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# THE LATTICE OF SUBVARIETIES OF THE BIREGULARIZATION OF THE VARIETY OF BOOLEAN ALGEBRAS

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**Abstract**

Let $\tau : F \to N$ be a type of algebras, where $F$ is a set of fundamental operation symbols and $N$ is the set of all positive integers. An identity $\varphi \approx \psi$ is called *biregular* if it has the same variables in each of it sides and it has the same fundamental operation symbols in each of it sides. For a variety $V$ of type $\tau$ we denote by $V_b$ the *biregularization* of $V$, i.e. the variety of type $\tau$ defined by all biregular identities from $Id(V)$.

Let $B$ be the variety of Boolean algebras of type $\tau_b : \{+, \cdot, '\} \to N$, where $\tau_b(+) = \tau_b(\cdot) = 2$ and $\tau_b(') = 1$. In this paper we characterize the lattice $\mathcal{L}(B_b)$ of all subvarieties of the biregularization of the variety $B$.

**Keywords:** subdirectly irreducible algebra, lattice of subvarieties, Boolean algebra, biregular identity.

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## 0. Preliminaries

We shall consider algebras of type $\tau : F \to N$, where $F$ is the set of all fundamental operation symbols and $N$ is the set of all positive integers (see [3]). If $\varphi$ is a term of type $\tau$ we denote by $Var(\varphi)$ the set of all variables occurring in $\varphi$ and by $F(\varphi)$ – the set of fundamental operation symbols occurring in $\varphi$. Writing $\varphi(x_{i_1}, \ldots, x_{i_n})$ instead of $\varphi$ we shall mean that $Var(\varphi) = \{x_{i_1}, \ldots, x_{i_n}\}$. An identity $\varphi \approx \psi$ of type $\tau$ is called *regular* (see

[8]) if $Var(\varphi) = Var(\psi)$. An identity $\varphi \approx \psi$ is called *biregular* if it is regular and $F(\varphi) = F(\psi)$. Regular identities and constructions connected with them were considered in [4]–[6], [8], [9], [16] and biregular identities were considered in [10]–[12], [14], [15], [18].

For a variety $V$ of type $\tau$ we denote by $Id(V)$ the set of all identities of type $\tau$ satisfied in every algebra from $V$. For a variety $V$ of type $\tau$ we denote by $V_r$ the variety of type $\tau$ defined by all regular identities from $Id(V)$ and we denote by $V_b$ the variety of type $\tau$ defined by the set $B(V)$ of all biregular identities from $Id(V)$. Obviously $B(V)$ is always an equational theory, so $Id(V_b) = B(V)$. The variety $V_b$ is called the biregularization of $V$. We denote by $\mathcal{L}(V)$ the lattice of all subvarieties of $V$. Studying identities of some special structural forms is useful for examining lattices of subvarieties. Let $B$ be the variety of Boolean algebras of type $\tau_b : \{+, \cdot, '\} \to N$, where $\tau_b(+) = \tau_b(\cdot) = 2$ and $\tau_b(') = 1$. In this paper we describe the lattice $\mathcal{L}(B_b)$.

Recall that an algebra $\mathfrak{A}$ is subdirectly irreducible if its lattice of congruences has exactly one atom (see [7]). If an algebra $\mathfrak{A}$ is subdirectly irreducible, we shall write shortly $\mathfrak{A}$ is an s.i. algebra. The notation $\mathfrak{A} \simeq \mathfrak{A}'$ will stand for "$\mathfrak{A}$ is isomorphic to $\mathfrak{A}'$".

## 1. Subdirectly irreducible algebras in $B_b$

Let us consider the following 14 algebras of type $\tau_b$.

$$\mathfrak{A}_1 = (\{a_1, b_1\}; +, \cdot, '), \quad \text{where} \quad x + y = \begin{cases} b_1, & \text{if } b_1 \in \{x, y\}, \\ a_1 & \text{otherwise}, \end{cases}$$

$$x \cdot y = \begin{cases} a_1, & \text{if } a_1 \in \{x, y\}, \\ b_1 & \text{otherwise}, \end{cases}$$

$$a_1' = b_1, \quad b_1' = a_1;$$

$$\mathfrak{A}_2 = (\{a_2, b_2\}; +, \cdot, '), \quad \text{where} \quad x + y = \begin{cases} b_2 & \text{if } b_2 \in \{x, y\} \\ a_2 & \text{otherwise}, \end{cases}$$

$$x \cdot y = x' = b_2 \quad \text{for every} \quad x, y \in \{a_2, b_2\};$$

$$\mathfrak{A}_3 = (\{a_3, b_3\};\, +, \cdot, {}'), \quad \text{where} \quad x \cdot y \;=\; \begin{cases} b_3, & \text{if} \;\; b_3 \in \{x, y\}, \\ a_3 & \text{otherwise}, \end{cases}$$

$$x + y = x' = b_3 \quad \text{for every} \quad x, y \in \{a_3, b_3\};$$

$\mathfrak{A}_4 = (\{a_4, b_4\};\, +, \cdot, {}')$, where

$$x + y = x \cdot y = x' = b_4 \quad \text{for every} \quad x, y \in \{a_4, b_4\};$$

$\mathfrak{A}_5 = (\{a_5, b_5\};\, +, \cdot, {}')$, where

$$x + y = x \cdot y = b_5, \;\; x' = x \quad \text{for every} \quad x, y \in \{a_5, b_5\};$$

$\mathfrak{A}_6 = (\{a_6, c_6, b_6\};\, +, \cdot, {}')$, where

$$x + y = x \cdot y = b_6, \quad \text{for every} \quad x, y \in \{a_6, c_6, b_6\},$$

$$a_6' = c_6, \;\; c_6' = a_6, \;\; b_6' = b_6;$$

$\mathfrak{A}_7 = (\{a_7, b_7\};\, +, \cdot, {}')$, where

$$x + y = x \cdot y \;=\; \begin{cases} b_7, & \text{if} \;\; b_7 \in \{x, y\}, \\ a_7 & \text{otherwise}, \end{cases}$$

$$x' = x \quad \text{for every} \quad x \in \{a_7, b_7\};$$

$\mathfrak{A}_8 = (\{a_8, c_8, b_8\};\, +, \cdot, {}')$, where

$$x + y \;=\; \begin{cases} b_8, & \text{if} \;\; b_8 \in \{x, y\}, \\ c_8, & \text{if} \;\; c_8 \in \{x, y\} \text{ and } b_8 \notin \{x, y\}, \\ a_8 & \text{otherwise}, \end{cases}$$

$$x \cdot y \;=\; \begin{cases} b_8, & \text{if} \;\; b_8 \in \{x, y\}, \\ a_8, & \text{if} \;\; a_8 \in \{x, y\} \text{ and } b_8 \notin \{x, y\}, \\ c_8 & \text{otherwise}, \end{cases}$$

$$a_8' = c_8, \;\; c_8' = a_8, \;\; b_8' = b_8;$$

$\mathfrak{A}_9 = (\{a_9, b_9\};\, +, \cdot, ')$, where

$$x + y = x \cdot y \quad = \quad \begin{cases} b_9, & \text{if } \ b_9 \in \{x, y\}, \\ a_9 & \text{otherwise,} \end{cases}$$

$$x' = b_9, \quad \text{for every} \quad x \in \{a_9, b_9\};$$

$\mathfrak{A}_{10} = (\{a_{10}, c_{10}, b_{10}\};\, +, \cdot, ')$, where

$$x + y \quad = \quad \begin{cases} b_{10}, & \text{if } \ b_{10} \in \{x, y\}, \\ c_{10}, & \text{if } \ c_{10} \in \{x, y\} \text{ and } b_{10} \notin \{x, y\}, \\ a_{10} & \text{otherwise,} \end{cases}$$

$$x \cdot y \quad = \quad \begin{cases} b_{10}, & \text{if } \ b_{10} \in \{x, y\}, \\ a_{10}, & \text{if } \ a_{10} \in \{x, y\} \text{ and } b_{10} \notin \{x, y\}, \\ c_{10} & \text{otherwise,} \end{cases}$$

$$x' = b_{10} \quad \text{for every} \ \ x \in \{a_{10}, c_{10}, b_{10}\};$$

$\mathfrak{A}_{11} = (\{a_{11}, b_{11}\};\, +, \cdot, ')$, where

$$x + y \quad = \quad \begin{cases} b_{11}, & \text{if } \ b_{11} \in \{x, y\}, \\ a_{11} & \text{otherwise,} \end{cases}$$

$$x \cdot y = b_{11} \quad \text{for every} \quad x, y \in \{a_{11}, b_{11}\},$$

$$x' = x \quad \text{for every} \quad x \in \{a_{11}, b_{11}\};$$

$\mathfrak{A}_{12} = (\{a_{12}, c_{12}, b_{12}\}; +, \cdot, ')$, where

$$x + y = \begin{cases} b_{12}, & \text{if } b_{12} \in \{x, y\}, \\ c_{12}, & \text{if } c_{12} \in \{x, y\} \text{ and } b_{12} \notin \{x, y\}, \\ a_{12} & \text{otherwise,} \end{cases}$$

$$x \cdot y = b_{12} \quad \text{for every} \quad x, y \in \{a_{12}, c_{12}, b_{12}\},$$

$$a'_{12} = c_{12}, \; c'_{12} = a_{12}, \; b'_{12} = b_{12};$$

$\mathfrak{A}_{13} = (\{a_{13}, b_{13}\}; +, \cdot, ')$, where

$$x + y = b_{13} \text{ for every } x, y \in \{a_{13}, b_{13}\},$$

$$x \cdot y = \begin{cases} b_{13}, & \text{if } b_{13} \in \{x, y\}, \\ a_{13} & \text{otherwise,} \end{cases}$$

$$x' = x \text{ for every } x \in \{a_{13}, b_{13}\};$$

$\mathfrak{A}_{14} = (\{a_{14}, c_{14}, b_{14}\}; +, \cdot, ')$, where

$$x + y = b_{14} \text{ for every } x, y \in \{a_{14}, c_{14}, b_{14}\},$$

$$x \cdot y = \begin{cases} b_{14}, & \text{if } b_{14} \in \{x, y\}, \\ a_{14}, & \text{if } a_{14} \in \{x, y\} \text{ and } b_{14} \notin \{x, y\}, \\ c_{14} & \text{otherwise,} \end{cases}$$

$$a'_{14} = c_{14}, \; c'_{14} = a_{14}, \; b'_{14} = b_{14}.$$

It is easy to check that none two of above 14 algebras are isomorphic.

**Theorem 1.1.** *Let* $\mathfrak{A} = (A; +, \cdot, ')$ *be an algebra of type* $\tau_b$. *Then* $\mathfrak{A}$ *is subdirectly irreducible and belongs to* $B_b$ *if and only if* $\mathfrak{A}$ *is isomorphic to one of the algebras* $\mathfrak{A}_1, \ldots, \mathfrak{A}_{14}$.

***Proof.*** For varieties $K_1, \ldots, K_n$ of the same type we denote by $K_1 \otimes \cdots \otimes K_n$ the class of all algebras isomorphic to a subdirect product of a family $\{\mathfrak{A}_1, \ldots, \mathfrak{A}_n\}$ of algebras, where $\mathfrak{A}_i$ runs over $K_i$ for every $i = 1, \ldots, n$.

For $\widetilde{F} \subseteq \{+, \cdot, '\}$, we denote by $B_{\widetilde{F}}$ the variety of type $\tau_b$ satisfying all regular identities $\varphi \approx \psi$ from $Id(B)$ with $F(\varphi) \cup F(\psi) \subseteq \widetilde{F}$ and satisfying all identities of type $\tau_b$ such that $F(\varphi) \cap (\{+, \cdot, '\} \setminus \widetilde{F}) \neq \varnothing \neq F(\psi) \cap (\{+, \cdot, '\} \setminus \widetilde{F})$. It was proved in [12], Theorem 9, that

$$B_b = B_r \otimes B_{\{+,\cdot\}} \otimes B_{\{+,'\}} \otimes B_{\{\cdot,'\}} \otimes B_{\{+\}} \otimes B_{\{\cdot\}} \otimes B_{\{'\}} \otimes B_\varnothing \tag{1.1}$$

Consequently to find all subdirectly irreducible algebras from $B_b$ it is enough to find all s.i. algebras from the varieties of the right side of (1.1).

It was proved in [6] that $\mathfrak{A}$ is s.i. and $\mathfrak{A} \in B_r$ iff $\mathfrak{A}$ is isomorphic to one of the algebras $\mathfrak{A}_1$, $\mathfrak{A}_7$ or $\mathfrak{A}_8$. It was proved in [13] that $\mathfrak{A}$ is s.i. and belongs to $B_{\{+\}}$ iff $\mathfrak{A}$ is isomorphic to $\mathfrak{A}_2$; $\mathfrak{A}$ is s.i. and $\mathfrak{A} \in B_{\{\cdot\}}$ iff $\mathfrak{A} \simeq \mathfrak{A}_3$; $\mathfrak{A}$ is s.i. and $\mathfrak{A} \in B_\varnothing$ iff $\mathfrak{A} \simeq \mathfrak{A}_4$ (cf. also [2]); $\mathfrak{A}$ is s.i. and $\mathfrak{A} \in B_{\{'\}}$ iff $\mathfrak{A} \simeq \mathfrak{A}_5$ or $\mathfrak{A} \simeq \mathfrak{A}_6$. It was proved in [19] (see Section 3, Examples 3.3–3.5) that $\mathfrak{A}$ is s.i. and $\mathfrak{A} \in B_{\{+,\cdot\}}$ iff $\mathfrak{A} \simeq \mathfrak{A}_9$ or $\mathfrak{A} \simeq \mathfrak{A}_{10}$ and $\mathfrak{A} \in B_{\{+,'\}}$ iff $\mathfrak{A} \simeq \mathfrak{A}_{11}$ or $\mathfrak{A} \simeq \mathfrak{A}_{12}$; $\mathfrak{A}$ is s.i. and $\mathfrak{A} \in B_{\{\cdot,'\}}$ iff $\mathfrak{A} \simeq \mathfrak{A}_{13}$ or $\mathfrak{A} \simeq \mathfrak{A}_{14}$. ∎

## 2. The lattice of subvarieties of $B_b$

Denote $Ir(B_b) = \{\mathfrak{A}_1, \ldots, \mathfrak{A}_{14}\}$. For a variety $V \subseteq B_b$ we denote $Ir(V) = \{\mathfrak{A}_k \in Ir(B_b) : \mathfrak{A}_k \in V\}$. Consequently, to describe the lattice $\mathcal{L}(B_b)$ we have to find all subsets $T$ of $Ir(B_b)$ being of the form $Ir(V)$ for some $V \subseteq B_b$. Apriory we have $2^{14}$ possibilities. However due to the lemmas below we can essentially reduce this amount.

**Lemma 2.1.** $\mathfrak{A}_1 \in HSP(\mathfrak{A}_8)$.

***Proof.*** Observe that the subalgebra $(\{a_8, c_8\}; \{+, \cdot, '\}|_{\{a_8,c_8\}})$ of $\mathfrak{A}_8$ is isomorphic to $\mathfrak{A}_1$. ∎

**Lemma 2.2.** $\mathfrak{A}_{2n-1} \in HSP(\mathfrak{A}_{2n})$ *for* $3 \leq n \leq 7$.

***Proof.*** Put $h(a_{2n}) = h(c_{2n}) = a_{2n-1}$, $h(b_{2n}) = b_{2n-1}$. Thus $h$ is a homomorphism. ∎

**Lemma 2.3.** $\mathfrak{A}_{2n} \in HSP(\{\mathfrak{A}_1, \mathfrak{A}_{2n-1}\})$ *for* $3 \leq n \leq 7$.

***Proof.*** In the direct product $\mathfrak{A}_1 \times \mathfrak{A}_{2n-1}$ put $h(\langle a_1, a_{2n-1}\rangle) = a_{2n}$, $h(\langle b_1, a_{2n-1}\rangle) = c_{2n}$, $h(\langle a_1, b_{2n-1}\rangle) = h(\langle b_1, b_{2n-1}\rangle) = b_{2n}$. ■

**Lemma 2.4.** $\mathfrak{A}_2 \in HSP(\{\mathfrak{A}_9, \mathfrak{A}_{11}\})$.

***Proof.*** In the direct product $\mathfrak{A}_9 \times \mathfrak{A}_{11}$ put $h(\langle a_9, a_{11}\rangle) = a_2$, $h(\langle a_9, b_{11}\rangle) = h(\langle b_9, a_{11}\rangle) = h(\langle b_9, b_{11}\rangle) = b_2$. ■

**Lemma 2.5.** $\mathfrak{A}_3 \in HSP(\{\mathfrak{A}_9, \mathfrak{A}_{13}\})$.

***Proof.*** The proof is analogous to that of Lemma 2.4. ■

**Lemma 2.6.** $\mathfrak{A}_5 \in HSP(\{\mathfrak{A}_{11}, \mathfrak{A}_{13}\})$.

***Proof.*** The proof is analogous to that of Lemma 2.4. ■

**Lemma 2.7.** $\mathfrak{A}_6 \in HSP(\{\mathfrak{A}_5, \mathfrak{A}_{12}\})$.

***Proof.*** In the direct product $\mathfrak{A}_5 \times \mathfrak{A}_{12}$ put $h(\langle a_5, a_{12}\rangle) = a_6$, $h(\langle a_5, c_{12}\rangle) = c_6$, $h(\langle x, y\rangle) = b_6$ otherwise. ■

**Lemma 2.8.** $\mathfrak{A}_6 \in HSP(\{\mathfrak{A}_5, \mathfrak{A}_{14}\})$.

***Proof.*** The proof is analogous to that of Lemma 2.7. ■

**Lemma 2.9.** $\mathfrak{A}_6 \in HSP(\{\mathfrak{A}_{11}, \mathfrak{A}_{14}\})$.

***Proof.*** In the direct product $\mathfrak{A}_{11} \times \mathfrak{A}_{14}$ put $h(\langle a_{11}, a_{14}\rangle) = a_6$, $h(\langle a_{11}, c_{14}\rangle) = c_6$ and $h(\langle x, y\rangle) = b_6$ otherwise. ■

**Lemma 2.10.** $\mathfrak{A}_6 \in HSP(\{\mathfrak{A}_{12}, \mathfrak{A}_{13}\})$.

***Proof.*** The proof is analogous to that of Lemma 2.9. ■

**Lemma 2.11.** $\mathfrak{A}_4$ *belongs to each of the sets* $HSP(\{\mathfrak{A}_2, \mathfrak{A}_3\})$, $HSP(\{\mathfrak{A}_2, \mathfrak{A}_5\})$, $HSP(\{\mathfrak{A}_3, \mathfrak{A}_5\})$, $HSP(\{\mathfrak{A}_2, \mathfrak{A}_{13}\})$, $HSP(\{\mathfrak{A}_3, \mathfrak{A}_{11}\})$, $HSP(\{\mathfrak{A}_5, \mathfrak{A}_9\})$.

***Proof.*** The proof is easy and it is left to the reader. ■

A set $T \subseteq Ir(B_b)$ will be called $B_b$*–closed* or briefly *closed* if it satisfies the following conditions (c$_1$)–(c$_{11}$):

(c$_1$) if $\mathfrak{A}_8 \in T$, then $\mathfrak{A}_1 \in T$;

(c$_2$) if $3 \le n \le 7$ and $\mathfrak{A}_{2n} \in T$, then $\mathfrak{A}_{2n-1} \in T$;

(c$_3$) if $3 \le n \le 7$ and $\{\mathfrak{A}_1, \mathfrak{A}_{2n-1}\} \subseteq T$, then $\mathfrak{A}_{2n} \in T$;

(c$_4$) if $\{\mathfrak{A}_9, \mathfrak{A}_{11}\} \subseteq T$, then $\mathfrak{A}_2 \in T$;

(c$_5$) if $\{\mathfrak{A}_9, \mathfrak{A}_{13}\} \subseteq T$, then $\mathfrak{A}_3 \in T$;

(c$_6$) if $\{\mathfrak{A}_{11}, \mathfrak{A}_{13}\} \subseteq T$, then $\mathfrak{A}_5 \in T$;

(c$_7$) if $\{\mathfrak{A}_5, \mathfrak{A}_{12}\} \subseteq T$, then $\mathfrak{A}_6 \in T$;

(c$_8$) if $\{\mathfrak{A}_5, \mathfrak{A}_{14}\} \subseteq T$, then $\mathfrak{A}_6 \in T$;

(c$_9$) if $\{\mathfrak{A}_{11}, \mathfrak{A}_{14}\} \subseteq T$, then $\mathfrak{A}_6 \in T$;

(c$_{10}$) if $\{\mathfrak{A}_{12}, \mathfrak{A}_{13}\} \subseteq T$, then $\mathfrak{A}_6 \in T$;

$$(c_{11}) \begin{cases} \text{if } \{\mathfrak{A}_2, \mathfrak{A}_3\} \subseteq T, \text{ then } \mathfrak{A}_4 \in T; & \text{if } \{\mathfrak{A}_2, \mathfrak{A}_5\} \subseteq T, \text{ then } \mathfrak{A}_4 \in T; \\ \text{if } \{\mathfrak{A}_3, \mathfrak{A}_5\} \subseteq T, \text{ then } \mathfrak{A}_4 \in T; & \text{if } \{\mathfrak{A}_2, \mathfrak{A}_{13}\} \subseteq T, \text{ then } \mathfrak{A}_4 \in T; \\ \text{if } \{\mathfrak{A}_3, \mathfrak{A}_{11}\} \subseteq T, \text{ then } \mathfrak{A}_4 \in T; & \text{if } \{\mathfrak{A}_5, \mathfrak{A}_9\} \subseteq T, \text{ then } \mathfrak{A}_4 \in T. \end{cases}$$

**Lemma 2.12.** *If $T \subseteq Ir(B_b)$, $T$ is $B_b$–*closed *and $\mathfrak{A}_k \notin T$ for some $k \in \{1, \ldots, 14\}$, then $\mathfrak{A}_k \notin HSP(T)$.*

***Proof.*** Let $k = 1$. Then $T \subseteq \{\mathfrak{A}_2, \ldots, \mathfrak{A}_{14}\}$. By (c$_1$) $\mathfrak{A}_8 \notin T$. Thus $T \subseteq \{\mathfrak{A}_2, \ldots, \mathfrak{A}_{14}\} \smallsetminus \{\mathfrak{A}_8\}$. Take the identity

$$\Big(\big((x+y)\cdot(x+y)\big)'\Big)' \approx \Big(\big((x\cdot y)+(x\cdot y)\big)'\Big)'. \tag{2.1}$$

Then we check that (2.1) is satisfied in every algebra $\mathfrak{A}_i$ for $i \in \{2, \ldots, 14\} \smallsetminus \{8\}$, so (2.1) is satisfied in $HSP(T)$ but (2.1) is not satisfied in $\mathfrak{A}_1$. Consequently $\mathfrak{A}_1 \notin HSP(T)$.

Let $k = 2$. Then none of the sets $\{\mathfrak{A}_9, \mathfrak{A}_{11}\}$, $\{\mathfrak{A}_9, \mathfrak{A}_{12}\}$, $\{\mathfrak{A}_{10}, \mathfrak{A}_{11}\}$, $\{\mathfrak{A}_{10}, \mathfrak{A}_{12}\}$ is included in $T$. In fact, by (c$_2$), if one of the sets is included in $T$, then $\{\mathfrak{A}_9, \mathfrak{A}_{11}\} \subseteq T$ and by (c$_4$) $\mathfrak{A}_2 \in T$, a contradiction. So, it must be

(2.2) $T \cap \{\mathfrak{A}_{11}, \mathfrak{A}_{12}\} = \varnothing$

or

(2.3) $T \cap \{\mathfrak{A}_9, \mathfrak{A}_{10}\} = \varnothing.$

If (2.2) holds, then take the identity

$$x + x \approx (x + x) \cdot (x + x).$$

Then every algebra from $T$ satisfies this identity, so it is satisfied in $HSP(T)$ but $\mathfrak{A}_2$ does not satisfy it. In case (2.3) we take the identity

$$x + x \approx ((x + x)')'.$$

Let $k = 3$. Then by ($c_5$) and ($c_2$) none of the sets $\{\mathfrak{A}_9, \mathfrak{A}_{13}\}$, $\{\mathfrak{A}_9, \mathfrak{A}_{14}\}$, $\{\mathfrak{A}_{10}, \mathfrak{A}_{13}\}$, $\{\mathfrak{A}_{10}, \mathfrak{A}_{14}\}$ can be included in $T$. If $T \cap \{\mathfrak{A}_{13}, \mathfrak{A}_{14}\} = \varnothing$, we take the identity

$$x \cdot x \approx (x \cdot x) + (x \cdot x).$$

If $T \cap \{\mathfrak{A}_9, \mathfrak{A}_{10}\} = \varnothing$, we take the identity

$$((x \cdot x)')' \approx x \cdot x.$$

Let $k = 4$. By ($c_2$) and ($c_{11}$) $T$ must be included in one of the sets:

$$\{\mathfrak{A}_1, \mathfrak{A}_2, \mathfrak{A}_7, \mathfrak{A}_8, \mathfrak{A}_9, \mathfrak{A}_{10}, \mathfrak{A}_{11}, \mathfrak{A}_{12}\}, \ \{\mathfrak{A}_1, \mathfrak{A}_3, \mathfrak{A}_7, \mathfrak{A}_8, \mathfrak{A}_9, \mathfrak{A}_{10}, \mathfrak{A}_{13}, \mathfrak{A}_{14}\}, \\ \{\mathfrak{A}_1, \mathfrak{A}_5, \mathfrak{A}_6, \mathfrak{A}_7, \mathfrak{A}_8, \mathfrak{A}_{11}, \mathfrak{A}_{12}, \mathfrak{A}_{13}, \mathfrak{A}_{14}\}.$$

We take the identities $x + x \approx x$, $x \cdot x \approx x$, $(x')' \approx x$, respectively.

Let $k = 5$. By ($c_2$), $\mathfrak{A}_6 \notin T$ and, by ($c_6$) and ($c_2$), none of the sets $\{\mathfrak{A}_{11}, \mathfrak{A}_{13}\}$, $\{\mathfrak{A}_{11}, \mathfrak{A}_{14}\}$, $\{\mathfrak{A}_{12}, \mathfrak{A}_{13}\}$, $\{\mathfrak{A}_{12}, \mathfrak{A}_{14}\}$ is included in $T$. If $T \cap \{\mathfrak{A}_{13}, \mathfrak{A}_{14}\} = \varnothing$, we take

$$(x')' \approx (x')' + (x')'.$$

If $T \cap \{\mathfrak{A}_{11}, \mathfrak{A}_{12}\} = \varnothing$, we take the identity

$$(x')' \approx (x')' \cdot (x')'.$$

Let $k = 6$. If $\mathfrak{A}_5 \in T$, then $T \cap \{\mathfrak{A}_8, \mathfrak{A}_1, \mathfrak{A}_{12}, \mathfrak{A}_{14}\} = \varnothing$ by ($c_3$), ($c_1$), ($c_7$), ($c_8$). We take the identity $(x')' \approx x'$. Let $\mathfrak{A}_5 \notin T$. If $\mathfrak{A}_1 \in T$, then, by ($c_2$), ($c_3$), ($c_9$), ($c_{10}$), it must be

$$T \cap \{\mathfrak{A}_{13}, \mathfrak{A}_{14}\} = \varnothing \tag{2.4}$$

or

$$T \cap \{\mathfrak{A}_{11}, \mathfrak{A}_{12}\} = \varnothing. \tag{2.5}$$

If (2.4) holds, we take the identity

$$(x')' \approx ((x+x)')'.$$

If (2.5) holds, we take the identity

$$(x')' \approx ((x \cdot x)')'.$$

If $\mathfrak{A}_5 \notin T$ and $\mathfrak{A}_1 \notin T$, then $\mathfrak{A}_8 \notin T$ by ($c_1$). Then, by ($c_9$), ($c_{10}$) and ($c_6$) we have two possibilities: (2.4), (2.5). We take the identities $(x')' \approx ((x+x)')'$, $(x')' \approx ((x \cdot x)')'$, respectively.

Let $k = 7$. Then by ($c_2$) $\mathfrak{A}_8 \notin T$. We take the identity

$$\big((x + (x \cdot y))'\big)' \approx \big((x + (x \cdot z))'\big)'. \tag{2.6}$$

Let $k = 8$. Then $T$ does not contain both $\mathfrak{A}_1$ and $\mathfrak{A}_7$ by ($c_3$). If $\mathfrak{A}_7 \notin T$, we take the identity (2.6). If $\mathfrak{A}_1 \notin T$ we take the identity (2.1).

Let $k = 9$. Then $\mathfrak{A}_{10} \notin T$ by ($c_2$). We take

$$\big(((x+y) \cdot (x+y))'\big)' \approx (x+y) \cdot (x+y). \tag{2.7}$$

Let $k = 10$. If $\mathfrak{A}_9 \notin T$, then we take the identity (2.7). If $\mathfrak{A}_9 \in T$, then $\{\mathfrak{A}_1, \mathfrak{A}_8\} \not\subseteq T$ by ($c_3$) and ($c_1$). We take

$$(x \cdot y) + (x \cdot y) \approx (x+y) \cdot (x+y).$$

Let $k = 11$. Then $\mathfrak{A}_{12} \notin T$ by ($c_2$). We take

$$\big(((x+y) \cdot (x+y))'\big)' \approx ((x+y)')'. \tag{2.8}$$

Let $k = 12$. If $\mathfrak{A}_{11} \notin T$, then we take the identity (2.8). If $\mathfrak{A}_{11} \in T$, then $\mathfrak{A}_1, \mathfrak{A}_8 \notin T$ by ($c_3$) and ($c_1$). Then we take

$$((x+y)')' \approx (x+y)'.$$

Let $k = 13$. Then $\mathfrak{A}_{14} \notin T$ by ($c_2$). We take

$$\big(((x \cdot y) + (x \cdot y))'\big)' \approx ((x \cdot y)')'. \tag{2.9}$$

Let $k = 14$. If $\mathfrak{A}_{13} \notin T$, we take the identity (2.9). If $\mathfrak{A}_{13} \in T$, then $\mathfrak{A}_1$, $\mathfrak{A}_8 \notin T$ by ($c_3$) and ($c_1$). We take

$$((x \cdot y)')' \approx (x \cdot y)'.$$

■

**Lemma 2.13.** *If a variety $V$ belongs to $\mathcal{L}(B_b)$ and $\mathfrak{A} \in V$, then $\mathfrak{A}$ is isomorphic to a subdirect product of a family of subdirectly irreducible algebras belonging to $Ir(V)$.*

***Proof.*** By Birkhoff's Subdirect Representation Theorem (see [1]), if $\mathfrak{A} \in V$, then it is isomorphic to an algebra $\mathfrak{A}'$ being a subdirect product of a family $\{\mathfrak{A}_j\}_{j\in J}$ of subdirectly irreducible algebras from $V$. By Theorem 1.1, each $\mathfrak{A}_j$ is isomorphic to an algebra $\mathfrak{A}_j^*$ from $Ir(B_b)$. Thus $\mathfrak{A}_j^*$ belongs to $V$ and belongs to $Ir(B_b)$, hence $\mathfrak{A}_j^*$ belongs to $Ir(V)$. Consequently, $\mathfrak{A}'$ is isomorphic to an algebra $\mathfrak{A}^*$ being a subdirect product of the family $\{\mathfrak{A}_j^*\}_{j\in J}$ and $\mathfrak{A}$ is isomorphic to $\mathfrak{A}^*$. ■

We denote by $\boldsymbol{T}(B_b)$ the set of all $B_b$-closed sets.

**Lemma 2.14.** *We have:*

(i) *For every variety $V \in \mathcal{L}(B_b)$, the set $Ir(V)$ is $B_b$-closed;*

(ii) *For every variety $V \in \mathcal{L}(B_b)$, we have $V = HSP(Ir(V))$;*

(iii) *If $T \in \boldsymbol{T}(B_b)$, then $T = Ir(HSP(T))$;*

(iv) *If $V_1$, $V_2 \in \mathcal{L}(B_b)$, then $V_1 \subseteq V_2$ iff $Ir(V_1) \subseteq Ir(V_2)$.*

***Proof.*** (i): If $\mathfrak{A}_8 \in Ir(V)$, then, by Lemma 2.1, we have $\mathfrak{A}_1 \in HSP(\mathfrak{A}_8) \subseteq HSP(Ir(V)) \subseteq V$, but $\mathfrak{A}_1 \in Ir(B_b)$, so $\mathfrak{A}_1 \in V \cap Ir(B_b) = Ir(V)$. Consequently, the set $Ir(V)$ satisfies ($c_1$). Similarly, using Lemmas 2.2–2.11, we show that $Ir(V)$ satisfies ($c_2$)–($c_{11}$).

(ii): Since $Ir(V) \subseteq V$, $HSP(Ir(V)) \subseteq V$. The converse inclusion follows at once from Lemma 2.13.

(iii): If an algebra $\mathfrak{A}$ belongs to $T$, then $\mathfrak{A} \in HSP(T)$. But $\mathfrak{A} \in Ir(B_b)$ since $T \subseteq Ir(B_b)$, so $\mathfrak{A} \in Ir(HSP(T))$. If $\mathfrak{A} \notin T$, then $\mathfrak{A} \notin HSP(T)$ by Lemma 2.12, hence $\mathfrak{A} \notin Ir(HSP(T))$.

(iv): If $V_1 \subseteq V_2$, then $Ir(V_1) \subseteq Ir(V_2)$ by the definition of $Ir(V)$. The converse implication follows at once from Lemma 2.13. ■

**Theorem 2.15.** *The set $T \subseteq Ir(B_b)$ is equal to $Ir(V)$ for some variety $V \in \mathcal{L}(B_b)$ iff $T$ is $B_b$-closed. There are 490 $B_b$-closed sets.*

***Proof.*** The first statement follows from Lemma 2.14 (i) and (iii).

Using a computer and transforming our considerations to indices of algebras $\mathfrak{A}_k$ from $Ir(B_b)$ one can find out $|\boldsymbol{T}(B_b)| = 490$. ■

**Theorem 2.16.** *The lattice $(\mathcal{L}(B_b); \subseteq)$ as a poset is isomorphic to the poset $(\boldsymbol{T}(B_b); \subseteq)$. Therefore the lattice $(\mathcal{L}(B_b); \subseteq)$ is isomorphic to the lattice $(\boldsymbol{T}(B_b); \subseteq)$ and $card(\mathcal{L}(B_b)) = 490$.*

***Proof.*** For $V \in \mathcal{L}(B_b)$ put $\varphi(V) = Ir(V)$. Then $\varphi$ is well defined by the definition of $Ir(V)$ and, by Lemma 2.14 (i), $\varphi$ maps $\mathcal{L}(B_b)$ into $\boldsymbol{T}(B_b)$. If $Ir(V_1) = Ir(V_2)$, then, by Lemma 2.14 (ii), $V_1 = HSP(Ir(V_1)) = HSP(Ir(V_2)) = V_2$. Thus $\varphi$ is 1-1. By Lemma 2.14 (iii), $\varphi$ is onto. If $V_1 \subseteq V_2$, then $Ir(V_1) \subseteq Ir(V_2)$ by the definition of $Ir(V)$. The converse inclusion follows at once from Lemma 2.13. ■

**Remark 2.17.** Results of this paper were presented on the conference "9th Workshop in Mathematics", organized by Technical University of Zielona Góra at September 2001, in Gronów (Poland).

The statement $card(\mathcal{L}(B_b)) = 490$ was confirmed on this conference by Peter Burmeister (Darmstadt, Germany) using his *ConImp* computer program based on the Formal Concept Analysis. For the documentation of the program see P. Burmeister, *ConImp – Ein Programm zur Formalen Begriffsanalyse* in: G. Stumme and R. Wille (Eds.), *Begriffliche Wissensverarbeitung: Methoden und Anwendungen*, Springer-Verlag, Berlin 2000, pp. 25–56; extended English version – *Formal Concept Analysis with ConImp: Introduction to the basic features* – one can find on the WWW-server:
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