to E$ defined by $\pi(d_i) = e_i$ for $d_i \neq -\infty$ witnesses $D \leq E$. (Note that $d_i \neq -\infty$ implies $e_i \neq -\infty$.)

Conversely, if $D \leq E$, then Lemma 2.3 and Fact 2.4 show that $(d_1, \ldots, d_n) \leq (e_1, \ldots, e_n)$. ■

3. SUMS OF CHAINS

Definition 3.1. Let $(Q_1, \leq_1)$ and $(Q_2, \leq_2)$ be disjoint partially ordered sets. The "horizontal sum" of Q_1 and Q_2 is the following partial order $(Q, \leq)$:

$Q = Q_1 \cup Q_2$, and $\leq \; = \; \leq_1 \cup \leq_2$, i.e., $x \leq y$ in Q iff for some

$\ell \in \{1, 2\}$ we have: $x, y \in Q_\ell$ and $x \leq_\ell y$.

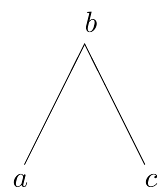
We write $(Q_1, \leq_1) + (Q_2, \leq_2)$ [or just $Q_1 + Q_2$] for the horizontal sum of Q_1 and Q_2.

Fact 3.2. *Let $Q = Q_1 + Q_2$. Then the partial order $\mathfrak{P}(Q)$ is naturally isomorphic to the product $\mathfrak{P}(Q_1) \times \mathfrak{P}(Q_2)$ (with the pointwise or "product" partial order).*

Proof. The map $(E_1, E_2) \mapsto E_1 \cup E_2$ is a bijection from $\mathfrak{P}(Q_1) \times \mathfrak{P}(Q_2)$ onto $\mathfrak{P}(Q_1 + Q_2)$, and it is easy to check that it is also an order isomorphism. ■

Definition 3.3. We write V for the 3-element partial order with a unique minimal and two maximal elements, and Λ for the dual order.

Lemma 3.4. *If Q is a partial order containing an isomorphic copy of Λ, then the power-ordered set $\mathfrak{P}(Q)$ is not a lattice.*



Proof. Let $a < b, c < b$ in Q, a and c be incomparable. We will show that in the partial order $\mathfrak{P}(Q)$ the elements $\{a, c\}$ and $\{b\}$ have no least upper bound.

Assume $E = \{a, c\} \vee \{b\}$. So, we have:

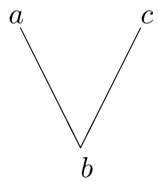
1. $\{a, c\} \leq E$.
2. $\{b\} \leq E$.
3. $E \leq \{a, b\}$ as $\{a, b\}$ is also an upper bound.
4. $E \leq \{c, b\}$, similarly.
5. By (1) and (3), E has exactly 2 elements.
6. By (3), both elements of E are $\leq b$, so by (2), $b \in E$.
7. Let $E = \{b, e\}$, $e \neq b$.
8. $e \leq a$, as $\{b, e\} \leq \{a, b\}$ (by (3)).
9. $e \leq c$, similarly. Hence $e < a$, $e < c$.
10. $a \leq e$ or $c \leq e$, as $\{a, c\} \leq \{b, e\}$ (by (1)).

Now (9) and (10) yield the desired contradiction. ∎

Lemma 3.5. *If Q is a finite partial order containing an isomorphic copy of V, then $\mathfrak{P}(Q)$ is either not a lattice, or a nondistributive lattice.*

Proof. Assume that $\mathfrak{P}(Q)$ is a lattice. By Lemma 3.4, every principal ideal $(a]$ in Q is linearly ordered (and finite, since Q is finite). Hence, for any $a, c \in Q$, $(a] \cap (c]$ is either empty or has a greatest element, in other words: if a and c have a common lower bound, then they have a greatest lower bound.

Assume that V embeds into Q, then there are incomparable elements a, c in Q with a greatest lower bound $b = a \wedge c$. As Λ does not embed into Q, a and c have no common upper bound, hence in $\mathfrak{P}(Q)$ we have



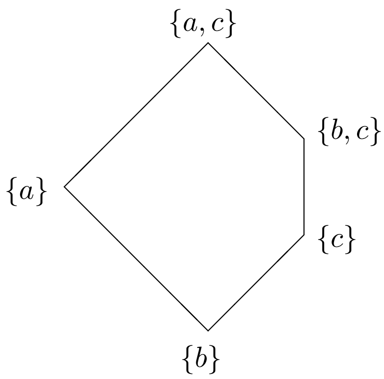
$$\{a\} \vee \{c\} = \{a, c\}$$

Also, $b = a \wedge c$ in Q implies that in the lattice $\mathfrak{P}(Q)$ we have

$$\{a\} \wedge \{b, c\} = \{b\}.$$

Proof: If $\{x\} \leq \{a\}$ and $\{x\} \leq \{b, c\}$, then $x \leq a$ and $x \leq c$, so $x \leq b$, $\{x\} \leq \{b\}$.

Hence the pentagon



is a sublattice of $\mathfrak{P}(Q)$, so $\mathfrak{P}(Q)$ is not distributive. ∎

Remark 3.6. $\mathfrak{P}(\mathrm{V})$ is in fact a lattice. In contrast, $\mathfrak{P}(\Lambda)$ is *not* a lattice.

Conclusion 3.7. *Let Q be a partial order. The following are equivalent:*

1. *Comparability is an equivalence relation on Q;*
2. *Q is a horizontal sum of chains;*
3. *Neither* V *nor* Λ *embeds into Q;*
4. *$\mathfrak{P}(Q)$ is a distributive lattice.*

Proof. (1) $\Leftrightarrow$ (2): The chains are just the equivalence classes.
(1) $\Leftrightarrow$ (3) is clear.
(2) $\Rightarrow$ (4) was proved in 2.5.
(4) $\Rightarrow$ (3) follows from 3.4 and 3.5. ■

4. Complements

Fact 4.1. *Let Q be a partial order, $A, B \subseteq Q$. Then:*

$$A \leq B \text{ iff } A \setminus B \leq B \setminus A.$$

Proof. Let $A_0 = A \setminus B = A \setminus (A \cap B)$, $B_0 = B \setminus A$.

If $\pi_0 : A_0 \to B_0$ witnesses $A_0 \leq B_0$, then we can extend π_0 by the identity function on $A \cap B$ to a map $\pi : A \to B$ witnessing $A \leq B$.

Conversely, let $\pi : A \to B$ witness $A \leq B$. Let π^n be the n-fold iterate of π (a *partial* function from A to B; e.g., $\pi^2(a)$ is only defined if $\pi(a) \in A \cap B$).

For each $a \in A_0 = A \setminus B$ let $n_a \geq 1$ be the first natural number such that $\pi^{n_a}(a) \notin A$. [Why does n_a exist? Note that a is not a fixpoint of π, $\pi(a) \neq a$, so no $\pi^n(a)$ can be a fixpoint of π, hence all $\pi^n(a)$ are distinct: $a < \pi(a) < \cdots$. But A is finite, so for some n we must have $\pi^n(a) \notin A$.]

Now define (for each $a \in A_0$): $\hat{\pi}(a) = \pi^{n_a}(a)$. Clearly $\hat{\pi} : A_0 \to B_0$, and $a < \hat{\pi}(a)$. To show that $\hat{\pi}$ is 1-1, assume $\hat{\pi}(a) = \hat{\pi}(a')$, and $n_{a'} = n_a + \ell$ for some $\ell \geq 0$. Since π is 1-1, $\pi^{n_a}(a) = \pi^{n_a+\ell}(a')$ implies $a = \pi^\ell(a')$, so since $a \notin B$ we must have $\ell = 0$, $a = a'$. ■

Lemma 4.2. *Let Q be a finite partial order. We will write $-X$ for $Q \setminus X$. Let $A, B \subseteq Q$. Then: $A \leq B$ iff $-B \leq -A$.*

Proof. By fact 4.1,

$$-B \leq -A \;\Leftrightarrow\; -B\smallsetminus(-A) \leq -A\smallsetminus(-B).$$

Now $-B\smallsetminus(-A) = A\smallsetminus B$, similarly $-A\smallsetminus(-B) = B\smallsetminus A$, so we can rewrite this as

$$-B \leq -A \;\Leftrightarrow\; A\smallsetminus B \leq B\smallsetminus A.$$

Again using Fact 4.1, we see that this is equivalent to $A \leq B$. ■

Hence the complement operation is an involutory anti-automorphism of $\mathfrak{P}(Q)$. If Q is an antichain, then $A \leq B$ iff $A \subseteq B$, so the power-ordered set $\mathfrak{P}(Q)$ is a Boolean algebra.

In general, the equation $A \wedge (-A) = \emptyset$ need not hold in the power-ordered set $\mathfrak{P}(Q)$. Indeed, if $a < b$ in Q, then $\{a\} \leq \{b\} \leq -\{a\}$.

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