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RING-LIKE STRUCTURES WITH UNIQUE SYMMETRIC DIFFERENCE RELATED TO QUANTUM LOGIC

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Abstract

Ring-like quantum structures generalizing Boolean rings and having the property that the terms corresponding to the two normal forms of the symmetric difference in Boolean algebras coincide are investigated. Subclasses of these structures are algebraically characterized and related to quantum logic. In particular, a physical interpretation of the proposed model following Mackey's approach to axiomatic quantum mechanics is given.

Keywords: generalized Boolean quasiring, symmetric difference, quantum logic.

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1. Introduction

It is well-known that in a Boolean algebra $(R; \vee, \wedge, ', 0, 1)$ the symmetric difference $\triangle$ is unique but can be defined by two equivalent formulas

$$x \triangle y := (x \wedge y') \vee (x' \wedge y), \text{ respectively} \tag{1}$$

$$x \triangle y := (x \vee y) \wedge (x' \vee y'). \tag{2}$$

This allows us to pass from Boolean algebras to Boolean rings $(R, +, \cdot)$ by identifying $+$ with $\triangle$ and $\cdot$ with $\wedge$. In our previous papers ([1] – [5]) we have considered ring-like structures generalizing Boolean rings, namely the quasirings $(R, +, \cdot)$, where $+$ has been a generalization of the two possible ways for expressing $\triangle$ in Boolean algebras given above. In these ring-like structures (1) and (2) correspond to the canonical operations

$$x +_1 y = 1 + (1 + x(1 + y))(1 + (1 + x)y), \text{ respectively} \tag{3}$$

$$x +_2 y = (1 + (1 + x)(1 + y))(1 + xy) \tag{4}$$

which will be different in general. (But it always holds $x +_1 y \leq x +_2 y$.) For example, if our ring-like structure corresponds to an orthomodular lattice not being a Boolean algebra (e. g. to the lattice of closed linear subspaces of a Hilbert space of dimension > 1) then it is known that $+_1 \neq +_2$. These structures generalizing Boolean rings with the possibility that $+_1 \neq +_2$ have been well investigated. But there is another way to generalize Boolean rings, namely to consider these ring-like structures for which $+_1 = +_2$. Then the operation $+$ can be uniquely defined as $+ := +_1 = +_2$. Such structures comprise Boolean rings but also ring-like structures corresponding to chains with an involutory antiautomorphism as well as many other structures. We will call such structures *ring-like structures with unique symmetric difference*. In this paper we investigate algebraic properties of these structures as well as interpret the unique operation $+$ in respect to its meaning for quantum logic. In particular, our approach will indicate other possibilities passing from classical logic to quantum logics different from the standard ones based on the Hilbert space logic. These non-classical logics which arise from our ring-like structures with unique symmetric difference will be called quantum logics with unique symmetric difference. Since the symmetric difference can be interpreted as the logical operation of "exclusive or" its uniqueness is equivalent to the assumption that our logic (possibly non-classical) still satisfies the classical law of negation of equivalence

$$(\neg(x \leftrightarrow y)) \leftrightarrow (x \leftrightarrow (\neg y)).$$

This property will correspond to the so-called weak associativity of our ring-like structures which generalizes but is not equivalent to full associativity inherent in Boolean rings. We hope that quantum logics with unique symmetric difference will provide another possibility for considering models for quantum mechanics not involving the necessity of employing Hilbert spaces or orthomodular structures.

Our paper will consist of five sections. In Section 2 after this introduction we recall the basic notions and theorems relevant to the theory of ring-like structures called generalized Boolean quasirings ($GBQR$s). In Section 3 we prove various characterizations of $GBQR$s with unique symmetric difference and consequences of the assumption that a $GBQR$ admits an operation $+$ such that $x + y$ is compatible (in the lattice-theoretic sense) to $x +_1 y$ and $x +_2 y$, respectively. We will give various characterizations of this property. Though we have not been able to prove a full representation theorem for $GBQR$s with unique symmetric difference, we will show in Section 4 that this class is large enough to admit non-classical models. Further, we will provide in Section 4 examples of $GBQR$s with unique symmetric difference and present a method for constructing those structures. Moreover, we will introduce the notion of a $[0, 1]$-valued generalized metric on an arbitrary $GBQR$ and show that Boolean rings (= classical logics) as generalized Boolean quasirings only admit two-valued generalized metrics. Finally, in Section 5 we give a physical interpretation of our ring-like structures with unique symmetric difference as derived from Mackey's probability function.

2. BASIC DEFINITIONS AND THEOREMS FROM THE THEORY OF GENERALIZED BOOLEAN QUASIRINGS

In order to present our approach we have to recall the basic notions and some results of the theory of generalized Boolean quasirings developed in the previous papers [1] – [5]. For notions concerning axiomatic quantum mechanics not defined in this paper, cf. e. g. [8].

A *generalized Boolean quasiring* ($GBQR$ for short) is an algebra $(R; +, \cdot)$ of type $(2, 2)$ possessing two elements 0 and 1 such that the following identities hold:

(i) $x + y = y + x$,

(ii) $0 + x = x$,

(iii) $(xy)z = x(yz)$,

(iv) $xy = yx$,

(v) $xx = x$,

(vi) $x0 = 0$,

(vii) $x1 = x$ and

(viii) $1 + (1 + xy)(1 + x) = x$.

Omitting (i) one may consider $+$ as a partial operation $\oplus$ on R with domain $\{0, 1\} \times R$. This way one obtains a partial algebra $(R; \oplus, \cdot)$ of type $(2, 2)$ which is called a *partial generalized Boolean quasiring* ($pGBQR$).

If $\mathcal{R} = (R, \oplus, \cdot)$ is a $pGBQR$ and one defines

$$\begin{aligned} x \vee y &:= 1 \oplus (1 \oplus x)(1 \oplus y), \\ x \wedge y &:= xy \text{ and} \\ x^* &:= 1 \oplus x \end{aligned}$$

for all $x, y \in R$, then $\mathbf{L}(\mathcal{R}) := (R; \vee, \wedge, ^*, 0, 1)$ is a bounded lattice with an involutory antiautomorphism *. Conversely, if $\mathcal{L} = (L; \vee, \wedge, ^*, 0, 1)$ is a bounded lattice with an involutory antiautomorphism and one defines

$$\begin{aligned} 0 \oplus x &:= x, \\ 1 \oplus x &:= x^* \text{ and} \\ xy &:= x \wedge y \end{aligned}$$

for all $x, y \in L$, then $\mathbf{R}(\mathcal{L}) := (L; \oplus, \cdot)$ is a $pGBQR$. For fixed base set, $\mathbf{L}$ and $\mathbf{R}$ are mutually inverse bijections between the set of all $pGBQR$s and the set of all bounded lattices with an involutory antiautomorphism.

Because of this one-to-one correspondence, a $pGBQR$ can also be considered as a lattice. In the following we will often use the operations $\vee, \wedge, ^*$ (as defined above) simultaneously with the operations $\oplus$ and $\cdot$. Moreover we will use the notations $x \leq y$ and $x \perp y$ ("x *orthogonal* to y") from lattice theory within $pGBQR$s. We then have $x \leq y \Leftrightarrow xy = x$ and $x \perp y \Leftrightarrow x \leq y^* \Leftrightarrow x^* \geq y$.

Any $p\,GBQR$ $(R;\oplus,\cdot)$ can be extended to a $GBQR$ $(R;+,\cdot)$ by defining $0+x = x+0 := 0\oplus x$ and $1+x = x+1 := 1\oplus x$ for $x\in R$, and $x+y = y+x \in R$ arbitrarily for $x,y\in R\smallsetminus\{0,1\}$.

However, there are the two canonical ways of extending $\oplus$ to a full operation $+$ which are suggested by the two ways of expressing the symmetric difference in Boolean algebras in normal forms:

$$x +_1 y := 1\oplus(1\oplus x(1\oplus y))(1\oplus(1\oplus x)y), \text{ respectively}$$
$$x +_2 y := (1\oplus(1\oplus x)(1\oplus y))(1\oplus xy),$$

in terms of lattice operations

$$x +_1 y = (x\wedge y^*)\vee(x^*\wedge y) \text{ and}$$
$$x +_2 y = (x\vee y)\wedge(x^*\vee y^*).$$

It is easy to see that for all elements x,y of a $GBQR$ $\mathcal{R}$ $x +_1 y \le x +_2 y$, $(x +_1 y)^* = x^* +_2 y = x +_2 y^*$ and $(x +_2 y)^* = x^* +_1 y = x +_1 y^*$.

Moreover, if for an extension $+$ it holds that $+_1 \le + \le +_2$, then

$$x\le y \Rightarrow x +_1 y = x+y = x +_2 y = x^*\wedge y \text{ and}$$
$$x\perp y \Rightarrow x +_1 y = x+y = x +_2 y = x\vee y.$$

A $GBQR$ $\mathcal{R}=(R;+,\cdot)$ is called *associative*, if the operation $+$ is associative, and $\mathcal{R}$ is called **-modular*, if for all $x,y\in R$ – expressed by lattice operations –

$$x\le y \Rightarrow (x\vee x^*)\wedge y = x\vee(x^*\wedge y)$$

which is equivalent to the fact that for all $x,y\in R$

$$x\le y \Rightarrow (x\vee y^*)\wedge y = x\vee(y^*\wedge y).$$

As shown in [4], for a $GBQR$ $\mathcal{R}=(R;+,\cdot)$ with $+_1\le+\le+_2$, the associativity of $+$ implies that $\mathcal{R}$ is *-modular.

We conclude our preliminary observations by the following remark: If $\mathcal{R}$ has *characteristic* 2, i. e. $x+x=0$ for all $x\in R$, the definition of *-modularity coincides with the definition of orthomodularity of the corresponding lattice $\mathbf{L}(\mathcal{R})$.

3. Structure theory of generalized Boolean quasirings with unique symmetric difference

In the following let $\mathcal{R} = (R; +, \cdot)$ denote an arbitrary, but fixed *GBQR*. $\mathcal{R}$ is called *weakly associative* if for all $x, y \in R$, $(1 + x) + y = 1 + (x + y)$ holds, i. e. if for every $x, y \in R$, $x^* + y = (x + y)^*$.

$\mathcal{R}$ is said to admit a *unique symmetric difference* or to be a *GBQR* with unique symmetric difference, if $+_1 = +_2$. Instead of "*GBQR* with unique symmetric difference" we will also say "*GBQR* with $+_1 = +_2$". That a *GBQR* admits a unique symmetric difference does not mean that $+_1$ or $+_2$ have to be related to the operation $+$ (though we will often make special assumptions about such a relation).

Since the logical connective $x \to y$ corresponds to $x^* \vee y$ and $x \leftrightarrow y$ corresponds to $((x \to y)$ and $(y \to x))$, i. e. to $(x^* \vee y) \wedge (y^* \vee x)$, the so-called law of negation of equivalence

$$(\neg(x \leftrightarrow y)) \leftrightarrow (x \leftrightarrow (\neg y))$$

corresponds to $((x^* \vee y) \wedge (y^* \vee x))^* = (x^* \vee y^*) \wedge (y \vee x)$, i. e. to the uniqueness of the symmetric difference.

Theorem 3.1. *A GBQR $\mathcal{R}$ admits a unique symmetric difference if and only if $+_1$ or $+_2$ is weakly associative.*

Proof. If $+_1 = +_2$, then because of $(x +_2 y)^* = x^* +_1 y$ we obtain $(1 +_1 x) +_1 y = x^* +_1 y = (x +_2 y)^* = (x +_1 y)^* = 1 +_1 (x +_1 y)$. Conversely, if $+_1$ is weakly associative

$$x +_1 y = (1 +_1 x^*) +_1 y = 1 +_1 (x^* +_1 y) = 1 +_1 (x +_2 y)^* = x +_2 y,$$

and the same argument applies to $+_2$ if it is assumed to be weakly associative. ∎

Theorem 3.2. *Let the operation $+$ of a GBQR $(R; +, \cdot)$ be weakly associative and comparable to $+_1$ or to $+_2$. If $+ \leq +_1$ or if $+ \geq +_2$, it follows that $+ = +_1 = +_2$. If $+ \geq +_1$ or if $+ \leq +_2$, it follows that $+_1 \leq + \leq +_2$.*

Proof. If $+ \leq +_1$, then

$$x +_1 y \leq x +_2 y = (x^* +_1 y)^* \leq (x^* + y)^* = x + y \leq x +_1 y$$

and if $+ \geq +_1$, then

$$x + y = (x^* + y)^* \leq (x^* +_1 y)^* = x +_2 y.$$

The remaining two cases follow from dual arguments. ∎

There are two characteristic subsets of a *GBQR* $\mathcal{R} = (R; +, \cdot)$:

The *orthogonal kernel* $\mathrm{OK}(\mathcal{R})$ of $\mathcal{R}$ (cf. [2]) which is defined by

$$\mathrm{OK}(\mathcal{R}) := \{x \in R \mid x(1+x) = x\}$$

and the *dual orthogonal kernel* $\mathrm{DOK}(\mathcal{R})$ of $\mathcal{R}$ (cf. [3]) which is defined by

$$\mathrm{DOK}(\mathcal{R}) := \{x \in R \mid (1+x)x = 1+x\}.$$

As one can see immediately, $\mathrm{OK}(\mathcal{R}) = \{x \wedge x^* \mid x \in R\}$ and $\mathrm{DOK}(\mathcal{R}) = \{x \vee x^* \mid x \in R\}$. We put

$$\overline{\mathrm{OK}}(\mathcal{R}) := \mathrm{OK}(\mathcal{R}) \cup \mathrm{DOK}(\mathcal{R}).$$

If any two distinct elements x, y of a subset S of R are orthogonal, i. e. $x \leq y^*$, then we call the *subset orthogonal.*

As well-known from the theory of orthomodular lattices two elements x and y of an orthomodular lattice $(L; \vee, \wedge, ', 0, 1)$ commute if and only if their commutator

$$c(x, y) := (x \wedge y) \vee (x \wedge y') \vee (x' \wedge y) \vee (x' \wedge y')$$

equals 1. The definition of $c(x, y)$ gives rise to define the *commutator* $c(x, y)$ of two elements $x, y \in R$ by

$$c(x, y) := 1 + (1 + xy)(1 + xy^*)(1 + x^*y)(1 + x^*y^*).$$

Lemma 3.1. *Let $\mathcal{R} = (R; +, \cdot)$ be a GBQR. For the statements*

(i) $+_1 = +_2$

(ii) $c(x, y) \in \mathrm{DOK}(\mathcal{R})$ *for all* $x, y \in R$

(iii) $\mathrm{OK}(\mathcal{R})$ *is orthogonal.*

(iv) $\mathrm{OK}(\mathcal{R})$ *is a sublattice of* $(R; \vee, \wedge)$.

we have the following implications: (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Leftrightarrow$ (iv) *and* (iii) $\not\Rightarrow$ (ii).

Proof. (i) $\Rightarrow$ (ii):

$$c(x,y) = (x +_1 y) \vee (x +_2 y)^* = (x +_1 y) \vee (x +_1 y)^* \in DOK(\mathcal{R})$$

(ii) $\Rightarrow$ (iii): $x \wedge x^* \le (c(x,y))^* \le c(x,y) \le y \vee y^*$

(iii) $\Rightarrow$ (iv): If $\mathrm{OK}(\mathcal{R})$ is orthogonal, then $(x \wedge x^*) \vee (y \wedge y^*) \le (x \vee x^*) \wedge (y \vee y^*)$ and $(x \wedge x^*) \wedge (y \wedge y^*) \le (x \vee x^*) \vee (y \vee y^*)$.

(iv) $\Rightarrow$ (iii): If $\mathrm{OK}(\mathcal{R})$ is a sublattice of $(R; \vee, \wedge)$, then $x \wedge x^* \le (x \wedge x^*) \vee (y \wedge y^*) \le (x \vee x^*) \wedge (y \vee y^*) \le y \vee y^*$.

(iii) $\not\Rightarrow$ (ii): Consider a *GBQR* whose corresponding lattice is the 0-1–pasting of two four-element Boolean algebras. ∎

Theorem 3.3. *Assume the operation* $+$ *of a GBQR* $\mathcal{R} = (R; +, \cdot)$ *is associative and* $+_1 \le + \le +_2$*. If* $\mathcal{R}$ *has characteristic* 2 *then* $+_1 = +_2$ *if and only if* $c(x,y) \in \mathrm{DOK}(\mathcal{R})$ *for all* $x, y \in R$ (*which then means that* $\mathcal{R}$ *has to be a Boolean ring*).

Proof. As mentioned in Section 2, the associativity of $+$ together with the assumption $+_1 \le + \le +_2$ implies that $\mathcal{R}$ is *-modular. If $\mathcal{R}$ has characteristic 2, $\mathbf{L}(\mathcal{R})$ is orthomodular and $\mathrm{DOK}(\mathcal{R}) = \{1\}$. In this case, the fact that $c(x,y) \in \mathrm{DOK}(\mathcal{R})$ for all $x, y \in R$ means that any two elements of R commute, from which we can conclude that $\mathbf{L}(\mathcal{R})$ is a Boolean algebra and hence $+_1 = +_2$.

Conversely, if $+_1 = +_2$, it follows that $c(x,y) \in \mathrm{DOK}(\mathcal{R})$ for all $x, y \in R$ by Lemma 3.1. ∎

Remark. To see that for a *-modular $\mathcal{R}$ the assumption $c(x,y) \in \mathrm{DOK}(\mathcal{R})$ for all $x, y \in R$ implies $+_1 = +_2$ if $\mathcal{R}$ has characteristic 2, can be also achieved by avoiding the argument that if any two elements of an orthomodular lattice commute, the lattice has to be a Boolean algebra. We give an alternative proof which only relies on the definitions of $c(x,y)$, $+_1$ and $+_2$ and shows at which points properties of $x \wedge x^*$ come in.

Assume that for two elements a, b of R it holds that $a \le b$ and $a^* \wedge b \le a \vee b^*$. Because of the *-modularity we then obtain $a^* \wedge b \le (a \vee b^*) \wedge b = a \vee (b^* \wedge b)$, which yields $(a \vee a^*) \wedge b = a \vee (a^* \wedge b) \le a \vee (b \wedge b^*)$. If $\mathcal{R}$ has characteristic 2, it follows that $b \le a$, hence $a = b$. Putting $a = x +_1 y$ and $b = x +_2 y$ and assuming $c(x,y) \in \mathrm{DOK}(\mathcal{R})$, we obtain $a \le b$ and

$$a^* \wedge b = (x +_1 y)^* \wedge (x +_2 y) = (c(x,y))^* \le c(x,y) = (x +_1 y) \vee (x +_2 y)^* = a \vee b^*,$$

from which we can conclude $a = b$, i. e. $x +_1 y = x +_2 y$.

Theorem 3.4. *Assume that* $\mathcal{R} = (R; +, \cdot)$ *has the property that* $y \perp xz^* \Rightarrow x(1+yz) = x(1+y)$ *for all* $x, y, z \in R$, *i. e.* $\mathbf{L}(\mathcal{R})$ *is distributive* (*cf.* [4]). *Then* $+_1 = +_2$ *if and only if* $\mathrm{OK}(\mathcal{R})$ *is orthogonal. If this is the case, the operation* $+_1 = +_2$ *has to be associative.*

Proof. If $+_1 = +_2$, then $\mathrm{OK}(\mathcal{R})$ is orthogonal according to Lemma 3.1. Conversely, if $\mathrm{OK}(\mathcal{R})$ is orthogonal, then for $x, y \in R$

$$x +_1 y \leq x +_2 y = (x \vee y) \wedge (x^* \vee y^*) = (x \wedge x^*) \vee (x \wedge y^*) \vee (x^* \wedge y) \vee (y \wedge y^*) \leq$$
$$\leq (x \vee x^*) \wedge (x \vee y) \wedge (x^* \vee y^*) \wedge (y \vee y^*) = (x \wedge y^*) \vee (x^* \wedge y) = x +_1 y,$$

and hence $x +_1 y = x +_2 y$. Now, assuming $+_1 = +_2$, then according to Theorem 4.1 in [3], $+_1$ is associative if and only if $+_1$ is weakly associative and fulfils the condition $(x +_1 y) \wedge z \leq x \vee (y +_1 z)^*$ for all $x, y, z \in R$. We obserwe that $+_1$ is weakly associative by Theorem 3.1. Further

$$\begin{aligned}(x +_1 y) \wedge z &= ((x \wedge y^*) \vee (x^* \wedge y)) \wedge z = (x \wedge y^* \wedge z) \vee (x^* \wedge y \wedge z) \text{ and} \\ x \vee (y +_1 z)^* &= x \vee ((y \wedge z^*) \vee (y^* \wedge z))^* = x \vee ((y^* \vee z) \wedge (y \vee z^*)) = \\ &= (x \vee y^* \vee z) \wedge (x \vee y \vee z^*).\end{aligned}$$

Because $x \wedge y^* \wedge z \leq (x \vee y^* \vee z) \wedge (x \vee y \vee z^*)$ and $x^* \wedge y \wedge z \leq (x \vee y^* \vee z) \wedge (x \vee y \vee z^*)$ we therefore obtain $(x +_1 y) \wedge z \leq x \vee (y +_1 z)^*$. ∎

Corollary 3.1. *Assume that the GBQR* $(R; +, \cdot)$ *has the property that* $y \perp xz^* \Rightarrow x(1+yz) = x(1+y)$ *for all* $x, y, z \in R$. *If* $+$ *is weakly associative and* $+ \leq +_1$ *or* $+ \geq +_2$, *then* $+ = +_1 = +_2$ *and* $+$ *is associative.*

Proof. The assertion of Corollary 3.1 immediately follows from Theorems 3.2 and 3.4. ∎

Corollary 3.2. *Assume that the GBQR* $\mathcal{R} = (R; +, \cdot)$ *has the property that the equation* $x(1+yz) = x(1+y)$ *holds for all* $x, y, z \in \mathrm{OK}(\mathcal{R})$ *and also for those* $x, y, z \in R$ *for which* $y \perp xz^*$. *If in addition* $+$ *is weakly associative and* $+ \geq +_1$ *or* $+ \leq +_2$, *then* $+ = +_1 = +_2$ *and* $+$ *is associative.*

Proof. For $z = 0$, the equation $x(1+yz) = x(1+y)$ yields $x \leq y^*$ for all $x, y \in \mathrm{OK}(\mathcal{R})$, hence $\mathrm{OK}(\mathcal{R})$ is orthogonal. Moreover, because $x(1+yz) = x(1+y)$ for all $x, y, z \in R$ with $y \perp xz^*$, we obtain that $\mathbf{L}(\mathcal{R})$ is distributive (cf. [4]). Therefore $+_1 = +_2$ by Theorem 3.4. Finally, because $+$ is weakly associative and $+ \geq +_1$ or $+ \leq +_2$, it follows by Theorem 3.2 that $+_1 \leq + \leq +_2$, hence $+ = +_1 = +_2$, and again by Theorem 3.4 that $+$ is associative. ∎

4. Examples of generalized Boolean quasirings with unique symmetric difference

Theorem 3.4 provides two opposing classes of examples: On the one hand, because $\mathrm{OK}(\mathcal{R}) = \{0\}$ if * is an orthocomplementation, there are all the *Boolean algebras*. On the other hand, there are all *GBQR*s $\mathcal{R}$ such that $\mathbf{L}(\mathcal{R})$ is a chain. For shortness we will refer to those *GBQR*s as *chains*.

That the orthogonal kernel of a chain $\mathcal{R} = (R; +, \cdot)$ is orthogonal follows from the fact, that $x \leq y$ or $y \leq x$ for all $x, y \in R$. If $x \leq y$, we obtain $x \wedge x^* \leq x \leq y \leq y \vee y^*$, and if $y \leq x$, we have $x \wedge x^* \leq x^* \leq y^* \leq y \vee y^*$. Therefore, Theorem 3.4 can be applied.

Among the chains there is an important *GBQR*, namely $([0,1]; +_C, \cdot_C)$ with $[0,1] \subseteq \mathbf{R}$, $1 +_C x = 1 - x$ (difference within $\mathbf{R}$) and $+_C = +_1 = +_2$. We will call this *GBQR* the *canonical* $[0,1]$-*GBQR* and will denote it by $[0,1]_C$. The operations of $[0,1]_C$ are the following:

$$x +_C y = \left\{ \begin{array}{ll} \min(1-x, y), & \text{if } x \leq y \\ \min(x, 1-y), & \text{if } x \geq y \end{array} \right\} = \min(\max(x,y), 1 - \min(x,y)),$$
$$x \cdot_C y = \min(x, y)$$

(with min denoting the minimum in $\mathbf{R}$). One can easily check that

$$x +_C y = (|x - y| + 1 - |x + y - 1|)/2$$

for all $x, y \in [0,1]$.

As with all *GBQR*s with unique symmetric difference the operation $+_C$ can be interpreted as "exclusive or". But there is a further possibility.

A function d from the square M^2 of a set M to $[0, \infty)$ is called a *generalized distance function* on M, if for all $x, y, z \in M$ the following conditions hold:

(i) if $x \neq y$, then $d(x,y) > 0$,

(ii) $d(x,y) = d(y,x)$,

(iii) $d(x,z) \leq d(x,y) + d(y,z)$.

Hence, for a generalized distance function d on M, it may hold $d(x, x) > 0$ for some $x \in M$. The function $d(x, y) = x +_C y$ is a generalized distance function on $[0, 1]$ with $d(x, x) = 0$ if and only if $x \in \{0, 1\}$. To see this, the crucial point is to prove the triangle inequality. Firstly, one can show that $d_1 : [0, 1]^2 \to [0, 1]$, defined by $d_1(x, y) := 1 - |x + y - 1|$ for all $x, y \in [0, 1]$, satisfies the triangle inequality. If e. g. $x + z, x + y, y + z \leq 1$, then

$$\begin{aligned} d_1(x, z) &= 1 - |x + z - 1| = x + z \leq x + y + y + z = \\ &= 1 - |x + y - 1| + 1 - |y + z - 1| = d_1(x, y) + d_1(y, z). \end{aligned}$$

In an entirely analogous way the remaining seven cases can be proved.
Next we observe that $d_0 : [0, 1]^2 \to [0, 1]$, defined by $d_0(x, y) := |x - y|$ for all $x, y \in [0, 1]$, also fulfils the triangle inequality. Taking into account that $+_C = (d_0 + d_1)/2$, we can conclude from this that also $+_C$ has to satisfy the triangle inequality.

The interpretation of $+$ as generalized distance function gives rise to the following definition, which we will need in Section 5.

Let $\mathcal{R}$ be an arbitrary *GBQR*. Then any homomorphism from $\mathcal{R}$ to the canonical $[0, 1]$-*GBQR* preserving the unity will be called a $[0, 1]$-*valued generalized metric on* $\mathcal{R}$.
We already observe here:

Theorem 4.1. *If $\mathcal{R} = (R; +, \cdot)$ has characteristic 2, in particular, if $\mathcal{R}$ is a Boolean ring, every $[0, 1]$-valued generalized metric on $\mathcal{R}$ is two-valued.*

Proof. Let h be a $[0, 1]$-valued generalized metric on $\mathcal{R}$ and $x \in R$. Then

$$h(x) +_C h(x) = h(x + x) = h(0) = h(1 + 1) = h(1) +_C h(1) = 1 +_C 1 = 0.$$

Therefore $h(x) \in \{0, 1\}$. ∎

The structure of *GBQR*s that are chains suggests how to construct further classes of *GBQR*s $\mathcal{R}$ with unique symmetric difference for which $\mathbf{L}(\mathcal{R})$ is not necessarily a distributive lattice.

For this purpose we first agree to call two elements x, y *weakly comparable*, if $xy = x$ or $yx = y$ or $x(1 + y) = x$ or $(1 + y)x = 1 + y$.

Next we observe
If $\mathcal{R}$ is an arbitrary GBQR such that all its elements are pairwise weakly comparable, then $+_1 = +_2$.

This assertion follows from the fact, that $x +_1 y = x +_2 y$ if $x \leq y$ or $x \perp y$, and from $x +_i y = x^* +_i y^*$ for $i = 1, 2$, as we have pointed out in Section 2.

Now we propose the following method for constructing *GBQR*s $\mathcal{R}$ with $+_1 = +_2$ that have certain desired properties:

Starting with an arbitrary lattice $\mathcal{L}$ having a smallest element consider $\mathcal{L}$ as $\mathrm{OK}(\mathcal{R})$ and the dual of $\mathcal{L}$ with all elements x substituted by x^* as $\mathrm{DOK}(\mathcal{R})$. Then set

$$\mathbf{L}(\mathcal{R}) := \overline{\mathrm{OK}}(\mathcal{R}) = \mathrm{OK}(\mathcal{R}) \cup \mathrm{DOK}(\mathcal{R}),$$

extend * to an involutory antiautomorphism by defining $(x^*)^* = x$ and also extend the (partial) operation $\oplus$ which is given by the lattice operations and * to an arbitrary commutative operation $+$. (If $\mathcal{L}$ has a greatest element, then the greatest element of $\mathrm{OK}(\mathcal{R})$ and the smallest element of $\mathrm{DOK}(\mathcal{R})$ may also be identified.)

5. Physical interpretation

Similarly as in the standard approach to quantum mechanics in Hilbert space proposed by G. W. Mackey ([6]) we will show that Mackey's function can also be used to provide a model for systems of experimental propositions which leads to a ring-like structure with unique symmetric difference. In the standard approach Mackey's function is defined as a mapping

$$p : \mathcal{O} \times \mathcal{S} \times \mathcal{B}(\mathbf{R}) \to [0,1]$$

where $\mathcal{O}$ denotes the set of all observables of a physical system, $\mathcal{S}$ the set of all its states and $\mathcal{B}(\mathbf{R})$ the Boolean algebra of Borel sets of the real line $\mathbf{R}$. For each $(A, \alpha, E) \in \mathcal{O} \times \mathcal{S} \times \mathcal{B}(\mathbf{R})$, $p(A, \alpha, E)$ is interpreted as the probability that a measurement of A for the system in state α will lead to a value in E. It is then assumed that the mapping $E \mapsto p(A, \alpha, E)$ for fixed $(A, \alpha) \in \mathcal{O} \times \mathcal{S}$ is a probability measure on $\mathcal{B}(\mathbf{R})$. We will generalize this model by assuming that results of measurements are (possibly abstract) objects belonging to a *GBQR* $\mathcal{R} = (R, +, \cdot)$ with unique symmetric difference. For $\mathcal{R}$ we can take $\mathcal{B}(\mathbf{R})$ as well as other *GBQR*s with unique symmetric difference, e. g. a chain of finitely many Borel sets of the real line. We will assume that for each $(A, \alpha) \in \mathcal{O} \times \mathcal{S}$ the mapping $E \mapsto p(A, \alpha, E)$ from R to $[0,1]$ is a homomorphism from $\mathcal{R}$ to $[0,1]_C$ preserving the unity. Then for each $(A, E) \in \mathcal{O} \times R$ we can define a mapping $p_{A,E}$ from $\mathcal{S}$ to $[0,1]$ by

$$p_{A,E}(\alpha) := p(A, \alpha, E)$$

for all $\alpha \in \mathcal{S}$. Let L denote the set $\{p_{A,E} \mid (A,E) \in \mathcal{O} \times R\}$. Observe that $p_{A,E}$ can be identified with the equivalence class

$$[(A,E)] := \{(B,F) \in \mathcal{O} \times R \mid p(B,\alpha,F) = p(A,\alpha,E) \text{ for all } \alpha \in \mathcal{S}\}.$$

In L we can define the binary operations $+$ and $\cdot$ pointwise, i. e. for $f, g \in L$

$$(f+g)(\alpha) := f(\alpha) +_C g(\alpha) \text{ for all } \alpha \in \mathcal{S} \text{ and}$$

$$(f \cdot g)(\alpha) := f(\alpha) \cdot_C g(\alpha) \text{ for all } \alpha \in \mathcal{S}.$$

We assume an axiom stating that $\mathcal{L} := (L; +, \cdot)$ is a $GBQR$. This $GBQR$ will be called the *logic of* p. The elements of L (considered as equivalence classes of the form $[(A,E)]$) will be called *logical propositions*. The operations $+$ and $\cdot$ in L correspond to the logical connectives "exclusive or" and "and", respectively.

Now for every observable $A \in \mathcal{O}$ we can define a mapping μ_A from R to L by

$$\mu_A(E) := p_{A,E}$$

for all $E \in R$. This mapping is clearly a homomorphism from $\mathcal{R}$ to $\mathcal{L}$. Moreover, for every $\alpha \in \mathcal{S}$ we can define a mapping m_α from L to $[0,1]$ by

$$m_\alpha(p_{A,E}) := p(A,\alpha,E)$$

for all $p_{A,E} \in L$. This mapping is well-defined and we assume that it is a homomorphism from $\mathcal{L}$ to $[0,1]_C$. Now we have

$$p(A,\alpha,E) = m_\alpha(\mu_A(E))$$

for all $(A,\alpha,E) \in \mathcal{O} \times \mathcal{S} \times R$ similarly as in Mackey's standard approach. Hence, every observable corresponds to a homomorphism from $\mathcal{R}$ to $\mathcal{L}$ and every state to a homomorphism from $\mathcal{L}$ to $[0,1]_C$.

Now we can interpret the logical meaning of the operation $+_C$ in $[0,1]_C$. E. g. let $[(A,E)]$ and $[(B,F)]$ be two experimental propositions with

$$p(A,\alpha,E) = p(B,\alpha,F) = \frac{1}{2}$$

in some state α. Then for $[(A,E)] + [(B,F)] = [(C,G)]$ we have

$$p(A,\alpha,E) +_C p(B,\alpha,F) = p(C,\alpha,G) = \frac{1}{2} +_C \frac{1}{2} = \frac{1}{2},$$

i. e.

$$p_{A,E}(\alpha) +_C p_{B,F}(\alpha) = p_{C,G}(\alpha) = \frac{1}{2} +_C \frac{1}{2} = \frac{1}{2}.$$

This means that the probability that a measurement of C in the state α will give a result within G (which is equivalent to the measurement of the logical proposition "[(A, E)] exclusive or [(B, F)]") is $1/2$. Hence, the operation $+_C$ on $[0, 1]$ should be interpreted in terms of the logical connective "exclusive or", "$1/2$ exclusive or $1/2$ is also $1/2$".

Note that if $\mathbf{L}(\mathcal{R})$ is a Boolean algebra then, as we know from Section 4, every homomorphism from $\mathcal{R}$ to $[0, 1]_C$ preserving the unity is two-valued and we cannot have the situation that $p(A, \alpha, E) = 1/2$. Hence, our model does not comprise the situation in the Hilbert space H quantum mechanics where the probabilities correspond to the values of probability measures on the lattice of closed linear subspaces of H and are not two-valued. This means that our model with a ring-like structure with unique symmetric difference is a generalization of the classical model other than quantum logics based on Hilbert spaces or orthomodular lattices. In quantum logics based on a Hilbert space, we retain the orthomodular law as universally valid in all models, classical and non-classical. In our generalization, we retain the classical law of negation of equivalence as universally valid in our model, classical and non-classical. Note that in our model we may have the situation that $a^* = a$, i. e. $\neg a = a$ ("non a equal to a") which is excluded by the orthomodular models.

We hope that some experimental results will be found in order to confirm the possibility of applying our model in some non-classical situations which cannot be explained by standard quantum mechanics based on Hilbert space.

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