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CONGRUENCE CLASSES IN BROUWERIAN SEMILATTICES[1]

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Abstract

Brouwerian semilattices are meet-semilattices with 1 in which every element a has a relative pseudocomplement with respect to every element b, i. e. a greatest element c with $a \wedge c \leq b$. Properties of classes of reflexive and compatible binary relations, especially of congruences of such algebras are described and an abstract characterization of congruence classes via ideals is obtained.

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1. Introduction

Definition 1.1. Let $(S, \wedge)$ be a meet-semilattice and $a, b, c \in S$ and let $\leq$ denote its induced partial ordering relation. The element c is called a *relative pseudocomplement* of a with respect to b if c is the greatest element x of S satisfying $a \wedge x \leq b$. An algebra $(S, \wedge, *, 1)$ of type $(2, 2, 0)$ is called a *Brouwerian semilattice* if $(S, \wedge)$ is a meet-semilattice with greatest element 1 and, for every $a, b \in S$, $a * b$ is the relative pseudocomplement of a with respect to b.

Remark 1.1. Without loss of generality the greatest element 1 of $(S, \leq)$ can be included in the similarity type of a Brouwerian semilattice since it is an algebraic (i. e. an equationally definable) constant, namely $a * a = 1$ for each $a \in S$ (see Lemma 1.1).

Notational convention. Throughout the paper let $\mathcal{S} = (S, \wedge, *, 1)$ denote an arbitrary but fixed Brouwerian semilattice.

Remark 1.2. It is well-known that the class of all Brouwerian semilattices forms a variety.

Lemma 1.1. *For* $a, b, c \in S$ (i)–(xii) *hold:*

(i) $a * 1 = 1$;

(ii) $1 * a = a$;

(iii) $a * a = 1$;

(iv) $(a * a) * a = a$;

(v) $a \leq b * a$;

(vi) $a \wedge (a * b) = a \wedge b$;

(vii) $a \leq b$ *if and only if* $a * b = 1$;

(viii) *if* $b \leq c$, *then* $c * a \leq b * a$;

(ix) $b \leq (b * a) * a$;

(x) $((b * a) * a) * a = b * a$;

(xi) $((b \wedge c) * a) * a = ((b * a) * a) \wedge ((c * a) * a)$;

(xii) $(b \wedge c) * a = (b \wedge ((c * a) * a)) * a = (((b * a) * a) \wedge c) * a =$
$= (((b * a) * a) \wedge ((c * a) * a)) * a$.

Remark 1.3. Though the listed properties of Brouwerian semilattices are mostly known (cf. e. g. [3]), for the convenience of the reader we provide a proof.

***Proof of Lemma* 1.1.** (i)–(iii) are trivial.

(iv) follows from (iii) and (ii).

(v) follows from $b \wedge a \leq a$.

(vi): Since $a \wedge (a * b) \leq b$, it holds $a \wedge (a * b) \leq a \wedge b$. On the other hand (v) implies $a \wedge b \leq a \wedge (a * b)$.

(vii): If $a \leq b$, then $a * b = 1$. If, conversely, $a * b = 1$, then $a = a \wedge 1 = a \wedge (a * b) = a \wedge b \leq b$ according to (vi).

(viii): $b \wedge (c * a) \leq c \wedge (c * a) \leq a$ and, hence, $c * a \leq b * a$.

(ix): $(b * a) \wedge b = b \wedge (b * a) \leq a$ and, hence, $b \leq (b * a) * a$.

(x): $((b * a) * a) * a \leq b * a$ according to (ix) and (viii). On the other hand $b * a \leq ((b * a) * a) * a$ according to (ix).

(xi): From $b \wedge c \leq b, c$, it follows by applying (viii) twice $((b \wedge c) * a) * a \leq (b * a) * a, (c * a) * a$ and, hence, $((b \wedge c) * a) * a \leq ((b * a) * a) \wedge ((c * a) * a)$. On the other hand the following are equivalent:

$$b \wedge c \wedge ((b \wedge c) * a) \leq a,$$

$$c \wedge ((b \wedge c) * a) \leq b * a,$$

$$c \wedge ((b \wedge c) * a) \leq ((b * a) * a) * a,$$

$$c \wedge ((b \wedge c) * a) \wedge ((b * a) * a) \leq a,$$

$$((b \wedge c) * a) \wedge ((b * a) * a) \leq c * a,$$

$$((b \wedge c) * a) \wedge ((b * a) * a) \leq ((c * a) * a) * a,$$

$$((b \wedge c) * a) \wedge ((b * a) * a) \wedge ((c * a) * a) \leq a \text{ and}$$

$$((b * a) * a) \wedge ((c * a) * a) \leq ((b \wedge c) * a) * a.$$

(xii): According to (x) and (xi) one obtains

$$(b \wedge c) * a = (((b \wedge c) * a) * a) * a = (((b * a) * a) \wedge ((c * a) * a)) * a,$$

$$\begin{aligned}(b \wedge ((c * a) * a)) * a &= (((b \wedge ((c * a) * a)) * a) * a) * a = \\ &= (((b * a) * a) \wedge ((((c * a) * a) * a) * a)) * a = \\ &= (((b * a) * a) \wedge ((c * a) * a)) * a\end{aligned}$$

and

$$\begin{aligned}(((b * a) * a) \wedge c) * a &= (((((b * a) * a) \wedge c) * a) * a) * a = \\ &= (((((b * a) * a) * a) * a) \wedge ((c * a) * a)) * a = \\ &= (((b * a) * a) \wedge ((c * a) * a)) * a.\end{aligned}$$

■

Remark 1.4. In the following we often make use of Lemma 1.1 without explicitly mentioning it.

2. Reflexive and compatible binary relations in Brouwerian semilattices

Let R be a binary relation on S, $a \in R$ and n a positive integer. Then $[a]R := \{x \in S \mid x\, R\, a\}$, $R^n := R \circ R \circ \cdots \circ R$ with n factors and R is *called compatible with respect to* $\mathcal{S}$ if it has the substitution property with respect to both operations $\wedge$ as well as $*$.

In this section R, R_1, R_2 denote some arbitrary but fixed reflexive binary relations on S which are compatible with respect to $\mathcal{S}$.

Theorem 2.1. *If $a \in S$ is the least element of $[a]R_1$, then $[a](R_2 \circ R_1)^n \subseteq [a](R_1 \circ R_2)^n$ and $[a](R_1 \circ (R_2 \circ R_1)^n) \subseteq [a](R_2 \circ (R_1 \circ R_2)^n)$ for every positive integer n.*

Proof. Let n be a positive integer. If $b \in [a](R_2 \circ R_1)^n$, then there exist $a_1, \ldots, a_{2n} \in S$ with $b\, R_2\, a_1\, R_1\, \ldots\, R_1\, a_{2n} = a$ and hence

$$b = b \wedge 1 = b \wedge (a_1 * a_1)\, R_1\, b \wedge (a_1 * a)\, R_2\, a_1 \wedge (a_1 * a) = a_1 \wedge a = a$$

for $n = 1$ and

$$\begin{aligned} b &= b \wedge 1 = b \wedge (a_1 * a_1)\, R_1\, b \wedge (a_1 * a_2)\, R_2\, a_1 \wedge (a_1 * a_3) = a_1 \wedge a_3\, R_1 \dots \\ &\dots R_1\, a_{2n-2} \wedge a_{2n}\, R_2\, a_{2n-1} \wedge a = a \end{aligned}$$

for $n > 1$, and therefore $b \in [a](R_1 \circ R_2)^n$. If $b \in [a](R_1 \circ (R_2 \circ R_1)^n)$, then there exist $a_1, \dots, a_{2n+1} \in S$ with $b\, R_1\, a_1\, R_2 \dots R_2\, a_{2n}\, R_1\, a_{2n+1} = a$ and hence

$$\begin{aligned} b &= b \wedge 1 = b \wedge (a_1 * a_1)\, R_2\, b \wedge (a_1 * a_2)\, R_1\, a_1 \wedge (a_1 * a_3) = a_1 \wedge a_3\, R_2 \dots \\ &\dots R_2\, a_{2n-2} \wedge a_{2n}\, R_1\, a_{2n-1} \wedge a\, R_2\, a_{2n} \wedge a = a, \end{aligned}$$

and therefore $b \in [a](R_2 \circ (R_1 \circ R_2)^n)$. ■

Lemma 2.1. *If $a \in S$ is the least element of $[a]R$, then $[a]R \subseteq [a]R^{-1}$.*

Proof. If $b \in [a]R$, then one obtains

$$b = b \wedge 1 = b \wedge (a * a) \in [b \wedge (b * a)]R^{-1} = [b \wedge a]R^{-1} = [a]R^{-1}.$$

■

Remark 2.1. From Lemma 2.1 it follows that if $a \in S$ is the least element of both $[a]R$ and $[a]R^{-1}$, then $[a]R = [a]R^{-1}$.

Lemma 2.2. *If $a \in S$ is the least element of $[a]R$, then $[a]R^n = [a]R$ for every positive integer n.*

Proof. We use induction on n. The case $n = 1$ is trivial. Now assume $n \geq 1$ and $[a]R^n = [a]R$. The inclusion $[a]R \subseteq [a]R^{n+1}$ is trivial. If $b \in [a]R^{n+1}$, then there exist $c_1, \dots, c_{n+1} \in S$ with $b\, R\, c_1\, R \dots R\, c_n\, R\, c_{n+1} = a$ and therefore

$$b = b \wedge 1 = b \wedge (c_1 * c_1)\, R\, c_1 \wedge (c_1 * c_2) = c_1 \wedge c_2\, R \dots R\, c_n \wedge a = a,$$

whence $b \in [a]R^n$ which implies $[a]R^{n+1} \subseteq [a]R^n = [a]R$. ■

Definition 2.1. For every subset M of S^2 let $\Theta(M)$ denote the least congruence on $\mathcal{S}$ including M. $\Theta(M)$ is usually called the *congruence generated by M*.

Theorem 2.2. *If $a \in S$ is the least element of both $[a]R$ and $[a]R^{-1}$, then $[a]R = [a]\Theta(R)$.*

Proof. Let $b \in [a](R \circ R^{-1})$. Then there exists an element c of S with $bRcR^{-1}a$. Since a is the least element of $[a]R^{-1}$ it follows $c \geq a$. This implies $b \wedge aRc \wedge a = a$ and since a is the least element of $[a]R$, it follows $b \wedge a \geq a$. This shows $b \geq a$. Therefore, a is the least element of $[a](R \circ R^{-1})$ and it follows from Lemma 2.2 that $[a](R \circ R^{-1})^n = [a](R \circ R^{-1})$ for every positive integer n. Because of Remark 2.1 and Lemma 2.2, $[a]R = [a]R^{-1}$ and $[a](R \circ R) = [a]R$, and therefore, $[a](R \circ R^{-1}) = [a](R \circ R) = [a]R$. Now

$$[a]\Theta(R) = [a]\left(\bigcup_{n=1}^{\infty}(R \circ R^{-1})^n\right) = \bigcup_{n=1}^{\infty}[a](R \circ R^{-1})^n = \bigcup_{n=1}^{\infty}[a]R = [a]R.$$

■

3. Congruences and congruence classes in Brouwerian semilattices

Definition 3.1. An algebra $\mathcal{A}$ is called an *algebra with* 1 if 1 is a distinguished fixed element of the base set of $\mathcal{A}$. Let $\mathcal{A}$ be an algebra with 1. $\mathcal{A}$ is called *weakly regular* if, for all $\Theta, \Phi \in \mathrm{Con}\mathcal{A}$, $[1]\Theta = [1]\Phi$ implies $\Theta = \Phi$. $\mathcal{A}$ is called *permutable at* 1 if, for any $\Theta, \Phi \in \mathrm{Con}\mathcal{A}$, it holds $[1](\Theta \circ \Phi) = [1](\Phi \circ \Theta)$. $\mathcal{A}$ is called *distributive at* 1 if, for all $\Theta, \Phi, \Psi \in \mathrm{Con}\mathcal{A}$, it holds $[1]((\Theta \vee \Phi) \cap \Psi) = [1]((\Theta \cap \Psi) \vee (\Phi \cap \Psi))$. $\mathcal{A}$ is called *arithmetic at* 1 if it is both permutable at 1 and distributive at 1. A variety $\mathcal{V}$ is called a *variety with* 1 if 1 is an equationally definable constant of $\mathcal{V}$. A variety with 1 is called *weakly regular*, respectively *arithmetic at* 1, if each of its members has the corresponding property.

Proposition 3.1. *A variety $\mathcal{V}$ with 1 is weakly regular if and only if there exist a positive integer n and binary terms $t_1, \ldots, t_n$ of $\mathcal{V}$ such that the condition $t_1(x, y) = \cdots = t_n(x, y) = 1$ is equivalent to $x = y$, and $\mathcal{V}$ is arithmetic at 1 if and only if there exists a binary term t of $\mathcal{V}$ satisfying $t(x, x) = t(1, x) = 1$ and $t(x, 1) = x$.*

Proof. The first assertion was proved in [2] and the second one in [1]. ■

Theorem 3.1. *The variety of Brouwerian semilattices is weakly regular and arithmetic at 1.*

Proof. This follows from Proposition 3.1 by taking $n := 2$, $t_1(x, y) := x * y$ and $t_2(x, y) = t(x, y) := y * x$. ■

Theorem 3.2. *For $a, b, c, d \in S$ and $\Theta \in \mathrm{Con}\mathcal{S}$,* (i)–(ix) *hold:*

(i) *If $b, c \in [a]\Theta$, then $b \wedge c \in [a]\Theta$.*

(ii) *If $b \geq a$, then $b \in [a]\Theta$ if and only if $(b * a) * a \in [a]\Theta$.*

(iii) *If $b, c \in [a]\Theta$, then $(b * c) * a \in [a]\Theta$.*

(iv) *If $b, c \in [a]\Theta$, then $((b * a) \wedge (c * a)) * a \in [a]\Theta$.*

(v) *If $b \,\Theta\, c$, then $(b * a) \wedge ((c * a) * a) \in [a]\Theta$.*

(vi) *If $b \leq c$, then $b * a \,\Theta\, c * a$ if and only if $(b * a) \wedge ((c * a) * a) \in [a]\Theta$.*

(vii) *If $b \leq c$ and $(b * a) \wedge ((c * a) * a) \in [a]\Theta$, then*
$((b \wedge d) * a) \wedge (((c \wedge d) * a) * a) \in [a]\Theta$.

(viii) *If $b \leq c$ and $(b*a) \wedge ((c*a)*a) \in [a]\Theta$, then $((b*a)*a) \wedge (c*a) \in [a]\Theta$.*

(ix) *If $b, c \wedge (b * a) \in [a]\Theta$, then $c \in [a]\Theta$.*

Proof. (i): $b \wedge c \in [a \wedge a]\Theta = [a]\Theta$.

(ii): If $b \in [a]\Theta$, then $(b * a) * a \in [(a * a) * a]\Theta = [a]\Theta$. If, conversely, $(b * a) * a \in [a]\Theta$, then $b = b \wedge 1 = b \wedge (a * a) \in [b \wedge (((b * a) * a) * a)]\Theta = [b \wedge (b * a)]\Theta = [b \wedge a]\Theta = [a]\Theta$.

(iii): $(b * c) * a \in [(a * a) * a]\Theta = [a]\Theta$.

(iv): $((b*a) \wedge (c*a)) * a \in [((a*a) \wedge (a*a)) * a]\Theta = [(a*a)*a]\Theta = [a]\Theta$.

(v): $(b*a) \wedge ((c*a)*a) \in [(b*a) \wedge ((b*a)*a)]\Theta = [(b*a) \wedge a]\Theta = [a]\Theta$.

(vi): If $b * a \,\Theta\, c * a$, then $(b * a) \wedge ((c * a) * a) \in [(b * a) \wedge ((b * a) * a)]\Theta = [(b * a) \wedge a]\Theta = [a]\Theta$. Assume, conversely, $(b * a) \wedge ((c * a) * a) \in [a]\Theta$. Then $[(c * a) * a]\Theta \wedge [b * a]\Theta = [a]\Theta$ and, hence, $[b * a]\Theta \leq [(c * a) * a]\Theta * [a]\Theta = [((c * a) * a) * a]\Theta = [c * a]\Theta$. On the other hand, $b \leq c$ implies $c * a \leq b * a$ and, hence, $[c * a]\Theta \leq [b * a]\Theta$. Together it follows $[b * a]\Theta = [c * a]\Theta$ and hence $b * a \,\Theta\, c * a$.

(vii): According to (vi), $b * a \,\Theta\, c * a$ which implies $(((b * a) * a) \wedge d) * a \,\Theta\, (((c*a)*a) \wedge d)*a$. On the other hand, $b \leq c$ implies $(b*a)*a \leq (c*a)*a$ and, hence, $((b * a) * a) \wedge d \leq ((c * a) * a) \wedge d$. Applying (vi) once more, one obtains

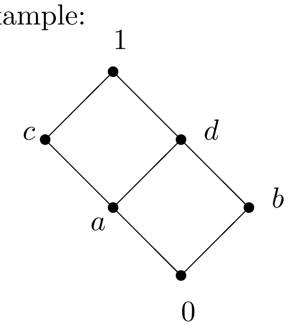
$$((b \wedge d)*a) \wedge (((c \wedge d)*a)*a) = ((((b*a)*a) \wedge d)*a) \wedge (((((c*a)*a) \wedge d)*a)*a) \in [a]\Theta.$$

(viii): According to (vi), $b*a\,\Theta\,c*a$, whence by (v):

$$((b*a)*a)\wedge(c*a)=((b*a)*a)\wedge(((c*a)*a)*a)\in[a]\Theta.$$

(ix): $c=c\wedge 1=c\wedge(a*a)\in[c\wedge(b*a)]\Theta=[a]\Theta.$ ■

Remark 3.1. The assumption $b\geq a$ in (ii) cannot be omitted as can be seen from the following example:



If Θ denotes the equivalence relation having the classes $\{0,b\}$, $\{a,d\}$ and $\{c,1\}$, then $(b*a)*a=c*a=d\in[a]\Theta$ but $b\notin[a]\Theta$. (Note that $b\not\geq a$.)

Corollary 3.1. *From* (vi), *it follows that for $a,b,c\in S$ with $b\leq c$ and, for $\Theta,\Phi\in\operatorname{Con}\mathcal{S}$ with $[a]\Theta=[a]\Phi$, $b*a\,\Theta\,c*a$ is equivalent to $b*a\,\Phi\,c*a$.*

Lemma 3.1. *If $a,b\in S$ and $\Theta\in\operatorname{Con}\mathcal{S}$, then $b\in[a]\Theta$ if and only if $a*b,b*a\in[1]\Theta$.*

Proof. If $b\in[a]\Theta$, then $a*b,b*a\in[a*a]\Theta=[1]\Theta$. If, conversely, $a*b,b*a\in[1]\Theta$, then $b=b\wedge 1\in[b\wedge(b*a)]\Theta=[b\wedge a]\Theta=[a\wedge b]\Theta=[a\wedge(a*b)]\Theta=[a\wedge 1]\Theta=[a]\Theta$. ■

Definition 3.2. A subset I of S is called an *ideal* of $\mathcal{S}$ if there exists a congruence Θ on $\mathcal{S}$ with $[1]\Theta=I$. Since the intersection of ideals of $\mathcal{S}$ is again an ideal of $\mathcal{S}$, there exists a smallest ideal of $\mathcal{S}$ including a given subset M of S. This ideal is called the *ideal of $\mathcal{S}$ generated by M* and it is denoted by $\mathrm{I}(M)$. For $a\in S$ and $M\subseteq S$ put $a*M:=\{a*x\mid x\in M\}$ and $M*a:=\{x*a\mid x\in M\}$.

Lemma 3.2. *For $a,b\in S$ and $\Theta\in\operatorname{Con}\mathcal{S}$ the following are equivalent:*

(i) $b\in[a]\Theta$;

(ii) $a*b,\ b*a\in(a*([a]\Theta))\cup(([a]\Theta)*a)$;

(iii) $a*b,\ b*a\in\mathrm{I}((a*([a]\Theta))\cup(([a]\Theta)*a))$;

(iv) $a*b,\ b*a\in[1]\Theta$.

Proof. The implications (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) are trivial.

(iii) $\Rightarrow$ (iv): $\mathrm{I}((a*([a]\Theta))\cup(([a]\Theta)*a))\subseteq[1]\Theta$ follows from Lemma 3.1.

(iv) $\Rightarrow$ (i): follows from Lemma 3.1. ∎

Corollary 3.2. *If $a\in S$, $\Theta,\Phi\in\mathrm{Con}\mathcal{S}$ and*

$$\mathrm{I}((a*([a]\Theta))\cup(([a]\Theta)*a))=\mathrm{I}((a*([a]\Phi))\cup(([a]\Phi)*a)),$$

then $[a]\Theta=[a]\Phi$.

Theorem 3.3. *A non-empty subset A of S is a class of some congruence on $\mathcal{S}$ if and only if $a\in A$, $b\in S$ and $a*b, b*a\in\mathrm{I}((a*A)\cup(A*a))$ together imply $b\in A$.*

Proof. Assume the condition of the theorem to hold. Let $c\in A$. Then there exists a congruence Ψ on $\mathcal{S}$ with $[1]\Psi=\mathrm{I}((c*A)\cup(A*c))$. If $d\in A$, then $c*d, d*c\in\mathrm{I}((c*A)\cup(A*c))=[1]\Psi$ and, hence, $d\in[c]\Psi$ according to Lemma 3.1. If, conversely, $e\in[c]\Psi$, then $c*e, e*c\in[1]\Psi=\mathrm{I}((c*A)\cup(A*c))$, because of Lemma 3.1, and, hence $e\in A$ according to the condition of the theorem. This shows $A=[c]\Psi$ and therefore A is a class of some congruence on $\mathcal{S}$. The rest of the proof follows from Lemma 3.2. ∎

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