# BALANCED $d$-LATTICES ARE COMPLEMENTED * 

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#### Abstract

We characterize $d$-lattices as those bounded lattices in which every maximal filter/ideal is prime, and we show that a $d$-lattice is complemented iff it is balanced iff all prime filters/ideals are maximal.


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According to Chajda and Eigenthaler ([1]), a d-lattice is a bounded lattice $L$ satisfying for all $a, c \in L$ the implications

[^0](i) $(a, 1) \in \theta(0, c) \rightarrow a \vee c=1$;
(ii) $(a, 0) \in \theta(1, c) \rightarrow a \wedge c=0$;
where $\theta(x, y)$ denotes the least congruence on $L$ containing the pair $(x, y)$. Every bounded distributive lattice is a $d$-lattice. The 5 -element nonmodular lattice $N_{5}$ is a $d$-lattice.

Theorem 1. A bounded lattice is a d-lattice if and only if all maximal ideals and maximal filters are prime.

Proof. Let $I$ be a maximal ideal in a $d$-lattice $L$. Let $x, y \in L \backslash I$. We need to show that $x \wedge y \in L \backslash I$. Since $I$ is maximal, there are $c_{1}, c_{2} \in I$ such that $c_{1} \vee x=c_{2} \vee y=1$. For $c=c_{1} \vee c_{2} \in I$ we have $c \vee x=c \vee y=1$. Then $(x, 1)=(0 \vee x, c \vee x) \in \theta(0, c)$ and similarly $(y, 1) \in \theta(0, c)$, hence $(x \wedge y, 1) \in \theta(0, c)$. By (i) we have $(x \wedge y) \vee c=1$, hence $x \wedge y \notin I$. The primality of maximal filters can be proved similarly.

Conversely, assume that all maximal ideals and filters in $L$ are prime. To show (i), assume that $a, c \in L, a \vee c \neq 1$. By the Zorn lemma, there exists a maximal ideal $I$ containing $a \vee c$. By our assumption, $I$ is prime. Then $\alpha=I^{2} \cup(L \backslash I)^{2}$ is a congruence on $L$. Since $c \in I$, we have $(0, c) \in \alpha$, which implies that $\theta(0, c) \subseteq \alpha$. Since $a \in I$, we have $(a, 1) \notin \alpha$, hence $(a, 1) \notin \theta(0, c)$. This shows (i). The proof of (ii) is similar.

By [1], a bounded lattice is called "balanced", if the 0 -class of any congruence determines the 1 -class, and conversely. They showed that complemented lattices are balanced, and they asked:
(*) Is there a $d$-lattice which is balanced but not complemented?

We use the above characterization of $d$-lattices to answer this question.
If $A$ is a subset of an algebra, write $\theta_{A}$ for the smallest congruence that identifies all elements of $A$; if $\phi$ is a congruence, $x$ an element, write $x / \phi$ for the $\phi$-congruence class of $x$.

Further, a congruence $\phi$ (on an algebra with constants 0 and 1 ) is called balanced if $0 / \phi=0 / \theta_{(1 / \phi)}$ and $1 / \phi=1 / \theta_{(0 / \phi)}$; an algebra is called balanced iff all its congruence relations are balanced, or equivalently if: for any congruence relations $\phi, \phi^{\prime}$ we have:

$$
0 / \phi=0 / \phi^{\prime} \text { iff } 1 / \phi=1 / \phi^{\prime}
$$

Fix a $d$-lattice $(L, \vee, \wedge, 0,1)$. For $a \in L$ we denote $F_{a}:=\{x: x \vee a=1\}$, and $I_{a}:=\{x: x \wedge a=0\}$.

Fact 2. $F_{a}$ is a filter, $I_{a}$ is an ideal.
Proof. Let $x, y \in F_{a}$. Similarly as in the proof of Theorem $1,(x, 1) \in$ $\theta(0, a),(y, 1) \in \theta(0, a)$, hence $(x \wedge y, 1) \in \theta(0, a)$, which by the definition of a $d$-lattice implies $x \wedge y \in F_{a}$. The proof for $I_{a}$ is similar.

Fact 3. If $I$ is an ideal disjoint to $F_{a}$, and $a \notin I$, then also the ideal generated by $I \cup\{a\}$ is disjoint to $F_{a}$.

Proof. If $x \leq i \vee a$ for some $i \in I$, and $x \in F_{a}$, then also $i \vee a \in F_{a}$, hence $i \vee a=(i \vee a) \vee a=1$. Thus, $i \in F_{a}$, so $F_{a} \cap I \neq \emptyset$.

Fact 4. If $f: L_{1} \rightarrow L_{2}$ is a homomorphism from $L_{1}$ onto $L_{2}$, and $L_{1}$ is balanced, then $L_{2}$ is balanced.

Proof. In fact, this holds "level-by-level": If $\phi$ is an unbalanced congruence on $L_{2}$, then the preimage of $\phi$ is unbalanced on $L_{1}$.

Theorem 5. The following are equivalent (for a d-lattice L):
(1) There is a maximal (hence prime) filter whose complement is not a maximal ideal.
(2) There is a maximal (hence prime) ideal whose complement is not a maximal filter.
(3) There are two prime ideals in $L$, one properly containing the other.
(3) There is a prime ideal in $L$ which is not maximal.
(4) There are two prime filters in $L$, one properly containing the other.
(4) ${ }^{\prime}$ There is a prime filter in $L$ which is not maximal.
(5) There is a homomorphism from $L$ onto the 3-element lattice $\{0, d, 1\}$.
(6) $L$ is not balanced.
(7) $L$ is not complemented.

In particular a d-lattice is balanced iff it is complemented.

## Proof.


$(1) \rightarrow(3):$ By Theorem 1, the complement of a maximal filter is a (necessarily prime) ideal. If this ideal is not maximal, it can be properly extended to a maximal (hence prime) ideal. The proof of $(2) \rightarrow(4)$ is similar (dual).
$(3) \rightarrow(3)^{\prime}$ is trivial, and $(3)^{\prime} \rightarrow(3)$ follows from Zorn's lemma and Theorem 1. Similarly we get $(4) \leftrightarrow(4)^{\prime}$.
$(3) \rightarrow(5):$ Let $I_{1} \subset I_{2} \subset L$ be prime ideals. Map $I_{1}$ to $0, I_{2} \backslash I_{1}$ to $d$, and $L \backslash I_{2}$ to 1 . Check that this is a lattice homomorphism. The proof of $(4) \rightarrow(5)$ is dual.
$(5) \rightarrow(6)$ follows from Fact 4 , since the three-element lattice is not balanced.
$(6) \rightarrow(7)$ is from [1].

Now we show $(7) \rightarrow(1)$. (Again, $(7) \rightarrow(2)$ is dual.) Assume that $L$ is not complemented, so there is some $a$ such that $F_{a} \cap I_{a}=\emptyset$. Let $F^{\prime}$ be the filter generated by $F_{a} \cup\{a\}$. We have $F^{\prime} \cap I_{a}=\emptyset$ by the dual of Fact 3 , so $F^{\prime}$ is proper. By the Zorn lemma, $F^{\prime}$ can be extended to a maximal filter $F$. Let $I^{\prime}=L \backslash F$. It is enough to see that $I^{\prime}$ is not maximal. Let $I$ be the ideal generated by $I^{\prime} \cup\{a\}$. By Fact $3, I \cap F_{a}=\emptyset$, so $I$ is a proper ideal properly extending $I^{\prime}$.

## References

[1] I. Chajda and G. Eigenthaler, Balanced congruences, Discuss. Math. - Gen. Algebra App. 21 (2001), 105-114.
[2] G. Grätzer, General Lattice Theory (the second edition), Birkhäuser Verlag, Basel 1998.

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