

## CONGRUENCE CLASSES IN BROUWERIAN SEMILATTICES<sup>1</sup>

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### Abstract

Brouwerian semilattices are meet-semilattices with 1 in which every element  $a$  has a relative pseudocomplement with respect to every element  $b$ , i. e. a greatest element  $c$  with  $a \wedge c \leq b$ . Properties of classes of reflexive and compatible binary relations, especially of congruences of such algebras are described and an abstract characterization of congruence classes via ideals is obtained.

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## 1. INTRODUCTION

**Definition 1.1.** Let  $(S, \wedge)$  be a meet-semilattice and  $a, b, c \in S$  and let  $\leq$  denote its induced partial ordering relation. The element  $c$  is called a *relative pseudocomplement* of  $a$  with respect to  $b$  if  $c$  is the greatest element  $x$  of  $S$  satisfying  $a \wedge x \leq b$ . An algebra  $(S, \wedge, *, 1)$  of type  $(2, 2, 0)$  is called a *Brouwerian semilattice* if  $(S, \wedge)$  is a meet-semilattice with greatest element 1 and, for every  $a, b \in S$ ,  $a * b$  is the relative pseudocomplement of  $a$  with respect to  $b$ .

**Remark 1.1.** Without loss of generality the greatest element 1 of  $(S, \leq)$  can be included in the similarity type of a Brouwerian semilattice since it is an algebraic (i. e. an equationally definable) constant, namely  $a * a = 1$  for each  $a \in S$  (see Lemma 1.1).

**Notational convention.** Throughout the paper let  $\mathcal{S} = (S, \wedge, *, 1)$  denote an arbitrary but fixed Brouwerian semilattice.

**Remark 1.2.** It is well-known that the class of all Brouwerian semilattices forms a variety.

**Lemma 1.1.** For  $a, b, c \in S$  (i)–(xii) hold:

- (i)  $a * 1 = 1$ ;
- (ii)  $1 * a = a$ ;
- (iii)  $a * a = 1$ ;
- (iv)  $(a * a) * a = a$ ;
- (v)  $a \leq b * a$ ;
- (vi)  $a \wedge (a * b) = a \wedge b$ ;
- (vii)  $a \leq b$  if and only if  $a * b = 1$ ;
- (viii) if  $b \leq c$ , then  $c * a \leq b * a$ ;
- (ix)  $b \leq (b * a) * a$ ;
- (x)  $((b * a) * a) * a = b * a$ ;
- (xi)  $((b \wedge c) * a) * a = ((b * a) * a) \wedge ((c * a) * a)$ ;
- (xii)  $(b \wedge c) * a = (b \wedge ((c * a) * a)) * a = (((b * a) * a) \wedge c) * a =$   
 $= (((b * a) * a) \wedge ((c * a) * a)) * a.$

**Remark 1.3.** Though the listed properties of Brouwerian semilattices are mostly known (cf. e. g. [3]), for the convenience of the reader we provide a proof.

**Proof of Lemma 1.1.** (i)–(iii) are trivial.

(iv) follows from (iii) and (ii).

(v) follows from  $b \wedge a \leq a$ .

(vi): Since  $a \wedge (a * b) \leq b$ , it holds  $a \wedge (a * b) \leq a \wedge b$ . On the other hand (v) implies  $a \wedge b \leq a \wedge (a * b)$ .

(vii): If  $a \leq b$ , then  $a * b = 1$ . If, conversely,  $a * b = 1$ , then  $a = a \wedge 1 = a \wedge (a * b) = a \wedge b \leq b$  according to (vi).

(viii):  $b \wedge (c * a) \leq c \wedge (c * a) \leq a$  and, hence,  $c * a \leq b * a$ .

(ix):  $(b * a) \wedge b = b \wedge (b * a) \leq a$  and, hence,  $b \leq (b * a) * a$ .

(x):  $((b * a) * a) * a \leq b * a$  according to (ix) and (viii). On the other hand  $b * a \leq ((b * a) * a) * a$  according to (ix).

(xi): From  $b \wedge c \leq b, c$ , it follows by applying (viii) twice  $((b \wedge c) * a) * a \leq (b * a) * a, (c * a) * a$  and, hence,  $((b \wedge c) * a) * a \leq ((b * a) * a) \wedge ((c * a) * a)$ . On the other hand the following are equivalent:

$$b \wedge c \wedge ((b \wedge c) * a) \leq a,$$

$$c \wedge ((b \wedge c) * a) \leq b * a,$$

$$c \wedge ((b \wedge c) * a) \leq ((b * a) * a) * a,$$

$$c \wedge ((b \wedge c) * a) \wedge ((b * a) * a) \leq a,$$

$$((b \wedge c) * a) \wedge ((b * a) * a) \leq c * a,$$

$$((b \wedge c) * a) \wedge ((b * a) * a) \leq ((c * a) * a) * a,$$

$$((b \wedge c) * a) \wedge ((b * a) * a) \wedge ((c * a) * a) \leq a \text{ and}$$

$$((b * a) * a) \wedge ((c * a) * a) \leq ((b \wedge c) * a) * a.$$

(xii): According to (x) and (xi) one obtains

$$(b \wedge c) * a = (((b \wedge c) * a) * a) * a = (((b * a) * a) \wedge ((c * a) * a)) * a,$$

$$\begin{aligned} (b \wedge ((c * a) * a)) * a &= (((b \wedge ((c * a) * a)) * a) * a) * a = \\ &= (((b * a) * a) \wedge (((c * a) * a) * a)) * a = \\ &= (((b * a) * a) \wedge ((c * a) * a)) * a \end{aligned}$$

and

$$\begin{aligned} (((b * a) * a) \wedge c) * a &= (((((b * a) * a) \wedge c) * a) * a) * a = \\ &= (((((b * a) * a) * a) * a) \wedge ((c * a) * a)) * a = \\ &= (((b * a) * a) \wedge ((c * a) * a)) * a. \end{aligned}$$

■

**Remark 1.4.** In the following we often make use of Lemma 1.1 without explicitly mentioning it.

## 2. REFLEXIVE AND COMPATIBLE BINARY RELATIONS IN BROUWERIAN SEMILATTICES

Let  $R$  be a binary relation on  $S$ ,  $a \in R$  and  $n$  a positive integer. Then  $[a]R := \{x \in S \mid x R a\}$ ,  $R^n := R \circ R \circ \dots \circ R$  with  $n$  factors and  $R$  is called *compatible with respect to  $\mathcal{S}$*  if it has the substitution property with respect to both operations  $\wedge$  as well as  $*$ .

In this section  $R, R_1, R_2$  denote some arbitrary but fixed reflexive binary relations on  $S$  which are compatible with respect to  $\mathcal{S}$ .

**Theorem 2.1.** *If  $a \in S$  is the least element of  $[a]R_1$ , then  $[a](R_2 \circ R_1)^n \subseteq [a](R_1 \circ R_2)^n$  and  $[a](R_1 \circ (R_2 \circ R_1)^n) \subseteq [a](R_2 \circ (R_1 \circ R_2)^n)$  for every positive integer  $n$ .*

**Proof.** Let  $n$  be a positive integer. If  $b \in [a](R_2 \circ R_1)^n$ , then there exist  $a_1, \dots, a_{2n} \in S$  with  $b R_2 a_1 R_1 \dots R_1 a_{2n} = a$  and hence

$$b = b \wedge 1 = b \wedge (a_1 * a_1) R_1 b \wedge (a_1 * a) R_2 a_1 \wedge (a_1 * a) = a_1 \wedge a = a$$

for  $n = 1$  and

$$\begin{aligned} b &= b \wedge 1 = b \wedge (a_1 * a_1) R_1 b \wedge (a_1 * a_2) R_2 a_1 \wedge (a_1 * a_3) = a_1 \wedge a_3 R_1 \dots \\ &\dots R_1 a_{2n-2} \wedge a_{2n} R_2 a_{2n-1} \wedge a = a \end{aligned}$$

for  $n > 1$ , and therefore  $b \in [a](R_1 \circ R_2)^n$ . If  $b \in [a](R_1 \circ (R_2 \circ R_1)^n)$ , then there exist  $a_1, \dots, a_{2n+1} \in S$  with  $b R_1 a_1 R_2 \dots R_2 a_{2n} R_1 a_{2n+1} = a$  and hence

$$\begin{aligned} b &= b \wedge 1 = b \wedge (a_1 * a_1) R_2 b \wedge (a_1 * a_2) R_1 a_1 \wedge (a_1 * a_3) = a_1 \wedge a_3 R_2 \dots \\ &\dots R_2 a_{2n-2} \wedge a_{2n} R_1 a_{2n-1} \wedge a R_2 a_{2n} \wedge a = a, \end{aligned}$$

and therefore  $b \in [a](R_2 \circ (R_1 \circ R_2)^n)$ . ■

**Lemma 2.1.** *If  $a \in S$  is the least element of  $[a]R$ , then  $[a]R \subseteq [a]R^{-1}$ .*

**Proof.** If  $b \in [a]R$ , then one obtains

$$b = b \wedge 1 = b \wedge (a * a) \in [b \wedge (b * a)]R^{-1} = [b \wedge a]R^{-1} = [a]R^{-1}. \quad \blacksquare$$

**Remark 2.1.** From Lemma 2.1 it follows that if  $a \in S$  is the least element of both  $[a]R$  and  $[a]R^{-1}$ , then  $[a]R = [a]R^{-1}$ .

**Lemma 2.2.** *If  $a \in S$  is the least element of  $[a]R$ , then  $[a]R^n = [a]R$  for every positive integer  $n$ .*

**Proof.** We use induction on  $n$ . The case  $n = 1$  is trivial. Now assume  $n \geq 1$  and  $[a]R^n = [a]R$ . The inclusion  $[a]R \subseteq [a]R^{n+1}$  is trivial. If  $b \in [a]R^{n+1}$ , then there exist  $c_1, \dots, c_{n+1} \in S$  with  $b R c_1 R \dots R c_n R c_{n+1} = a$  and therefore

$$b = b \wedge 1 = b \wedge (c_1 * c_1) R c_1 \wedge (c_1 * c_2) = c_1 \wedge c_2 R \dots R c_n \wedge a = a,$$

whence  $b \in [a]R^n$  which implies  $[a]R^{n+1} \subseteq [a]R^n = [a]R$ . ■

**Definition 2.1.** For every subset  $M$  of  $S^2$  let  $\Theta(M)$  denote the least congruence on  $S$  including  $M$ .  $\Theta(M)$  is usually called the *congruence generated by  $M$* .

**Theorem 2.2.** *If  $a \in S$  is the least element of both  $[a]R$  and  $[a]R^{-1}$ , then  $[a]R = [a]\Theta(R)$ .*

**Proof.** Let  $b \in [a](R \circ R^{-1})$ . Then there exists an element  $c$  of  $S$  with  $bRcR^{-1}a$ . Since  $a$  is the least element of  $[a]R^{-1}$  it follows  $c \geq a$ . This implies  $b \wedge aRc \wedge a = a$  and since  $a$  is the least element of  $[a]R$ , it follows  $b \wedge a \geq a$ . This shows  $b \geq a$ . Therefore,  $a$  is the least element of  $[a](R \circ R^{-1})$  and it follows from Lemma 2.2 that  $[a](R \circ R^{-1})^n = [a](R \circ R^{-1})$  for every positive integer  $n$ . Because of Remark 2.1 and Lemma 2.2,  $[a]R = [a]R^{-1}$  and  $[a](R \circ R) = [a]R$ , and therefore,  $[a](R \circ R^{-1}) = [a](R \circ R) = [a]R$ . Now

$$[a]\Theta(R) = [a] \left( \bigcup_{n=1}^{\infty} (R \circ R^{-1})^n \right) = \bigcup_{n=1}^{\infty} [a](R \circ R^{-1})^n = \bigcup_{n=1}^{\infty} [a]R = [a]R. \quad \blacksquare$$

### 3. CONGRUENCES AND CONGRUENCE CLASSES IN BROUWERIAN SEMILATTICES

**Definition 3.1.** An algebra  $\mathcal{A}$  is called an *algebra with 1* if 1 is a distinguished fixed element of the base set of  $\mathcal{A}$ . Let  $\mathcal{A}$  be an algebra with 1.  $\mathcal{A}$  is called *weakly regular* if, for all  $\Theta, \Phi \in \text{Con}\mathcal{A}$ ,  $[1]\Theta = [1]\Phi$  implies  $\Theta = \Phi$ .  $\mathcal{A}$  is called *permutable at 1* if, for any  $\Theta, \Phi \in \text{Con}\mathcal{A}$ , it holds  $[1](\Theta \circ \Phi) = [1](\Phi \circ \Theta)$ .  $\mathcal{A}$  is called *distributive at 1* if, for all  $\Theta, \Phi, \Psi \in \text{Con}\mathcal{A}$ , it holds  $[1]((\Theta \vee \Phi) \cap \Psi) = [1]((\Theta \cap \Psi) \vee (\Phi \cap \Psi))$ .  $\mathcal{A}$  is called *arithmetic at 1* if it is both permutable at 1 and distributive at 1. A variety  $\mathcal{V}$  is called a *variety with 1* if 1 is an equationally definable constant of  $\mathcal{V}$ . A variety with 1 is called *weakly regular*, respectively *arithmetic at 1*, if each of its members has the corresponding property.

**Proposition 3.1.** A variety  $\mathcal{V}$  with 1 is weakly regular if and only if there exist a positive integer  $n$  and binary terms  $t_1, \dots, t_n$  of  $\mathcal{V}$  such that the condition  $t_1(x, y) = \dots = t_n(x, y) = 1$  is equivalent to  $x = y$ , and  $\mathcal{V}$  is arithmetic at 1 if and only if there exists a binary term  $t$  of  $\mathcal{V}$  satisfying  $t(x, x) = t(1, x) = 1$  and  $t(x, 1) = x$ .

**Proof.** The first assertion was proved in [2] and the second one in [1]. ■

**Theorem 3.1.** The variety of Brouwerian semilattices is weakly regular and arithmetic at 1.

**Proof.** This follows from Proposition 3.1 by taking  $n := 2$ ,  $t_1(x, y) := x * y$  and  $t_2(x, y) = t(x, y) := y * x$ . ■

**Theorem 3.2.** For  $a, b, c, d \in S$  and  $\Theta \in \text{Con}S$ , (i)–(ix) hold:

- (i) If  $b, c \in [a]\Theta$ , then  $b \wedge c \in [a]\Theta$ .
- (ii) If  $b \geq a$ , then  $b \in [a]\Theta$  if and only if  $(b * a) * a \in [a]\Theta$ .
- (iii) If  $b, c \in [a]\Theta$ , then  $(b * c) * a \in [a]\Theta$ .
- (iv) If  $b, c \in [a]\Theta$ , then  $((b * a) \wedge (c * a)) * a \in [a]\Theta$ .
- (v) If  $b \Theta c$ , then  $(b * a) \wedge ((c * a) * a) \in [a]\Theta$ .
- (vi) If  $b \leq c$ , then  $b * a \Theta c * a$  if and only if  $(b * a) \wedge ((c * a) * a) \in [a]\Theta$ .
- (vii) If  $b \leq c$  and  $(b * a) \wedge ((c * a) * a) \in [a]\Theta$ , then  
 $((b \wedge d) * a) \wedge (((c \wedge d) * a) * a) \in [a]\Theta$ .
- (viii) If  $b \leq c$  and  $(b * a) \wedge ((c * a) * a) \in [a]\Theta$ , then  $((b * a) * a) \wedge (c * a) \in [a]\Theta$ .
- (ix) If  $b, c \wedge (b * a) \in [a]\Theta$ , then  $c \in [a]\Theta$ .

**Proof.** (i):  $b \wedge c \in [a \wedge a]\Theta = [a]\Theta$ .

(ii): If  $b \in [a]\Theta$ , then  $(b * a) * a \in [(a * a) * a]\Theta = [a]\Theta$ . If, conversely,  $(b * a) * a \in [a]\Theta$ , then  $b = b \wedge 1 = b \wedge (a * a) \in [b \wedge ((b * a) * a)]\Theta = [b \wedge (b * a)]\Theta = [b \wedge a]\Theta = [a]\Theta$ .

(iii):  $(b * c) * a \in [(a * a) * a]\Theta = [a]\Theta$ .

(iv):  $((b * a) \wedge (c * a)) * a \in [((a * a) \wedge (a * a)) * a]\Theta = [(a * a) * a]\Theta = [a]\Theta$ .

(v):  $(b * a) \wedge ((c * a) * a) \in [(b * a) \wedge ((b * a) * a)]\Theta = [(b * a) \wedge a]\Theta = [a]\Theta$ .

(vi): If  $b * a \Theta c * a$ , then  $(b * a) \wedge ((c * a) * a) \in [(b * a) \wedge ((b * a) * a)]\Theta = [(b * a) \wedge a]\Theta = [a]\Theta$ . Assume, conversely,  $(b * a) \wedge ((c * a) * a) \in [a]\Theta$ . Then  $[(c * a) * a]\Theta \wedge [b * a]\Theta = [a]\Theta$  and, hence,  $[b * a]\Theta \leq [(c * a) * a]\Theta * [a]\Theta = [((c * a) * a) * a]\Theta = [c * a]\Theta$ . On the other hand,  $b \leq c$  implies  $c * a \leq b * a$  and, hence,  $[c * a]\Theta \leq [b * a]\Theta$ . Together it follows  $[b * a]\Theta = [c * a]\Theta$  and hence  $b * a \Theta c * a$ .

(vii): According to (vi),  $b * a \Theta c * a$  which implies  $((b * a) * a) \wedge d * a \Theta (((c * a) * a) \wedge d) * a$ . On the other hand,  $b \leq c$  implies  $(b * a) * a \leq (c * a) * a$  and, hence,  $((b * a) * a) \wedge d \leq ((c * a) * a) \wedge d$ . Applying (vi) once more, one obtains

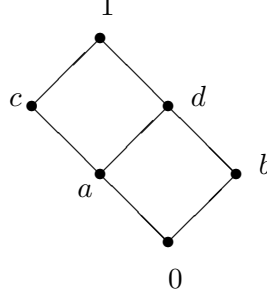
$$((b \wedge d) * a) \wedge (((c \wedge d) * a) * a) = (((b * a) * a) \wedge d) * a \wedge (((c * a) * a) \wedge d) * a \in [a]\Theta.$$

(viii): According to (vi),  $b * a \Theta c * a$ , whence by (v):

$$((b * a) * a) \wedge (c * a) = ((b * a) * a) \wedge (((c * a) * a) * a) \in [a]\Theta.$$

(ix):  $c = c \wedge 1 = c \wedge (a * a) \in [c \wedge (b * a)]\Theta = [a]\Theta$ . ■

**Remark 3.1.** The assumption  $b \geq a$  in (ii) cannot be omitted as can be seen from the following example:



If  $\Theta$  denotes the equivalence relation having the classes  $\{0, b\}$ ,  $\{a, d\}$  and  $\{c, 1\}$ , then  $(b * a) * a = c * a = d \in [a]\Theta$  but  $b \notin [a]\Theta$ . (Note that  $b \not\geq a$ .)

**Corollary 3.1.** From (vi), it follows that for  $a, b, c \in S$  with  $b \leq c$  and, for  $\Theta, \Phi \in \text{Con}\mathcal{S}$  with  $[a]\Theta = [a]\Phi$ ,  $b * a \Theta c * a$  is equivalent to  $b * a \Phi c * a$ .

**Lemma 3.1.** If  $a, b \in S$  and  $\Theta \in \text{Con}\mathcal{S}$ , then  $b \in [a]\Theta$  if and only if  $a * b, b * a \in [1]\Theta$ .

**Proof.** If  $b \in [a]\Theta$ , then  $a * b, b * a \in [a * a]\Theta = [1]\Theta$ . If, conversely,  $a * b, b * a \in [1]\Theta$ , then  $b = b \wedge 1 \in [b \wedge (b * a)]\Theta = [b \wedge a]\Theta = [a \wedge b]\Theta = [a \wedge (a * b)]\Theta = [a \wedge 1]\Theta = [a]\Theta$ . ■

**Definition 3.2.** A subset  $I$  of  $S$  is called an *ideal* of  $\mathcal{S}$  if there exists a congruence  $\Theta$  on  $\mathcal{S}$  with  $[1]\Theta = I$ . Since the intersection of ideals of  $\mathcal{S}$  is again an ideal of  $\mathcal{S}$ , there exists a smallest ideal of  $\mathcal{S}$  including a given subset  $M$  of  $S$ . This ideal is called the *ideal of  $\mathcal{S}$  generated by  $M$*  and it is denoted by  $I(M)$ . For  $a \in S$  and  $M \subseteq S$  put  $a * M := \{a * x \mid x \in M\}$  and  $M * a := \{x * a \mid x \in M\}$ .

**Lemma 3.2.** For  $a, b \in S$  and  $\Theta \in \text{Con}\mathcal{S}$  the following are equivalent:

- (i)  $b \in [a]\Theta$ ;
- (ii)  $a * b, b * a \in (a * ([a]\Theta)) \cup (([a]\Theta) * a)$ ;
- (iii)  $a * b, b * a \in I((a * ([a]\Theta)) \cup (([a]\Theta) * a))$ ;
- (iv)  $a * b, b * a \in [1]\Theta$ .



**Proof.** The implications (i)  $\Rightarrow$  (ii)  $\Rightarrow$  (iii) are trivial.

(iii)  $\Rightarrow$  (iv):  $I((a * ([a]\Theta)) \cup (([a]\Theta) * a)) \subseteq [1]\Theta$  follows from Lemma 3.1.

(iv)  $\Rightarrow$  (i): follows from Lemma 3.1. ■

**Corollary 3.2.** *If  $a \in S$ ,  $\Theta, \Phi \in \text{Con}\mathcal{S}$  and*

$$I((a * ([a]\Theta)) \cup (([a]\Theta) * a)) = I((a * ([a]\Phi)) \cup (([a]\Phi) * a)),$$

*then  $[a]\Theta = [a]\Phi$ .*

**Theorem 3.3.** *A non-empty subset  $A$  of  $S$  is a class of some congruence on  $\mathcal{S}$  if and only if  $a \in A$ ,  $b \in S$  and  $a * b, b * a \in I((a * A) \cup (A * a))$  together imply  $b \in A$ .*

**Proof.** Assume the condition of the theorem to hold. Let  $c \in A$ . Then there exists a congruence  $\Psi$  on  $\mathcal{S}$  with  $[1]\Psi = I((c * A) \cup (A * c))$ . If  $d \in A$ , then  $c * d, d * c \in I((c * A) \cup (A * c)) = [1]\Psi$  and, hence,  $d \in [c]\Psi$  according to Lemma 3.1. If, conversely,  $e \in [c]\Psi$ , then  $c * e, e * c \in [1]\Psi = I((c * A) \cup (A * c))$ , because of Lemma 3.1, and, hence  $e \in A$  according to the condition of the theorem. This shows  $A = [c]\Psi$  and therefore  $A$  is a class of some congruence on  $\mathcal{S}$ . The rest of the proof follows from Lemma 3.2. ■

#### REFERENCES

- [1] J. Duda, *Arithmeticity at 0*, Czechoslovak Math. J. **37** (1987), 197–206.
- [2] K. Fichtner, *Eine Bemerkung über Mannigfaltigkeiten universeller Algebren mit Idealen*, Monatsb. Deutsch. Akad. Wiss. Berlin **12** (1970), 21–25.
- [3] P. Köhler, *Brouwerian semilattices: the lattice of total subalgebras*, Banach Center Publ. **9** (1982), 47–56.
- [4] W.C. Nemitz, *Implicative semi-lattices*, Trans. Amer. Math. Soc. **117** (1965), 128–142.

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