

BALANCED d -LATTICES ARE COMPLEMENTED *

MARTIN GOLDSTERN

Technische Universität Wien
Institut für Algebra und Computermathematik
Wiedner Hauptstraße 8–10/118, A–1040 Wien, Austria

e-mail: martin.goldstern@tuwien.ac.at

<http://www.tuwien.ac.at/goldstern/>

AND

MIROSLAV PLOŠČICA[†]

Mathematical Institute, Slovak Academy of Sciences
Grešákova 6, 04001 Košice, Slovakia

e-mail: ploscica@saske.sk

<http://www.saske.sk/MI/eng/ploscica.htm>

Abstract

We characterize d -lattices as those bounded lattices in which every maximal filter/ideal is prime, and we show that a d -lattice is complemented iff it is balanced iff all prime filters/ideals are maximal.

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According to Chajda and Eigenthaler ([1]), a d -lattice is a bounded lattice L satisfying for all $a, c \in L$ the implications

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- (i) $(a, 1) \in \theta(0, c) \rightarrow a \vee c = 1$;
- (ii) $(a, 0) \in \theta(1, c) \rightarrow a \wedge c = 0$;

where $\theta(x, y)$ denotes the least congruence on L containing the pair (x, y) . Every bounded distributive lattice is a d -lattice. The 5-element nonmodular lattice N_5 is a d -lattice.

Theorem 1. *A bounded lattice is a d -lattice if and only if all maximal ideals and maximal filters are prime.*

Proof. Let I be a maximal ideal in a d -lattice L . Let $x, y \in L \setminus I$. We need to show that $x \wedge y \in L \setminus I$. Since I is maximal, there are $c_1, c_2 \in I$ such that $c_1 \vee x = c_2 \vee y = 1$. For $c = c_1 \vee c_2 \in I$ we have $c \vee x = c \vee y = 1$. Then $(x, 1) = (0 \vee x, c \vee x) \in \theta(0, c)$ and similarly $(y, 1) \in \theta(0, c)$, hence $(x \wedge y, 1) \in \theta(0, c)$. By (i) we have $(x \wedge y) \vee c = 1$, hence $x \wedge y \notin I$. The primality of maximal filters can be proved similarly.

Conversely, assume that all maximal ideals and filters in L are prime. To show (i), assume that $a, c \in L$, $a \vee c \neq 1$. By the Zorn lemma, there exists a maximal ideal I containing $a \vee c$. By our assumption, I is prime. Then $\alpha = I^2 \cup (L \setminus I)^2$ is a congruence on L . Since $c \in I$, we have $(0, c) \in \alpha$, which implies that $\theta(0, c) \subseteq \alpha$. Since $a \in I$, we have $(a, 1) \notin \alpha$, hence $(a, 1) \notin \theta(0, c)$. This shows (i). The proof of (ii) is similar. ■

By [1], a bounded lattice is called “balanced”, if the 0-class of any congruence determines the 1-class, and conversely. They showed that complemented lattices are balanced, and they asked:

(*) Is there a d -lattice which is balanced but not complemented?

We use the above characterization of d -lattices to answer this question.

If A is a subset of an algebra, write θ_A for the smallest congruence that identifies all elements of A ; if ϕ is a congruence, x an element, write x/ϕ for the ϕ -congruence class of x .

Further, a congruence ϕ (on an algebra with constants 0 and 1) is called balanced if $0/\phi = 0/\theta_{(1/\phi)}$ and $1/\phi = 1/\theta_{(0/\phi)}$; an algebra is called balanced iff all its congruence relations are balanced, or equivalently if: for any congruence relations ϕ, ϕ' we have:

$$0/\phi = 0/\phi' \text{ iff } 1/\phi = 1/\phi'.$$

Fix a d -lattice $(L, \vee, \wedge, 0, 1)$. For $a \in L$ we denote $F_a := \{x : x \vee a = 1\}$, and $I_a := \{x : x \wedge a = 0\}$.

Fact 2. F_a is a filter, I_a is an ideal.

Proof. Let $x, y \in F_a$. Similarly as in the proof of Theorem 1, $(x, 1) \in \theta(0, a)$, $(y, 1) \in \theta(0, a)$, hence $(x \wedge y, 1) \in \theta(0, a)$, which by the definition of a d -lattice implies $x \wedge y \in F_a$. The proof for I_a is similar. ■

Fact 3. If I is an ideal disjoint to F_a , and $a \notin I$, then also the ideal generated by $I \cup \{a\}$ is disjoint to F_a .

Proof. If $x \leq i \vee a$ for some $i \in I$, and $x \in F_a$, then also $i \vee a \in F_a$, hence $i \vee a = (i \vee a) \vee a = 1$. Thus, $i \in F_a$, so $F_a \cap I \neq \emptyset$. ■

Fact 4. If $f : L_1 \rightarrow L_2$ is a homomorphism from L_1 onto L_2 , and L_1 is balanced, then L_2 is balanced.

Proof. In fact, this holds “level-by-level”: If ϕ is an unbalanced congruence on L_2 , then the preimage of ϕ is unbalanced on L_1 . ■

Theorem 5. The following are equivalent (for a d -lattice L):

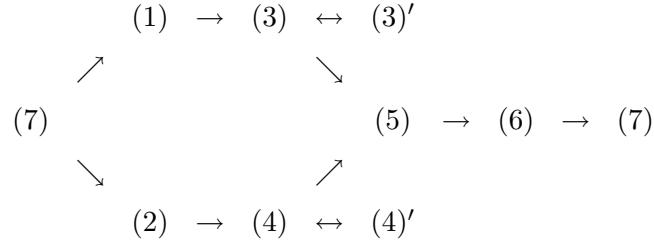
- (1) There is a maximal (hence prime) filter whose complement is not a maximal ideal.
- (2) There is a maximal (hence prime) ideal whose complement is not a maximal filter.
- (3) There are two prime ideals in L , one properly containing the other.
- (3)' There is a prime ideal in L which is not maximal.
- (4) There are two prime filters in L , one properly containing the other.
- (4)' There is a prime filter in L which is not maximal.
- (5) There is a homomorphism from L onto the 3-element lattice $\{0, d, 1\}$.

(6) L is not balanced.

(7) L is not complemented.

In particular a d -lattice is balanced iff it is complemented.

Proof.



(1) \rightarrow (3): By Theorem 1, the complement of a maximal filter is a (necessarily prime) ideal. If this ideal is not maximal, it can be properly extended to a maximal (hence prime) ideal. The proof of (2) \rightarrow (4) is similar (dual).

(3) \rightarrow (3)' is trivial, and (3)' \rightarrow (3) follows from Zorn's lemma and Theorem 1. Similarly we get (4) \leftrightarrow (4)'.

(3) \rightarrow (5): Let $I_1 \subset I_2 \subset L$ be prime ideals. Map I_1 to 0, $I_2 \setminus I_1$ to d , and $L \setminus I_2$ to 1. Check that this is a lattice homomorphism. The proof of (4) \rightarrow (5) is dual.

(5) \rightarrow (6) follows from Fact 4, since the three-element lattice is not balanced.

(6) \rightarrow (7) is from [1].

Now we show (7) \rightarrow (1). (Again, (7) \rightarrow (2) is dual.) Assume that L is not complemented, so there is some a such that $F_a \cap I_a = \emptyset$. Let F' be the filter generated by $F_a \cup \{a\}$. We have $F' \cap I_a = \emptyset$ by the dual of Fact 3, so F' is proper. By the Zorn lemma, F' can be extended to a maximal filter F . Let $I' = L \setminus F$. It is enough to see that I' is not maximal. Let I be the ideal generated by $I' \cup \{a\}$. By Fact 3, $I \cap F_a = \emptyset$, so I is a proper ideal properly extending I' . ■

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