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BALANCED *d*-LATTICES ARE COMPLEMENTED *

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Abstract

We characterize d-lattices as those bounded lattices in which every maximal filter/ideal is prime, and we show that a d-lattice is complemented iff it is balanced iff all prime filters/ideals are maximal.

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According to Chajda and Eigenthaler ([1]), a d-lattice is a bounded lattice L satisfying for all $a, c \in L$ the implications

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(i) $(a,1) \in \theta(0,c) \;\to\; a \vee c = 1$;

(ii) $(a,0) \in \theta(1,c) \;\to\; a \wedge c = 0$;

where $\theta(x,y)$ denotes the least congruence on L containing the pair (x,y). Every bounded distributive lattice is a d-lattice. The 5-element nonmodular lattice N_5 is a d-lattice.

Theorem 1. *A bounded lattice is a d-lattice if and only if all maximal ideals and maximal filters are prime.*

Proof. Let I be a maximal ideal in a d-lattice L. Let $x,y \in L\smallsetminus I$. We need to show that $x \wedge y \in L\smallsetminus I$. Since I is maximal, there are $c_1, c_2 \in I$ such that $c_1 \vee x = c_2 \vee y = 1$. For $c = c_1 \vee c_2 \in I$ we have $c \vee x = c \vee y = 1$. Then $(x,1) = (0 \vee x, c \vee x) \in \theta(0,c)$ and similarly $(y,1) \in \theta(0,c)$, hence $(x \wedge y, 1) \in \theta(0,c)$. By (i) we have $(x \wedge y) \vee c = 1$, hence $x \wedge y \notin I$. The primality of maximal filters can be proved similarly.

Conversely, assume that all maximal ideals and filters in L are prime. To show (i), assume that $a, c \in L$, $a \vee c \neq 1$. By the Zorn lemma, there exists a maximal ideal I containing $a \vee c$. By our assumption, I is prime. Then $\alpha = I^2 \cup (L\smallsetminus I)^2$ is a congruence on L. Since $c \in I$, we have $(0,c) \in \alpha$, which implies that $\theta(0,c) \subseteq \alpha$. Since $a \in I$, we have $(a,1) \notin \alpha$, hence $(a,1) \notin \theta(0,c)$. This shows (i). The proof of (ii) is similar. ∎

By [1], a bounded lattice is called "balanced", if the 0-class of any congruence determines the 1-class, and conversely. They showed that complemented lattices are balanced, and they asked:

($*$) Is there a d-lattice which is balanced but not complemented?

We use the above characterization of d-lattices to answer this question.

If A is a subset of an algebra, write θ_A for the smallest congruence that identifies all elements of A; if ϕ is a congruence, x an element, write x/ϕ for the ϕ-congruence class of x.

Further, a congruence ϕ (on an algebra with constants 0 and 1) is called balanced if $0/\phi = 0/\theta_{(1/\phi)}$ and $1/\phi = 1/\theta_{(0/\phi)}$; an algebra is called balanced iff all its congruence relations are balanced, or equivalently if: for any congruence relations ϕ, ϕ' we have:

$$0/\phi = 0/\phi' \text{ iff } 1/\phi = 1/\phi'.$$

Fix a d-lattice $(L, \vee, \wedge, 0, 1)$. For $a \in L$ we denote $F_a := \{x : x \vee a = 1\}$, and $I_a := \{x : x \wedge a = 0\}$.

Fact 2. *F_a is a filter, I_a is an ideal.*

Proof. Let $x, y \in F_a$. Similarly as in the proof of Theorem 1, $(x, 1) \in \theta(0, a)$, $(y, 1) \in \theta(0, a)$, hence $(x \wedge y, 1) \in \theta(0, a)$, which by the definition of a d-lattice implies $x \wedge y \in F_a$. The proof for I_a is similar. ■

Fact 3. *If I is an ideal disjoint to F_a, and $a \notin I$, then also the ideal generated by $I \cup \{a\}$ is disjoint to F_a.*

Proof. If $x \leq i \vee a$ for some $i \in I$, and $x \in F_a$, then also $i \vee a \in F_a$, hence $i \vee a = (i \vee a) \vee a = 1$. Thus, $i \in F_a$, so $F_a \cap I \neq \emptyset$. ■

Fact 4. *If $f : L_1 \to L_2$ is a homomorphism from L_1 onto L_2, and L_1 is balanced, then L_2 is balanced.*

Proof. In fact, this holds "level-by-level": If ϕ is an unbalanced congruence on L_2, then the preimage of ϕ is unbalanced on L_1. ■

Theorem 5. *The following are equivalent (for a d-lattice L):*

(1) *There is a maximal (hence prime) filter whose complement is not a maximal ideal.*

(2) *There is a maximal (hence prime) ideal whose complement is not a maximal filter.*

(3) *There are two prime ideals in L, one properly containing the other.*

(3)′ *There is a prime ideal in L which is not maximal.*

(4) *There are two prime filters in L, one properly containing the other.*

(4)′ *There is a prime filter in L which is not maximal.*

(5) *There is a homomorphism from L onto the 3-element lattice $\{0, d, 1\}$.*

(6) *L is not balanced.*

(7) *L is not complemented.*

In particular a d-lattice is balanced iff it is complemented.

Proof.

$$
\begin{array}{ccccccccccc}
 & & (1) & \rightarrow & (3) & \leftrightarrow & (3)' & & & & \\
 & \nearrow & & & & & & \searrow & & & \\
(7) & & & & & & & & (5) \rightarrow (6) \rightarrow (7) & & \\
 & \searrow & & & & & & \nearrow & & & \\
 & & (2) & \rightarrow & (4) & \leftrightarrow & (4)' & & & &
\end{array}
$$

(1) $\rightarrow$ (3): By Theorem 1, the complement of a maximal filter is a (necessarily prime) ideal. If this ideal is not maximal, it can be properly extended to a maximal (hence prime) ideal. The proof of (2) $\rightarrow$ (4) is similar (dual).

(3) $\rightarrow$ (3)$'$ is trivial, and (3)$'$ $\rightarrow$ (3) follows from Zorn's lemma and Theorem 1. Similarly we get (4) $\leftrightarrow$ (4)$'$.

(3) $\rightarrow$ (5): Let $I_1 \subset I_2 \subset L$ be prime ideals. Map I_1 to 0, $I_2 \smallsetminus I_1$ to d, and $L \smallsetminus I_2$ to 1. Check that this is a lattice homomorphism. The proof of (4) $\rightarrow$ (5) is dual.

(5) $\rightarrow$ (6) follows from Fact 4, since the three-element lattice is not balanced.

(6) $\rightarrow$ (7) is from [1].

Now we show (7) $\rightarrow$ (1). (Again, (7) $\rightarrow$ (2) is dual.) Assume that L is not complemented, so there is some a such that $F_a \cap I_a = \emptyset$. Let F' be the filter generated by $F_a \cup \{a\}$. We have $F' \cap I_a = \emptyset$ by the dual of Fact 3, so F' is proper. By the Zorn lemma, F' can be extended to a maximal filter F. Let $I' = L \smallsetminus F$. It is enough to see that I' is not maximal. Let I be the ideal generated by $I' \cup \{a\}$. By Fact 3, $I \cap F_a = \emptyset$, so I is a proper ideal properly extending I'. ∎

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