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# THE SEMANTICAL HYPERUNIFICATION PROBLEM

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**Abstract**

A hypersubstitution of a fixed type $\tau$ maps $n$-ary operation symbols of the type to $n$-ary terms of the type. Such a mapping induces a unique mapping defined on the set of all terms of type $\tau$. The kernel of this induced mapping is called the kernel of the hypersubstitution, and it is a fully invariant congruence relation on the (absolutely free) term algebra $\mathcal{F}_\tau(X)$ of the considered type ([2]). If $V$ is a variety of type $\tau$, we consider the composition of the natural homomorphism with the mapping induced by a hypersubstitution. The kernel of this mapping is called the *semantical kernel* of the hypersubstitution with respect to the given variety. If the pair $(s,t)$ of terms belongs to the semantical kernel of a hypersubstitution, then this hypersubstitution equalizes $s$ and $t$ with respect to the variety. Generalizing the concept of a unifier, we define a semantical hyperunifier for a pair of terms with respect to a variety. The problem of finding a semantical hyperunifier with respect to a given variety for any two terms is then called the *semantical hyperunification problem.*

We prove that the semantical kernel of a hypersubstitution is a fully invariant congruence relation on the absolutely free algebra of

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the given type. Using this kernel, we define three relations between sets of hypersubstitutions and sets of varieties and introduce the Galois correspondences induced by these relations. Then we apply these general concepts to varieties of semigroups.

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## 1. Introduction

Let $\{f_i \mid i \in I\}$ be an indexed set of operation symbols of type $\tau = (n_i)_{i\in I}$, where $f_i$ is $n_i$-ary for $n_i \in \mathbb{N}\backslash\{0\}$, and let $W_\tau(X)$ be the set of all terms built up from elements of the alphabet $X = \{x_1, x_2, \ldots\}$ and operation symbols from $\{f_i \mid i \in I\}$. An arbitrary mapping

$$\sigma : \{f_i|i \in I\} \longrightarrow W_\tau(X)$$

which preserves the arity, that is, which maps every $n_i$-ary operation symbol of type $\tau$ to an $n_i$-ary term of the same type, is called a *hypersubstitution* of type $\tau$. Any hypersubstitution $\sigma$ induces a mapping

$$\hat{\sigma} : W_\tau(X) \to W_\tau(X)$$

in the following inductive way:

(i) $\hat{\sigma}[x_i] := x_i \in X$,

(ii) $\hat{\sigma}[f_i(t_1, \ldots, t_{n_i})] := \sigma(f_i)(\hat{\sigma}[t_1], \ldots, \hat{\sigma}[t_{n_i}])$.

The right hand side of (ii) is the superposition of the term $\sigma(f_i)$ with the terms $\hat{\sigma}[t_1], \ldots, \hat{\sigma}[t_{n_i}]$. This extension is uniquely determined and allows us to define a multiplication, denoted by $\circ_h$, on the set $Hyp(\tau)$ of all hypersubstitutions of type $\tau$, by

$$\sigma_1 \circ_h \sigma_2 = \hat{\sigma}_1 \circ \sigma_2,$$

where $\circ$ is the usual composition of functions. This multiplication is associative, and if we denote by $\sigma_{id}$ the identity hypersubstitution which maps each $n_i$-ary operation symbol $f_i$ to the term $f_i(x_1, \ldots, x_{n_i})$, we obtain a monoid $(Hyp(\tau); \circ_h, \sigma_{id})$.

Hypersubstitutions can be used to define the concept of a hyperidentity in a variety $V$ of algebras of type $\tau$. An equation $s \approx t$ consisting of terms of type $\tau$ forms a *hyperidentity* in $V$ if for all $\sigma \in Hyp(\tau)$ the equations $\hat{\sigma}[s] \approx \hat{\sigma}[t]$ are satisfied as identities in $V$. The identity $s \approx t$ is called an *S-hyperidentity of* $V$, for some subset $S \subseteq Hyp(\tau)$, if $\hat{\sigma}[s] \approx \hat{\sigma}[t]$ are identities of $V$ for all $\sigma \in S$. A variety $V$ is called *S-solid* if every identity in $V$ is an $S$-hyperidentity of $V$. In the special case that $S = Hyp(\tau)$, we speak of a *hyperidentity of the variety* $V$ and of a *solid variety.* For more information on hyperidentities and solidity, we refer the reader to [4].

If $\sigma$ is a hypersubstitution of type $\tau$, it is very natural to ask for its kernel,

$$ker\ \sigma := \{(t,t')|t,t' \in W_\tau(X) \text{ and } \hat{\sigma}[t] = \hat{\sigma}[t']\}.$$

By definition, the kernel of a hypersubstitution is an equivalence relation on the set $W_\tau(X)$. In fact this kernel turns out to be a fully invariant congruence.

**Proposition 1.1** ([4])**.** *The kernel of a hypersubstitution $\sigma$ of type $\tau$ is a fully invariant congruence relation on the absolutely free algebra $\mathcal{F}_\tau(X) = (W_\tau(X); (\overline{f}_i)_{i\in I})$ (where the operations $\overline{f}_i$ are defined by*

$$\overline{f}_i : W_\tau(X)^{n_i} \longrightarrow W_\tau(X) \ \textit{ with } \ \overline{f}_i(t_1,\ldots,t_{n_i}) := f_i(t_1,\ldots,t_{n_i})). \qquad \blacksquare$$

Let $V$ be an arbitrary variety of algebras of type $\tau$ and let $IdV$ be the set of all identities satisfied in $V$. Then we generalize the concept of a kernel of a hypersubstitution of type $\tau$ in the following way:

**Definition 1.2.** The set

$$ker_V\sigma := \{(t,t') \mid t,t' \in W_\tau(X) \text{ and } \hat{\sigma}[t] \approx \hat{\sigma}[t'] \in IdV\}$$

will be called the *kernel of $\sigma$ with respect to $V$* or the *semantical kernel* of $\sigma$. The kernel $ker\sigma$ of a hypersubstitution $\sigma$ will be called the *syntactical kernel.*

It is well known that the variety $Alg(\tau)$ of all algebras of type $\tau$ has the property that an identity $s \approx t$ holds in it iff $s = t$. This means that the syntactical kernel is in fact the semantical kernel with respect to the variety $Alg(\tau)$.

For the solution of the word problem in a variety $V$, the concept of a unifier is important. A substitution is any mapping $s : X \longrightarrow W_\tau(X)$, and any such substitution has a unique extension $\bar{s} : W_\tau(X) \longrightarrow W_\tau(X)$.

A substitution $s$ for which $\overline{s}(t) \approx \overline{s}(t')$ is an identity in $V$ is called a unifier for $t$ and $t'$ with respect to the variety $V$. If $V$ is the variety of all algebras of type $\tau$, this happens only if $\overline{s}(t) = \overline{s}(t')$, and in this case we call $s$ a syntactical unifier for $t$ and $t'$. To solve the unification problem (see [6]) means to decide whether for two given terms there exists a unifier or not. The concept of a unifier can be generalized to the concept of a hyperunifier and the unification problem to the hyperunification problem, by considering hypersubstitutions instead of substitutions.

**Definition 1.3.** Let $V$ be a variety of type $\tau$. A hypersubstitution $\sigma \in Hyp(\tau)$ is called a *hyperunifier with respect to $V$ for the terms $t, t' \in W_\tau(X)$* if $\hat{\sigma}[t] \approx \hat{\sigma}[t'] \in IdV$, that is if $(t, t') \in ker_V\sigma$. When such a hyperunifier exists, the terms $t$ and $t'$ are called *hyperunifiable* in the variety $V$. If $V$ is the variety $Alg(\tau)$ of all algebras of type $\tau$, the unifier is called a *syntactical hyperunifier*, otherwise it is called a *semantical hyperunifier*.

The *semantical hyperunification problem* for a variety $V$ is then the problem of deciding, for any two distinct terms, whether the terms are hyperunifiable in $V$ or not. Since by definition our $ker_V\sigma$ is the set of all pairs of terms for which $\sigma$ is a semantical hyperunifier with respect to the variety $V$, we can use such kernels in solving the semantical hyperunification problem.

When a hyperunifier exists for two terms, we want to compare all such hyperunifiers. In order to do this, we use the product $\circ_h$ to define a binary relation on $Hyp(\tau)$:

**Definition 1.4.** Let $\sigma_1$ and $\sigma_2$ be hypersubstitutions of type $\tau$. Then $\sigma_1 \preceq \sigma_2$ if there is a hypersubstitution $\lambda$ of type $\tau$ such that $\sigma_1 = \lambda \circ_h \sigma_2$.

Clearly, the relation $\preceq$ is reflexive (using the identity hypersubstitution $\sigma_{id}$) and transitive (since the product of two hypersubstitutions of type $\tau$ is again a hypersubstitution of type $\tau$). The intersection of a reflexive and transitive relation (that is, a quasiorder) with its inverse relation gives an equivalence relation on $Hyp(\tau)$, defined by

$$\sigma_1 \simeq \sigma_2 :\Longleftrightarrow \sigma_1 \preceq \sigma_2 \text{ and } \sigma_2 \preceq \sigma_1.$$

This relation is the well-known Green's relation $\mathcal{L}$, which is a right congruence on the monoid $Hyp(\tau)$ (see [3]).

Now we can define a relation $\leq$ on the quotient set $Hyp(\tau)/\simeq$ by setting

$$[\sigma_1]_\simeq \leq [\sigma_2]_\simeq :\Longleftrightarrow \sigma_1 \preceq \sigma_2.$$

It is well-known and easy to check that this definition gives an order relation on $Hyp(\tau)/\simeq$.

Using the quasiorder $\preceq$ on the set $Hyp(\tau)$ one can also generalize the concept of a most general unifier of the terms $t$ and $t'$ to the concept of a most general hyperunifier of $t$ and $t'$:

**Definition 1.5.** Let $t$ and $t'$ be two terms of type $\tau$ and let $\sigma_1$ and $\sigma_2$ be two hyperunifiers of $t$ and $t'$. Then $\sigma_1$ is more general than $\sigma_2$ if $\sigma_1 \preceq \sigma_2$. A hyperunifier $\sigma$ of $t$ and $t'$ is called a *most general* (or minimal) hyperunifier of $t$ and $t'$ if $\sigma \preceq \sigma'$ for all hyperunifiers $\sigma'$ of $t$ and $t'$.

## 2. The semantical kernel of a hypersubstitution

The syntactical kernel is the (semantical) kernel with respect to the variety $Alg(\tau)$ of all algebras of type $\tau$. The semantical and syntactical kernels are closely related to each other. Let $\mathcal{F}_\tau(X)$ be the absolutely free algebra of type $\tau$ and let $F_V(X)$ be the relatively free algebra with respect to the variety $V$ of type $\tau$. We denote by $natIdV$ the natural homomorphism

$$natIdV : \mathcal{F}_\tau(X) \longrightarrow \mathcal{F}_V(X)$$

which maps each term $t$ of type $\tau$ to the class $[t]_{IdV}$. Then we have:

**Proposition 2.1.** *Let $V$ be a variety and $\sigma$ a hypersubstitution, both of type $\tau$. Then*

$$ker_V\sigma = ker(natIdV \circ \hat{\sigma}).$$

***Proof.*** For any terms $t$ and $t'$, we have

$$(t,t') \in ker(natIdV \circ \hat{\sigma}) \Longleftrightarrow (natIdV \circ \hat{\sigma})(t) = (natIdV \circ \hat{\sigma})(t')$$

$$\Longleftrightarrow natIdV(\hat{\sigma}[t]) = natIdV(\hat{\sigma}[t']) \Longleftrightarrow [\hat{\sigma}[t]]_{IdV} = [\hat{\sigma}[t']]_{IdV}$$

$$\Longleftrightarrow \hat{\sigma}[t] \approx \hat{\sigma}[t'] \in IdV \Longleftrightarrow (t,t') \in ker_V\sigma.$$

■

**Proposition 2.2.** *For any type $\tau$, the semantical kernel of the hypersubstitution $\sigma$ with respect to the variety $V$ is a congruence relation on $\mathcal{F}_\tau(X)$.*

***Proof.*** Clearly, $ker_V\sigma$ is an equivalence relation on $\mathcal{F}_\tau(X)$. Assume that $f_i$ is an $n_i$-ary operation symbol of type $\tau$ and that $(t_j, t'_j) \in ker_V\sigma$ for $j = 1, \ldots, n_i$. Then $\hat{\sigma}[t_j] \approx \hat{\sigma}[t'_j] \in IdV$ for $j = 1, \ldots, n_i$. It follows from this (by induction on the complexity of terms) that $t(\hat{\sigma}[t_1], \ldots, \hat{\sigma}[t_{n_i}]) \approx t(\hat{\sigma}[t'_1], \ldots, \hat{\sigma}[t'_{n_i}]) \in IdV$ for any $n_i$-ary term $t$, and in particular that $\sigma(f_i)(\hat{\sigma}[t_1], \ldots, \hat{\sigma}[t_{n_i}]) \approx \sigma(f_i)(\hat{\sigma}[t'_1], \ldots, \hat{\sigma}[t'_{n_i}]) \in IdV$. This means that $\hat{\sigma}[f_i(t_1, \ldots, t_{n_i})] \approx \hat{\sigma}[f_i(t'_1, \ldots, t'_{n_i})] \in IdV$, and $(f_i(t_1, \ldots, t_{n_i}), f_i(t'_1, \ldots t'_{n_i})) \in ker_V\sigma$. ■

Since the composition $natIdV \circ \hat{\sigma}$ is not a homomorphism from $\mathcal{F}_\tau(X)$ to $\mathcal{F}_V(X)$, we could not use for the proof the fact that the kernel of a homomorphism is a congruence relation.

Moreover, the kernels of semantical hypersubstitutions are fully invariant congruence relations. To prove this, we will use the fact that any substitution $s : X \longrightarrow W_\tau(X)$ can be uniquely extended to an endomorphism $\bar{s} : \mathcal{F}_\tau(X) \longrightarrow \mathcal{F}_\tau(X)$, defined by

$$\bar{s}(f_i(t_1, \ldots, t_{n_i})) = f_i(\bar{s}(t_1), \ldots, \bar{s}(t_{n_i})).$$

Using induction on the complexity of the term $t = f(r_1, \ldots r_n)$, it can be shown that this last equation is valid for arbitrary terms as well as for the operation symbol $f_i$, so that

$$\bar{s}(t(t_1, \ldots, t_{n_i})) = t(\bar{s}(t_1), \ldots, \bar{s}(t_{n_i}))$$

for every term $t$. Here $t(t_1, \ldots, t_{n_i})$ means the composition (superposition) of terms.

**Theorem 2.3.** *Let $\sigma$ be a hypersubstitution of type $\tau = (n_i)_{i\in I}$, with $n_i \geq 1$ for all $i \in I$. Then $ker_V\sigma$ is a fully invariant congruence relation on the absolutely free algebra $\mathcal{F}_\tau(X)$.*

***Proof.*** By Proposition 2.2 we only have to show that $ker_V\sigma$ is fully invariant. Let $s : X \longrightarrow W_\tau(X)$ be a substitution and let $\bar{s}$ be its extension. Consider a mapping $s^* : X \longrightarrow W_\tau(X)$ defined by $s^*(x) := \hat{\sigma}[s(x)]$ for every $x \in X$. Since $s^*$ is also a substitution, it can be uniquely extended to an endomorphism $\bar{s^*} : \mathcal{F}_\tau(X) \longrightarrow \mathcal{F}_\tau(X)$.

We show by induction on the complexity of a term $t$ that

$$(*) \qquad \bar{s^*}(\hat{\sigma}[t]) = \hat{\sigma}[\bar{s}(t)]$$

for every $t \in W_\tau(X)$. First, if $t = x \in X$ is a variable then $\bar{s^*}(\hat{\sigma}[x]) = \bar{s^*}(x) = s^*(x) = \hat{\sigma}[s(x)] = \hat{\sigma}[\bar{s}(x)]$ by the definition of $s^*$. Now let $t = f_i(t_1, \ldots, t_{n_i})$ and suppose that $\bar{s^*}(\hat{\sigma}[t_j]) = \hat{\sigma}[\bar{s}(t_j)]$ for $j = 1, \ldots, n_i$. Then

$$\bar{s^*}(\hat{\sigma}[t]) = \bar{s^*}(\sigma(f_i)(\hat{\sigma}[t_1], \ldots, \hat{\sigma}[t_{n_i}])) = \sigma(f_i)(\bar{s^*}(\hat{\sigma}[t_1]), \ldots, \bar{s^*}(\hat{\sigma}[t_{n_i}]))$$

$$= \sigma(f_i)(\hat{\sigma}[\bar{s}(t_1)], \ldots, \hat{\sigma}[\bar{s}(t_{n_i})]) = \hat{\sigma}[\bar{s}(f_i(t_1, \ldots, t_{n_i}))] = \hat{\sigma}[\bar{s}(t)].$$

Now let $(t, t') \in ker_V\sigma$, and let $s$ be any substitution. Then $\hat{\sigma}[t] \approx \hat{\sigma}[t'] \in IdV$. Since $IdV$ is a fully invariant congruence relation on $\mathcal{F}_\tau(X)$, we have

$$\bar{s^*}(\hat{\sigma}[t]) \approx \bar{s^*}(\hat{\sigma}[t']) \in IdV,$$

and by $(*)$

$$\hat{\sigma}[\bar{s}(t)] \approx \hat{\sigma}[\bar{s}(t')] \in IdV.$$

This means that $(\bar{s}(t), \bar{s}(t')) \in ker_V\sigma$, and hence $ker_V\sigma$ is fully invariant. ■

Since $ker_V\sigma$ is fully invariant, it is an equational theory and therefore there is a variety $V'$ of type $\tau$ for which $ker_V\sigma = IdV'$. It is natural then to compare the varieties $V$ and $V'$, or dually the sets of identities $IdV$ and $ker_V\sigma$. We will consider the possibilities that $ker_V\sigma = IdV$, that $ker_V\sigma \subseteq IdV$ and that $ker_V\sigma \supseteq IdV$. These possibilities define three relations and three Galois correspondences which we will study in the next section.

## 3. Three Galois correspondences

Let $W$ be a given variety of type $\tau$. We will denote by $\mathcal{L}(W)$ the subvariety lattice of $W$. In this section we define and study three relations $KER$, $R$ and $R'$ between $Hyp(\tau)$ and $\mathcal{L}(W)$, based on the relationship between $ker_V\sigma$ and $IdV$. We also study the three Galois correspondences, between sets of hypersubstitutions and collections of varieties, induced by these three relations. Note that in order to compare kernels and sets of identities of the form $IdV$, we shall regard $IdV$ as a set consisting of pairs of terms of type $\tau$, by identifying the identity $p \approx q$ with the pair $(p, q) \in W_\tau(X)^2$.

**Definition 3.1.** Let $KER \subseteq Hyp(\tau) \times \mathcal{L}(W)$ be the relation defined by

$$(\sigma, V) \in KER :\ \Leftrightarrow\ ker_V\sigma = IdV.$$

In [1] the binary relation $R \subseteq Hyp(\tau) \times \mathcal{L}(W)$ defined by

$$(\sigma, V) \in R : \;\Leftrightarrow\; \sigma[V] \subseteq V$$

was considered. The set $\sigma[V]$ is defined as the set of all algebras which are derived from algebras from $V$, using the hypersubstitution $\sigma$. If $\mathcal{A} = (A; (f_i)_{i\in I})$ is an algebra, then the algebra $\sigma(\mathcal{A}) = (A; \sigma(f_i)_{i\in I})$ of the same type and having the same universe $A$ is called a *derived algebra* of $\mathcal{A}$. Hypersubstitutions satisfying $\sigma[V] \subseteq V$ are called *$V$-proper hypersubstitutions*. For more background see [4].

When $\sigma[V] \subseteq V$, then $Id\sigma[V] \supseteq IdV$, which means that if $s \approx t \in IdV$, then $s \approx t \in Id\ \sigma[V]$ and $\hat{\sigma}[s] \approx \hat{\sigma}[t] \in IdV$, using the so-called "conjugate property" (see [4]). But this means $IdV \subseteq ker_V\sigma$. Conversely, from $IdV \subseteq ker_V\sigma$, we get $\sigma[V] \subseteq V$. Altogether we see that the relation $R \subseteq Hyp(\tau) \times \mathcal{L}(W)$ can also be defined by

$$(\sigma, V) \in R : \;\Leftrightarrow\; IdV \subseteq ker_V\sigma.$$

We define a third relation by

**Definition 3.2.** Let $R' \subseteq Hyp(\tau) \times \mathcal{L}(W)$ be the relation defined by

$$(\sigma, V) \in R' : \;\Leftrightarrow\; ker_V\sigma \subseteq IdV.$$

This means that $(\sigma, V) \in R'$ if and only if whenever $\hat{\sigma}[s] \approx \hat{\sigma}[t] \in IdV$, it must be that $s \approx t \in IdV$.

Directly from the definitions, we obtain

$$KER = R \cap R'.$$

Each of these three relations defines a Galois correspondence between the sets $Hyp(\tau)$ and $\mathcal{L}(W)$. Let $S$ be a subset of $Hyp(\tau)$ and let $L$ be a subset of $\mathcal{L}(W)$. In [1], the Galois correspondence induced by the relation $R$ was defined as the pair $(\eta, \theta)$ with

$$\eta : \mathcal{P}(Hyp(\tau)) \longrightarrow \mathcal{P}(\mathcal{L}(W)), \quad \theta : \mathcal{P}(\mathcal{L}(W)) \longrightarrow \mathcal{P}(Hyp(\tau)),$$

defined by

$$\eta(S) := \{V \in \mathcal{L}(W) \mid \forall \sigma \in S((\sigma, V) \in R)\} \quad \text{and}$$

$$\theta(L) := \{\sigma \in Hyp(\tau) \mid \forall V \in L((\sigma, V) \in R)\}.$$

Now we define two more Galois correspondences $(\alpha, \beta)$ and $(\gamma, \delta)$, for the other two relations. We have

$$\alpha : \mathcal{P}(Hyp(\tau)) \longrightarrow \mathcal{P}(\mathcal{L}(W)), \quad \beta : \mathcal{P}(\mathcal{L}(W)) \longrightarrow \mathcal{P}(Hyp(\tau)), \quad \text{with}$$

$$\alpha(S) := \{V \in \mathcal{L}(W) \mid \forall \sigma \in S((\sigma, V) \in R')\} \quad \text{and}$$
$$\beta(L) := \{\sigma \in Hyp(\tau) \mid \forall V \in L((\sigma, V) \in R')\};$$

and

$$\gamma : \mathcal{P}(Hyp(\tau)) \longrightarrow \mathcal{P}(\mathcal{L}(W)), \quad \delta : \mathcal{P}(\mathcal{L}(W)) \longrightarrow \mathcal{P}(Hyp(\tau)), \quad \text{with}$$

$$\gamma(S) := \{V \in \mathcal{L}(W) \mid \forall \sigma \in S((\sigma, V) \in KER)\} \quad \text{and}$$
$$\delta(L) := \{\sigma \in Hyp(\tau) \mid \forall V \in L((\sigma, V) \in KER)\}.$$

The following result was proved in [1].

**Proposition 3.3** ([1])**.** *For any subset $L$ of $\mathcal{L}(W)$ and for any subset $S$ of $Hyp(\tau)$, the image $\theta(L)$ is a submonoid of $Hyp(\tau)$ and the image $\eta(S)$ is a sublattice of $\mathcal{L}(W)$.* ■

It is easy to see that the lattice $\eta(S)$ is in fact a complete sublattice of $\mathcal{L}(W)$. We want to check now whether images under $\beta$ and $\delta$ are also submonoids of $Hyp(\tau)$, and whether images under $\alpha$ and $\gamma$ are also sublattices of $\mathcal{L}(W)$. The following consequence of the fact that $KER = R \cap R'$ will be useful.

**Proposition 3.4.** *For every subset $S \subseteq Hyp(\tau)$ and every subset $L \subseteq \mathcal{L}(W)$, the operators $\gamma$, $\eta$, $\alpha$, $\delta$, $\theta$ and $\beta$ satisfy*

$$\gamma(S) = \eta(S) \cap \alpha(S) \quad \text{and} \quad \delta(L) = \theta(L) \cap \beta(L). \quad ■$$

For a singleton set $\{V\}$ consisting of one variety, the monoid $\theta(\{V\})$ is just the monoid of $V$-proper hypersubstitutions, usually denoted by $P(V)$ (see [5]). Płonka also defined another monoid $P_0(V)$ associated with $V$, the monoid of all inner hypersubstitutions. We will show that for single varieties our other two images, $\delta(\{V\})$ and $\beta(\{V\})$, are also monoids, and in analogy with $P(V)$ we will call them $P_1(V)$ and $P_2(V)$, respectively.

**Proposition 3.5.** *Let $W$ be a variety of type $\tau$ and let $V \in \mathcal{L}(W)$. Then $\beta(\{V\})$ and $\delta(\{V\})$ are submonoids of $Hyp(\tau)$.*

***Proof.*** From Proposition 3.4, we get $P_1(V) = P(V) \cap P_2(V)$. Since $P(V)$ is a submonoid of $Hyp(\tau)$ by Proposition 3.3, we only have to show that $P_2(V)$ is a submonoid of $Hyp(\tau)$. Let $\sigma_1$ and $\sigma_2$ be in $P_2(V)$, so that $(\sigma_1, V), (\sigma_2, V) \in R'$. Then $ker_V\sigma_1 \subseteq IdV$ and $ker_V\sigma_2 \subseteq IdV$. Therefore for every pair $(s, t)$ of terms of type $\tau$ we have:

$\hat{\sigma}_1[s] \approx \hat{\sigma}_1[t] \in IdV \Longrightarrow s \approx t \in IdV$ and $\hat{\sigma}_2[s] \approx \hat{\sigma}_2[t] \in IdV \Longrightarrow s \approx t \in IdV.$

Then

$$\hat{\sigma}_1[\hat{\sigma}_2[s]] \approx \hat{\sigma}_1[\hat{\sigma}_2[t]] \in IdV \Longrightarrow \hat{\sigma}_2[s] \approx \hat{\sigma}_2[t] \in IdV \Longrightarrow s \approx t \in IdV$$

and thus $\sigma_1 \circ_h \sigma_2 \in P_2(V)$. It is clear that $\sigma_{id} \in P_2(V)$. ∎

Now our claim holds for arbitrary subsets $L$ of $\mathcal{L}(W)$, since $\beta(L)$ is the intersection of $\beta(\{V\})$ for all $V \in L$, and similarly for $\delta$. This shows that for all three maps, any images under the maps are submonoids of $Hyp(\tau)$. Now we turn to the dual maps $\alpha$ and $\gamma$, to see if their images are always sublattices. The following lemma will help us to answer this question.

**Lemma 3.6.** *Let $\sigma$ be a hypersubstitution of type $\tau$ and let $V_1, V_2$ be varieties of type $\tau$. Then*

$$ker_{(V_1 \vee V_2)}\sigma = ker_{V_1}\sigma \cap ker_{V_2}\sigma.$$

***Proof.*** Using the definition and the fact that $Id(V_1 \vee V_2) = IdV_1 \cap IdV_2$, we see that

$$\begin{aligned} ker_{(V_1 \vee V_2)}\sigma &= \{(s,t) \mid \hat{\sigma}[s] \approx \hat{\sigma}[t] \in Id(V_1 \vee V_2)\} \\ &= \{(s,t) \mid \hat{\sigma}[s] \approx \hat{\sigma}[t] \in IdV_1\} \ \cap \ \{(s,t) \mid \hat{\sigma}[s] \approx \hat{\sigma}[t] \in IdV_2\} \\ &= ker_{V_1}\sigma \cap ker_{V_2}\sigma. \end{aligned}$$

∎

Let $Con_{inv}\mathcal{F}_\tau(X)$ be the lattice of all fully invariant congruence relations on the absolutely free algebra $\mathcal{F}_\tau(X)$ of type $\tau$. Since the kernels of hypersubstitutions are elements of the lattice $Con_{inv}\mathcal{F}_\tau(X)$, we may consider a mapping

$$\varphi : \mathcal{L}(Alg(\tau)) \times Hyp(\tau) \longrightarrow Con_{inv}\mathcal{F}_\tau(X),$$

which associates to each hypersubstitution $\sigma$ and to each variety $V$ of the type $\tau$ the kernel $ker_V\sigma$. If we fix the hypersubstitution $\sigma$, the resulting mapping on $\mathcal{L}(Alg(\tau))$ is anti-isotone.

**Corollary 3.7.** *If $V_1 \subseteq V_2$ are varieties of type $\tau$ and if $\sigma$ is a hypersubstitution of the same type, then $ker_{V_2}\sigma \subseteq ker_{V_1}\sigma$.*

***Proof.*** $V_1 \subseteq V_2$ gives $V_1 \vee V_2 = V_2$, so by the previous Lemma we get $ker_{V_2}\sigma = ker_{(V_1 \vee V_2)}\sigma = ker_{V_1}\sigma \cap ker_{V_2}\sigma$, and then $ker_{V_2}\sigma \subseteq ker_{V_1}\sigma$. ∎

Let $TR$ be the trivial variety of type $\tau$. It is easy to see that for any hypersubstitution $\sigma$, we have $ker_{TR}\sigma$ equal to the set $W_\tau(X)^2$ of all identities of type $\tau$, and equal to $IdTR$. Since $TR \subseteq V \subseteq Alg(\tau)$ for any variety $V$ of type $\tau$, we have $ker_{Alg(\tau)}\sigma = ker\sigma \subseteq ker_V\sigma \subseteq ker_{TR}\sigma$. This means that the syntactical kernel of a hypersubstitution is always a subset of any semantical kernel of that hypersubstitution.

The mapping $\varphi$ defined above is also surjective: for any fully invariant congruence relation $\Sigma$ of $\mathcal{F}_\tau(X)$, there is a variety $V$ such that $\Sigma = IdV$ and a hypersubstitution, namely $\sigma_{id}$, for which $IdV = ker_V\sigma_{id}$.

Now we can prove the following.

**Theorem 3.8.** *Let $W$ be a variety of type $\tau$. Then for any subset $S$ of $Hyp(\tau)$, the images $\alpha(S)$ and $\gamma(S)$ are complete join-subsemilattices of $\mathcal{L}(W)$.*

***Proof.*** Let $V_1, V_2 \in \alpha(S)$. Then for all $\sigma \in S$ we have $ker_{V_1}\sigma \subseteq IdV_1$ and $ker_{V_2}\sigma \subseteq IdV_2$, so that $ker_{V_1}\sigma \cap ker_{V_2}\sigma \subseteq IdV_1 \cap IdV_2 = Id(V_1 \vee V_2)$. The same argument extends to any family of varieties in $\mathcal{L}(W)$, showing that $\alpha(S)$ is closed under arbitrary joins of varieties in $\mathcal{L}(W)$. The same proof, but with set inclusion replaced by equality, holds for $\gamma(S)$. (But the result for $\gamma(S)$ is also a consequence of the result for $\alpha$ and Proposition 3.4.) ∎

We now present a counterexample showing that $\gamma(S)$ (and hence by Proposition 3.4 also $\alpha(S)$) is not in general a sublattice of $\mathcal{L}(W)$. We consider the type $\tau = (2)$, the two-element alphabet $X_2 = \{x_1, x_2\}$ and the hypersubstitution $\sigma$ which maps the binary operation symbol $f$ to the term $f(x_1, f(x_1, x_2))$. Instead of $f(x_1, x_2)$ we will write $x_1x_2$. We denote by $\langle\sigma\rangle$ the submonoid of $Hyp(2)$ generated by $\sigma$. If $\Sigma$ is a set of equations of type (2), we set $\chi_{\langle\sigma\rangle}[\Sigma]$ to be the set $\{\hat{\sigma}'[s] \approx \hat{\sigma}'[t] \mid s \approx t \in \Sigma \text{ and } \sigma' \in \langle\sigma\rangle\}$. Clearly, classes of the form $Mod\chi_{\langle\sigma\rangle}[\Sigma]$ are $\langle\sigma\rangle$-solid varieties.

We define two varieties $U_1$ and $U_2$ of type (2), by

$U_1 := Mod\chi_{\langle\sigma\rangle}[\{x_1(x_1x_2) \approx ((x_1x_1)x_1)x_2\}]$,

$U_2 := Mod\chi_{\langle\sigma\rangle}[\{x_1(x_1(x_1(x_1x_2))) \approx ((x_1x_1)x_1)x_2\}]$.

We need the following preliminary Lemma.

**Lemma 3.9.** *With $\sigma$, $U_1$ and $U_2$ as defined above,*

$$ker_{U_1}\sigma = IdU_1 \quad and \quad ker_{U_2}\sigma = IdU_2.$$

***Proof.*** Since $U_1$ and $U_2$ are $\langle\sigma\rangle$-solid, we have only to show that $ker_{U_i}\sigma \subseteq IdU_i$ for $i = 1, 2$. We will do this only for $U_1$, since the proof for $U_2$ is similar. We make the following observations regarding identities of $U_1$:

1 Every nontrivial identity $\hat{\sigma}[s] \approx \hat{\sigma}[t]$ in $U_1$ can be derived from the set $\chi_{\langle\sigma\rangle\setminus\{\sigma_{id}\}}[\{x_1(x_1x_2) \approx ((x_1x_1)x_1)x_2\}]$, since no subterm of $\hat{\sigma}[s]$ or of $\hat{\sigma}[t]$ has the form $((x_1x_1)x_1)x_2$. We set $U' := Mod\chi_{\langle\sigma\rangle\setminus\{\sigma_{id}\}}[\{x_1(x_1x_2) \approx ((x_1x_1)x_1)x_2\}]$.

2 Let W(X) denote the set of all terms of type (2), then let $\sigma[W(X)]$ be the set of all $\hat{\sigma}[w]$ with $w \in W(X)$. We show by induction on the complexity of the term $w$ that for all $\psi : X \longrightarrow W(X)$ and all $w \in W(X)$, and all $x_1 \in X$,

$$(\bar{\psi}(x_1) \notin \sigma[W(X)] \Longrightarrow \bar{\psi}(\hat{\sigma}[w]) \notin \sigma[W(X)]).$$

Here $\bar{\psi}$ is the unique extension of $\psi$ to the set $W(X)$. First, if $w = x_1 \in X$, then $\bar{\psi}(\hat{\sigma}[w]) = \bar{\psi}(x_1) \notin \sigma[W(X)]$.

Inductively, assume that $w = f(w_1, w_2)$ and that $w_1, w_2$ satisfy the implication. Then $\bar{\psi}(\hat{\sigma}[f(w_1, w_2)]) = \bar{\psi}(f(\hat{\sigma}[w_1], f(\hat{\sigma}[w_1], \hat{\sigma}[w_2]))) = f(\bar{\psi}(\hat{\sigma}[w_1]), f(\bar{\psi}(\hat{\sigma}[w_1]), \bar{\psi}(\hat{\sigma}[w_2])))$, where $\bar{\psi}(\hat{\sigma}[w_1]) \notin \sigma[W(X)]$ or $\bar{\psi}(\hat{\sigma}[w_2]) \notin \sigma[W(X)]$. This shows that $\bar{\psi}(\hat{\sigma}[f(w_1, w_2)]) \notin \sigma[W(X)]$.

3 Because of 2, we can modify the substitution rule in our case to

$$s \approx t \in IdU_1 \text{ and } \psi : X \longrightarrow \sigma[W(X)] \Longrightarrow \bar{\psi}(s) \approx \bar{\psi}(t) \in IdU_1.$$

This means we do not have to consider substitutions which map variables to terms outside of $\sigma[W(X)]$.

4 Since each identity $\hat{\sigma}[s] \approx \hat{\sigma}[t]$ in $U_1$ has the form $f(s_1, f(s_1, s_2)) \approx f(t_1, f(t_1, t_2))$ with $s_1, s_2, t_1, t_2 \in \sigma[W(X)]$, the compatibility rule can be modified as follows:

$$s_1 \approx t_1 \in IdU_1 \text{ and } s_2 \approx t_2 \in IdU_1 \implies$$
$$f(s_1, f(s_1, s_2)) \approx f(t_1, f(t_1, t_2)) \in IdU_1.$$

5 By 1, we know that $\hat{\sigma}[s] \approx \hat{\sigma}[t] \in IdU'$ implies $\hat{\sigma}[s] \approx \hat{\sigma}[t] \in IdU_1$. Now we show by induction on the length of a derivation that the following propositions are satisfied:

$$IdU' \subseteq \{s \approx t \mid s, t \in \sigma[W(X)]\}$$

and

$$\hat{\sigma}[s] \approx \hat{\sigma}[t] \in IdU' \Longrightarrow s \approx t \in IdU_1.$$

Clearly, $\chi_{\langle\sigma\rangle\setminus\{\sigma_{id}\}}[\{x_1(x_1x_2) \approx ((x_1x_1)x_1)x_2\}] \subseteq \{s \approx t \mid s, t \in \sigma[W(X)]\}$ and if $\hat{\sigma}[s] \approx \hat{\sigma}[t] \in \chi_{\langle\sigma\rangle\setminus\{\sigma_{id}\}}[\{x_1(x_1x_2) \approx ((x_1x_1)x_1)x_2\}]$, then $s \approx t \in IdU_1$ by definition.

We check each of the five derivation rules in turn.

Reflexivity: We have $\hat{\sigma}[s] \approx \hat{\sigma}[s] \in IdU'$, where $s \approx s \in IdU_1$ for all $s \in W(X)$.

Symmetry: If $\hat{\sigma}[s] \approx \hat{\sigma}[t] \in IdU'$ with $s \approx t \in IdU_1$, then $\hat{\sigma}[t] \approx \hat{\sigma}[s] \in IdU'$, where $t \approx s \in IdU_1$ follows from $s \approx t \in IdU_1$.

Transitivity: If $\hat{\sigma}[r] \approx \hat{\sigma}[s] \in IdU'$ and $\hat{\sigma}[s] \approx \hat{\sigma}[t] \in IdU'$ with $r \approx s \in IdU_1$ and $s \approx t \in IdU_1$, then $\hat{\sigma}[r] \approx \hat{\sigma}[t] \in IdU'$, where $r \approx s \in IdU_1$ and $s \approx t \in IdU_1$ implies $r \approx t \in IdU_1$.

Substitution rule: Here we use 2. Assume that $\hat{\sigma}[s] \approx \hat{\sigma}[t] \in IdU'$ with $s \approx t \in IdU_1$ and consider two substitutions $\varphi : X \longrightarrow \sigma[W(X)]$ and $\varphi^* : X \longrightarrow W(X)$ with $\varphi(x_1) = \hat{\sigma}[\varphi^*(x_1)]$, for $x_1 \in X$. We show by induction on the complexity of the term $w \in W(X)$ that

$$\bar{\varphi}(\hat{\sigma}[w]) = \hat{\sigma}[\bar{\varphi^*}(w)].$$

First, if $w = x_1$, then $\bar{\varphi}(\hat{\sigma}[x_1]) = \varphi(x_1) = \hat{\sigma}[\bar{\varphi^*}(x_1)]$. Here $\bar{\varphi}$ and $\bar{\varphi^*}$ are the unique extensions of $\varphi$ and $\varphi^*$, resp., to the set $W(X)$.
Inductively, suppose that $w = f(w_1, w_2)$ and that $\bar{\varphi}(\hat{\sigma}[w_i]) = \hat{\sigma}[\bar{\varphi^*}(w_i)]$, for $i = 1, 2$. Then

$\bar{\varphi}(\hat{\sigma}[w]) = \bar{\varphi}(\sigma(f)(\hat{\sigma}[w_1], \hat{\sigma}[w_2])) = \bar{\varphi}(f(\hat{\sigma}[w_1], f(\hat{\sigma}[w_1], \hat{\sigma}[w_2])))$

$= f(\bar{\varphi}(\hat{\sigma}[w_1]), \bar{\varphi}(f(\hat{\sigma}[w_1]), \hat{\sigma}[w_2]))) = f(\bar{\varphi}(\hat{\sigma}[w_1]), f(\bar{\varphi}(\hat{\sigma}[w_1]), \bar{\varphi}(\hat{\sigma}[w_2])))$

$= f(\hat{\sigma}[\bar{\varphi^*}(w_1)], f(\hat{\sigma}[\bar{\varphi^*}(w_1)], \hat{\sigma}[\bar{\varphi^*}(w_2)]))) = \hat{\sigma}[f(\bar{\varphi^*}(w_1), \bar{\varphi^*}(w_2))]$

$= \hat{\sigma}[\bar{\varphi^*}(f(w_1, w_2))] = \hat{\sigma}[\bar{\varphi^*}(w)]$.

Now we have $\bar{\varphi}(\hat{\sigma}[s]) \approx \bar{\varphi}(\hat{\sigma}[t]) \in IdU'$ with $\bar{\varphi}(\hat{\sigma}[s]) = \hat{\sigma}[\bar{\varphi^*}(s)]$ and $\bar{\varphi}(\hat{\sigma}[t]) = \hat{\sigma}[\bar{\varphi^*}(t)]$, where $\bar{\varphi^*}(s) \approx \bar{\varphi^*}(t) \in IdU_1$ because of $s \approx t \in IdU_1$.

Compatibility: We apply 4. If $\hat{\sigma}[s_1] \approx \hat{\sigma}[t_1] \in IdU'$ and

$\hat{\sigma}[s_2] \approx \hat{\sigma}[t_2] \in IdU'$ with $s_1 \approx t_1 \in IdU_1$ and $s_2 \approx t_2 \in IdU_1$, then

$f(\hat{\sigma}[s_1], f(\hat{\sigma}[s_1], \hat{\sigma}[s_2])) \approx f(\hat{\sigma}[t_1], f(\hat{\sigma}[t_1], \hat{\sigma}[t_2])) \in IdU'$ with

$f(\hat{\sigma}[s_1], f(\hat{\sigma}[s_1], \hat{\sigma}[s_2])) = \hat{\sigma}[f(s_1, s_2)]$ and $f(\hat{\sigma}[t_1], f(\hat{\sigma}[t_1], \hat{\sigma}[t_2])) =$

$\hat{\sigma}[f(t_1, t_2)]$, where $f(s_1, s_2) \approx f(t_1, t_2) \in IdU_1$ since $s_1 \approx t_1 \in IdU_1$

and $s_2 \approx t_2 \in IdU_1$.

Altogether we have the following result: If $s \approx t \in ker_{U_1}\sigma$, then $\hat{\sigma}[s] \approx \hat{\sigma}[t] \in IdU_1$. By 1, we have $\hat{\sigma}[s] \approx \hat{\sigma}[t] \in IdU'$. Because of 3 and 4, we have $\hat{\sigma}[s] \approx \hat{\sigma}[t] \in IdU_1 \cap \{s \approx t \mid s, t \in \sigma[W(X)]\}$ by the derivation rules which are considered in 5. Consequently 5 shows that $s \approx t \in IdU_1$. Altogether we have $ker_{U_1}\sigma \subseteq IdU_1$. ∎

**Lemma 3.10.** *For the hypersubstitution $\sigma$ of the example above, and type $(2)$, the set $\gamma(\{\sigma\})$ does not form a lattice.*

***Proof.*** The previous lemma shows that $ker_{U_i}\sigma = IdU_i$ for $i = 1, 2$. We consider the variety $U_1 \wedge U_2 = Mod(IdU_1 \cup IdU_2)$. We denote by $\ell(s)$ the length of a term $s$, i.e. the number of occurrences of variables in $s$. It is easy to check that for any nontrivial identity $s \approx t \in IdU_1 \cup IdU_2$ we get $\ell(s) \geq 3$ and $\ell(t) \geq 3$. Then for each identity $s \approx t \in IdMod(IdU_1 \cup IdU_2)$ we have also $\ell(s) \geq 3$ and $\ell(t) \geq 3$. This shows that $x_1x_2 \approx x_1(x_1x_2) \notin Id(U_1 \wedge U_2)$. From $x_1(x_1x_2) \approx ((x_1x_1)x_1)x_2 \in IdU_1$ and $x_1(x_1(x_1(x_1x_2))) \approx ((x_1x_1)x_1)x_2 \in IdU_2$ it follows that $x_1(x_1x_2) \approx x_1(x_1(x_1(x_1x_2))) \in Id(U_1 \wedge U_2)$. Also $\hat{\sigma}[x_1x_2] = x_1(x_1x_2)$ and $\hat{\sigma}[x_1(x_1x_2)] = x_1(x_1(x_1(x_1x_2)))$.

This shows that $\hat{\sigma}[x_1x_2] \approx \hat{\sigma}[x_1(x_1x_2)] \in Id(U_1 \wedge U_2)$, so that $x_1x_2 \approx x_1(x_1x_2) \in ker_{(U_1\wedge U_2)}\sigma$. Altogether we have $ker_{(U_1\wedge U_2)}\sigma \neq Id(U_1 \wedge U_2)$, and $U_1 \wedge U_2 \notin \gamma(\{\sigma\})$. ■

Now we want to calculate the images $\alpha(S), \eta(S)$, and $\gamma(S)$ when $S$ is the largest or smallest possible monoid of hypersubstitutions. We will take $W$ to be the largest variety $Alg(\tau)$. It is easy to see that for every variety $V$ of type $\tau$ we have $ker_V\sigma_{id} = IdV$. This means that the image of the smallest submonoid $S = \{\sigma_{id}\}$ under all three maps $\alpha$, $\eta$ and $\gamma$ is all of $Alg(\tau)$. The next two lemmas investigate the images of the largest monoid $S = Hyp(\tau)$.

**Lemma 3.11.** *Let $\tau = (n_i)_{i\in I}$ be a type with $n_i \geq 2$ for some $i \in I$. Then $\alpha(Hyp(\tau)) = \{TR\}$. (Here $TR$ is the trivial variety of type $\tau$.)*

***Proof.*** Since the trivial variety $TR$ satisfies all possible identities of type $\tau$, we have $ker_{TR}\sigma = IdTR$ for all hypersubstitutions $\sigma$ in $Hyp(\tau)$. This gives $TR \in \alpha(Hyp(\tau))$, i.e. $\{TR\} \subseteq \alpha(Hyp(\tau))$.

Conversely, let $V$ be a variety of type $\tau$ with $ker_V\sigma \subseteq IdV$ for all $\sigma \in Hyp(\tau)$. Let $f_j$ be an operation symbol of type $\tau$ with arity $n_j \geq 2$. We consider the hypersubstitutions $\sigma_1$ and $\sigma_2 \in Hyp(\tau)$ with $\sigma_1(f_j) = x_1$ and $\sigma_2(f_j) = x_2$. Then we have $(f_j(x_1, x_2, x_1, \ldots, x_1), x_1) \in ker_V\sigma_1$ and $(f_j(x_1, x_2, x_1, \ldots, x_1), x_2) \in ker_V\sigma_2$. But by assumption, we have $ker_V\sigma_1$ and $ker_V\sigma_2$ both subsets of $IdV$, so now we have $x_1 \approx f_j(x_1, x_2, x_1, \ldots, x_1) \approx x_2$ holding in $V$. This forces $V = TR$, and we have $\alpha(Hyp(\tau)) \subseteq \{TR\}$. Altogether $\alpha(Hyp(\tau)) = \{TR\}$. ■

Next we consider the remaining case, where our type contains only unary operation symbols.

**Lemma 3.12.** *Let $\tau = (n_i)_{i\in I}$ with $n_i = 1$ for all $i \in I$. Then $\alpha(Hyp(\tau)) = \{TR, B\}$, where $B = Mod\{f_i(x_1) \approx x_1 \mid i \in I\}$.*

***Proof.*** As before, we have $TR \in \alpha(Hyp(\tau))$, but now we show that also $B \in \alpha(Hyp(\tau))$. It is easy to see that any term over $B$ contains exactly one variable, and for two terms $s$ and $t$ we have $s \approx t \in IdB$ iff the variable in $s$ is the same as the variable in $t$. It follows from this that for any hypersubstitution $\sigma$, $\hat{\sigma}[s] \approx \hat{\sigma}[t]$ is in $IdB$ iff $s \approx t$ is in $IdB$. This makes $ker_B\sigma = IdB$ for all $\sigma$, so $B$ is in $\alpha(Hyp(\tau))$.

We now have $\{TR, B\} \subseteq \alpha(Hyp(\tau))$. For the opposite inclusion, we let $V$ be a variety of type $\tau$ for which $ker_V\sigma \subseteq IdV$ for all $\sigma \in Hyp(\tau)$. Let $\sigma$ be a hypersubstitution with $\sigma(f_i) = x_1$ for all $i \in I$. Then $f_i(x_1) \approx x_1 \in ker_V\sigma$ for all $i \in I$, so by our assumption on $V$ we have $f_i(x_1) \approx x_1 \in IdV$ for all $i \in I$. This shows that $V \subseteq B$. It is well-known that $B$ has no subvarieties other than $B$ and $TR$, so we have $V \in \{TR, B\}$. Thus $\alpha(Hyp(\tau)) \subseteq \{TR, B\}$, and altogether we have $\alpha(Hyp(\tau)) = \{TR, B\}$. ■

$V \in \eta(Hyp(\tau))$ means that $IdV \subseteq ker_V\sigma$ for all $\sigma \in Hyp(\tau)$, that is, that for any hypersubstitution $\sigma$ and any $s \approx t \in IdV$, the identity $\hat{\sigma}[s] \approx \hat{\sigma}[t]$ also holds in $V$. This says precisely that $V$ is solid, so that the image under $\eta$ of the monoid $Hyp(\tau)$ is the lattice $\mathcal{S}(\tau)$ of all solid varieties of type $\tau$.

Since $\gamma$ is the intersection of $\eta$ and $\alpha$, we can also determine the image $\gamma(Hyp(\tau))$. Since the trivial variety $TR$ and the variety $B$ considered in Lemma 3.12 are solid, we see that $\gamma(Hyp(\tau))$ is either $\{TR\}$, if there is an operation symbol $f_j$ with $n_j \geq 2$, or $\{TR, B\}$ otherwise.

Now for arbitrary subsets $S$ of $Hyp(\tau)$ we have either

$$\{TR, B\} = \gamma(Hyp(\tau)) \subseteq \gamma(S) \subseteq \gamma(\{\sigma_{id}\}) = Alg(\tau)$$

or

$$\{TR\} = \gamma(Hyp(\tau)) \subseteq \gamma(S) \subseteq \gamma(\{\sigma_{id}\}) = Alg(\tau),$$

depending on the type $\tau$, as above; and similarly for $\alpha$, either

$$\{TR, B\} = \alpha(Hyp(\tau)) \subseteq \alpha(S) \subseteq \alpha(\{\sigma_{id}\}) = Alg(\tau)$$

or

$$\{TR\} = \alpha(Hyp(\tau)) \subseteq \alpha(S) \subseteq \alpha(\{\sigma_{id}\}) = Alg(\tau).$$

For the operator $\eta$ we have

$$\mathcal{S}(\tau) = \eta(Hyp(\tau)) \subseteq \eta(S) \subseteq \eta(\{\sigma_{id}\}) = Alg(\tau).$$

## 4. Relations on sets of hypersubstitutions

In this section we show that to calculate the images of the operators introduced in the previous section we can restrict our efforts to certain "special" hypersubstitutions. This is also the case if we want to test whether an identity $s \approx t$ is a hyperidentity in the variety $V$. Then we can restrict our checking to a subset of the given set of hypersubstitutions. In [5] J. Płonka introduced the following equivalence relation on $Hyp(\tau)$: Let $V$ be a variety of type $\tau$. Two hypersubstitutions $\sigma_1$ and $\sigma_2$ of type $\tau$ are called $V$-equivalent, and we write $\sigma_1 \sim_V \sigma_2$, if for all operation symbols $f_i$ of the type the identities $\sigma_1(f_i) \approx \sigma_2(f_i)$ are satisfied in $V$.

The following lemma shows how the relation $\sim_V$ can be used.

**Lemma 4.1** ([5]). *Let $V$ be a variety of type $\tau$, and let $\sigma_1, \sigma_2 \in Hyp(\tau)$. Then the following two conditions* (i) *and* (ii) *are equivalent:*

(i) $\sigma_1 \sim_V \sigma_2$.

(ii) *For all $t \in W_\tau(X)$ the equation $\hat{\sigma}_1[t] \approx \hat{\sigma}_2[t]$ is an identity in $V$.*

*Moreover,*

(iii) *For all $s, t \in W_\tau(X)$, and for all $\sigma_1, \sigma_2 \in Hyp(\tau)$, if $\sigma_1 \sim_V \sigma_2$, then $\hat{\sigma}_1[s] \approx \hat{\sigma}_1[t] \in IdV$ iff $\hat{\sigma}_2[s] \approx \hat{\sigma}_2[t] \in IdV$.* ■

From the last condition it follows that the monoid $P(V)$ of all proper hypersubstitutions of the variety $V$ is the union of equivalence classes with respect to $\sim_V$. This means that to test if a given identity $s \approx t$ is a hyperidentity of $V$, we need only check the application of $\hat{\sigma}$ for one representative $\sigma$ from each $\sim_V$ equivalence class.

It is easy to see that $V$-equivalent hypersubstitutions induce equal kernels with respect to $V$.

**Proposition 4.2.** *Let $V$ be a variety of type $\tau$ and let $\sigma_1, \sigma_2 \in Hyp(\tau)$ with $\sigma_1 \sim_V \sigma_2$. Then*

(i) $ker_V\sigma_1 = ker_V\sigma_2$, *and*

(ii) *$P(V), P_1(V)$, and $P_2(V)$ are unions of $\sim_V$-classes.*

***Proof.*** (i): By Lemma 4.1 (iii) we have $(s,t) \in ker_V\sigma_1$ iff $(s,t) \in ker_V\sigma_2$, so the two kernels are equal.

(ii): For $P(V)$ this is clear (see for instance [4]). Let $\sigma_1 \in P_2(V)$ and $\sigma_1 \sim_V \sigma_2$. Then $ker_V\sigma_1 \subseteq IdV$ and $ker_V\sigma_1 = ker_V\sigma_2$ imply that also $ker_V\sigma_2 \subseteq IdV$, making $\sigma_2 \in P_2(V)$. The proof for $P_1(V)$ is similar. ■

Although $V$-equivalent hypersubstitutions produce the same kernels with respect to $V$, the converse is not always true: it is possible for $ker_V\sigma_1 = ker_V\sigma_2$ when $\sigma_1$ and $\sigma_2$ are not $V$-equivalent. As an example we consider the semigroup variety $V = Mod\{x_1(x_2x_3) \approx (x_1x_2)x_3,\ x_1x_2x_3 \approx x_1x_3\}$ and the hypersubstitutions $\sigma_{x_1}$ and $\sigma_{x_1^2}$ which map the binary operation symbol to the terms $x_1$ and $x_1^2$, respectively. Then $\sigma_{x_1} \not\sim_V \sigma_{x_1^2}$ since the idempotent identity $x_1^2 \approx x_1$ is not satisfied in $V$; but

$$\hat{\sigma}_{x_1}[s] \approx \hat{\sigma}_{x_1}[t] \in IdV \Longleftrightarrow \hat{\sigma}_{x_1^2}[s] \approx \hat{\sigma}_{x_1^2}[t] \in IdV,$$

and $ker_V\sigma_{x_1} = ker_V\sigma_{x_1^2}$.

Therefore, it makes sense to define the following relation on $Hyp(\tau)$:

**Definition 4.3.** Let $V$ be a variety of type $\tau$. Let $\sim_{ker_V}$ be the relation on $Hyp(\tau)$ defined by

$$\sigma_1 \sim_{ker_V} \sigma_2 \text{ iff } ker_V\sigma_1 = ker_V\sigma_2.$$

By Proposition 4.2, we see that $\sim_V \subseteq \sim_{ker_V}$ (although these relations need not be equal), and that $P(V), P_1(V), P_2(V)$ are unions of equivalence classes with respect to $\sim_{ker_V}$. It is easy to see that neither $\sim_V$ nor $\sim_{ker_V}$ is a congruence relation on $Hyp(\tau)$.

Another consequence of Proposition 4.2 is that if $\sigma_1 \sim_V \sigma_2$ and $\sigma_1$ is a hyperunifier with respect to $V$ of the terms $t$ and $t'$, then $\sigma_2$ is also a hyperunifier with respect to $V$ of $t$ and $t'$. This leads us to compare the relation $\sim_{ker_V}$ with the relation $\simeq$ (the Green's relation $\mathcal{L}$) introduced in Section 1.

**Lemma 4.4.** *In the syntactical case, that is if $V = Alg(\tau)$, we have*

$$ker\sigma \subseteq ker(\rho \circ_h \sigma)$$

*for all hypersubstitutions $\sigma$ and $\rho$ in $Hyp(\tau)$.*

***Proof.*** For any $(t,t') \in ker\sigma$, we have $\hat{\sigma}[t] = \hat{\sigma}[t']$ and hence $\hat{\rho}[\hat{\sigma}[t]]= \hat{\rho}[\hat{\sigma}[t']]$. This means that $(\rho \circ_h \sigma)\hat{}\,[t] = (\rho \circ_h \sigma)\hat{}\,[t']$, and, therefore, we have $(t,t') \in ker(\rho \circ_h \sigma)$. ∎

As a consequence of this, we have:

**Corollary 4.5.** *If $\sigma_1 \simeq \sigma_2$, then $ker\sigma_1 = ker\sigma_2$.*

***Proof.*** If $\sigma_1 \simeq \sigma_2$, then there are hypersubstitutions $\rho_1, \rho_2 \in Hyp(\tau)$ such that $\sigma_1 = \rho_1 \circ_h \sigma_2$ and $\sigma_2 = \rho_2 \circ_h \sigma_1$. Then we have

$$ker\sigma_1 = ker(\rho_1 \circ_h \sigma_2) \supseteq ker\sigma_2$$

and

$$ker\sigma_2 = ker(\rho_2 \circ_h \sigma_1) \supseteq ker\sigma_1.$$

So, $ker\sigma_1 = ker\sigma_2$. ∎

In the semantical case, when $V \subset Alg(\tau)$, Lemma 4.4 no longer holds without an additional restriction on $V$. Lemma 4.4 is a special case of the following.

**Lemma 4.6.** *Let $V$ be a variety of type $\tau$ and let $\sigma \in Hyp(\tau)$. Then*

$$ker_V\sigma \subseteq ker_V(\rho \circ_h \sigma)$$

*for all hypersubstitutions $\rho \in P(V)$.*

***Proof.*** For any $(t,t') \in ker_V\sigma$ we have $\hat{\sigma}[t] \approx \hat{\sigma}[t'] \in IdV$. Since $\rho$ is a $V$-proper hypersubstitution, this implies that $\hat{\rho}[\hat{\sigma}[t]] \approx \hat{\rho}[\hat{\sigma}[t']] \in IdV$, and so $(t,t') \in ker_V(\rho \circ_h \sigma)$. ∎

If $V$ is a solid variety, then the inclusion of Lemma 4.6 holds for all hypersubstitutions, giving the following special case.

**Corollary 4.7.** *When $V$ is a solid variety of type $\tau$, then*

$$\sigma \simeq \rho \implies ker_V\rho = ker_V\sigma.$$ ∎

Altogether for solid varieties $V$ the inclusion $\simeq \subseteq \sim_{ker_V}$ is satisfied, and this means that if $\sigma_1 \simeq \sigma_2$ and $\sigma_1$ is a hyperunifier with respect to $V$ of the terms $t$ and $t'$, then $\sigma_2$ is also a hyperunifier with respect to $V$ of $t$ and $t'$.

## 5. Applications to varieties of semigroups

In this section we determine images under $\alpha$ and $\beta$ of some varieties of semigroups. We now set our variety $W$ (see Definition 3.1) to be the variety $SEM$ of all semigroups, and consider $\alpha$ and $\beta$ as mappings between $\mathcal{P}(Hyp(2))$ and $\mathcal{P}(\mathcal{L}(SEM))$. We begin with some notation for identities and hypersubstitutions in this setting. As usual for semigroups, we write identities with the binary operation symbol $f$ replaced by juxtaposition, and with brackets omitted. Any hypersubstitution $\sigma$ is determined by the binary term $\sigma(f)$, and we denote by $\sigma_t$ the hypersubstitution which maps $f$ to $t$. For a set $S \subseteq Hyp(2)$ and a single hypersubstitution $\sigma \in Hyp(2)$, we denote by $\sigma \circ_h S$ the set of all products of the form $\sigma \circ_h \rho$ for $\rho \in S$. For any term $t$ of type (2), we use $var(t)$ for the set of all variables occurring in the term $t$. A hypersubstitution $\sigma$ of type (2) is called regular if $var(\sigma(f))$ $= var(f(x_1, x_2))$; that is, if the binary term $\sigma(f)$ uses both variables $x_1$ and $x_2$.

For convenience, we list here some sets of hypersubstitutions we shall need:

$Hyp :=$ the set of all hypersubstitutions of type (2),

$Reg :=$ the set of all regular hypersubstitutions of type (2),

$Left := \{\sigma \mid \sigma \in Hyp$ and the first variable of $\sigma(f)$ is $x_1\}$, the set of all leftmost hypersubstitutions of type (2),

$Right := \{\sigma \mid \sigma \in Hyp$ and the last variable of $\sigma(f)$ is $x_2\}$, the set of all rightmost hypersubstitutions of type (2),

$Out := \{\sigma \mid \sigma \in Hyp$ and the first variable of $\sigma(f)$ is $x_1$ and the last variable of $\sigma(f)$ is $x_2\}$, the set of all outermost hypersubstitutions of type (2),

$Pre := Hyp \setminus \{\sigma_{x_1}, \sigma_{x_2}\}$, the set of all pre-hypersubstitutions of type (2).

We shall also refer to the following varieties of semigroups:

$SEM$, the variety of all semigroups,

$TR$, the variety of trivial semigroups,

$RZ = Mod\{x_1x_2 \approx x_2\}$, the variety of all right-zero semigroups,

$LZ = Mod\{x_1x_2 \approx x_1\}$, the variety of all left-zero semigroups,

$RN = Mod\{x_1(x_2x_3) \approx (x_1x_2)x_3,\ x_1x_2x_3 \approx x_2x_1x_3,\ x^2 \approx x\}$, the variety of all right-normal idempotent semigroups,

$LN = Mod\{x_1(x_2x_3) \approx (x_1x_2)x_3,\ x_1x_2x_3 \approx x_1x_3x_2,\ x^2 \approx x\}$, the variety of all left-normal idempotent semigroups,

$Z = Mod\{x_1x_2 \approx x_3x_4\}$, the variety of all zero semigroups,

$LZ \vee Z = Mod\{x_1(x_2x_3) \approx (x_1x_2)x_3,\ x_1x_2 \approx x_1x_3\}$, the join of $LZ$ and $Z$,

$RZ \vee Z = Mod\{x_1(x_2x_3) \approx (x_1x_2)x_3,\ x_1x_2 \approx x_3x_2\}$, the join of $RZ$ and $Z$,

$SL = Mod\{(x_1x_2)x_3 \approx x_1(x_2x_3),\ x_1x_2 \approx x_2x_1,\ x_1^2 \approx x_1\}$, the variety of all semilattices

$B = Mod\{(x_1x_2)x_3 \approx x_1(x_2x_3),\ x_1^2 \approx x_1\}$, the variety of all bands.

$RB = Mod\{(x_1x_2)x_3 \approx x_1(x_2x_3) \approx x_1x_3,\ x_1^2 \approx x_1\}$, the variety of all rectangular bands.

$NB = Mod\{(x_1x_2)x_3 \approx x_1(x_2x_3),\ x_1x_2x_3x_4 \approx x_1x_3x_2x_4,\ x_1^2 \approx x_1\}$, the variety of all normal bands.

$RegB = Mod\{(x_1x_2)x_3 \approx x_1(x_2x_3),\ x_1x_2x_1x_3x_1 \approx x_1x_2x_3x_1,\ x_1^2 \approx x_1\}$, the variety of all regular bands.

We begin by finding the image $\beta(\{V\})$ for some of the varieties $V$ listed here. This means that for a given variety $V$, we want to find those hypersubstitutions $\sigma$ for which $ker_V\sigma \subseteq IdV$. Our work is simplified by the result of Proposition 4.2: since hypersubstitutions which are equivalent modulo the relation $\sim_V$ have the same kernels with respect to $V$, it is enough to consider one representative hypersubstitution from each equivalence class modulo $\sim_V$ in $Hyp$. The varieties appearing in our first Theorem have well-known finite sets of representatives.

**Theorem 5.1.** (i) $\beta(\{SL\}) = Reg$,

(ii) $\beta(\{LZ\}) = Left$,

(iii) $\beta(\{RZ\}) = Right$,

(iv) $\beta(\{RB\}) = Out \cup \sigma_{x_2x_1} \circ_h Out$,

(v) $\beta(\{Z\}) = Pre$,

(vi) $\beta(\{NB\}) = \beta(\{RegB\}) = Out \cup \sigma_{x_2x_1} \circ_h Out$,

(vii) $\beta(\{LN\}) = Left \cap Reg$,

(viii) $\beta(\{RN\}) = Right \cap Reg$.

***Proof.*** (i): Modulo the relation $\sim_{SL}$ on $Hyp$, any type (2) hypersubstitution is equivalent to one of $\sigma_{x_1}$, $\sigma_{x_2}$ and $\sigma_{x_1x_2}$. The last of these is the identity hypersubstitution, and any regular hypersubstitution of type (2) is equivalent to it. This shows that any regular hypersubstitution $\sigma$ has $ker_{SL}\sigma = ker_{SL}\sigma_{id} = IdSL$, so that $Reg \subseteq \beta(\{SL\})$.

For the opposite inclusion, we use the fact that any non-regular hypersubstitution $\sigma$ is equivalent, modulo $\sim_{SL}$, to one of $\sigma_{x_1}$ or $\sigma_{x_2}$. But we have $\hat{\sigma}_{x_1}[x_1] \approx \hat{\sigma}_{x_1}[x_1x_2] \in IdSL$ and $\hat{\sigma}_{x_2}[x_1] \approx \hat{\sigma}_{x_2}[x_2x_1] \in IdSL$, but neither $x_1 \approx x_1x_2$ nor $x_1 \approx x_2x_1$ holds in $SL$. This gives the inclusion $\beta(\{SL\}) \subseteq Reg$ and therefore equality.

(ii): We use the fact that $IdLZ$ consists of all type (2) equations $s \approx t$, where the leftmost variables in $s$ and in $t$ are the same. It follows from this that the set $\{\sigma_{x_1}, \sigma_{x_2}\}$ gives a system of representatives for $Hyp$ modulo the relation $\sim_{LZ}$. Of these two representatives, the first is leftmost, while the second is not. But also $\hat{\sigma}_{x_1}[s] \approx \hat{\sigma}_{x_1}[t] \in IdLZ$ implies $s \approx t \in IdLZ$, and we have $Left \subseteq \beta(\{LZ\})$. For the opposite inclusion we see that $\hat{\sigma}_{x_2}[x_2x_1] \approx \hat{\sigma}_{x_2}[x_1] \in IdLZ$, but $x_2x_1 \approx x_1 \notin IdLZ$.

(iii): This can be proved in a similar way.

(iv): It is well-known that $IdRB$ consists exactly of all type (2) equations $s \approx t$, where the leftmost variable in $s$ agrees with the leftmost variable in $t$ and the rightmost variable in $s$ agrees with the rightmost variable in $t$, and that the set $\{\sigma_{x_1}, \sigma_{x_2}, \sigma_{x_1x_2}, \sigma_{x_2x_1}\}$ gives a complete system of representatives of $Hyp$ modulo $\sim_{RB}$. We will show that $\beta(\{RB\})$ contains the last two representatives from this set, but not the first two. We always have $\sigma_{x_1x_2}$ in $\beta(\{RB\})$. Also if $\hat{\sigma}_{x_2x_1}[s] \approx \hat{\sigma}_{x_2x_1}[t] \in IdRB$ then $s \approx t$ in $IdRB$, so that $\sigma_{x_2x_1}$ is also in $\beta(\{RB\})$. For the converse inclusion, we notice that $\hat{\sigma}_{x_1}[x_1x_2] \approx \hat{\sigma}_{x_1}[x_1] \in IdRB$ and $\hat{\sigma}_{x_2}[x_1x_2] \approx \hat{\sigma}_{x_2}[x_2] \in IdRB$ but neither $x_1x_2 \approx x_1$ nor $x_1x_2 \approx x_2$ are identities in $RB$.

(v): In this case a complete system of representatives for $Hyp$ modulo $\sim_Z$ is given by $\{\sigma_{x_1}, \sigma_{x_2}, \sigma_{x_1x_2}\}$. The identity hypersubstitution $\sigma_{x_1x_2}$ is of course in $\beta(\{Z\})$, while $\hat{\sigma}_{x_1}[x_1^2] \approx \hat{\sigma}_{x_1}[x_1] \in IdZ$ and $\hat{\sigma}_{x_2}[x_1^2] \approx \hat{\sigma}_{x_2}[x_1] \in IdZ$ but $x_1^2 \approx x_1 \notin IdZ$ shows that the other two representatives are not. Moreover, the monoid $Pre$ consists exactly of the equivalence class of $\sigma_{x_1x_2}$ modulo $\sim_Z$ .

(vi): Let $V$ be either of the varieties $NB$ or $RegB$. Then the set $\{\sigma_{x_1}, \sigma_{x_2}, \sigma_{x_1x_2}, \sigma_{x_2x_1}, \sigma_{x_1x_2x_1}, \sigma_{x_2x_1x_2}\}$ is a complete set of representatives from $Hyp$ modulo $\sim_V$. We shall show that of these six, only $\sigma_{x_1x_2}$ and $\sigma_{x_2x_1}$ are in

$\beta(\{V\})$ in this case. There is nothing to prove for $\sigma_{x_1x_2}$. For $\sigma_{x_2x_1}$, we note that since the variety $V$ is solid, we have

$$\begin{aligned}\hat{\sigma}_{x_2x_1}[s] \approx \hat{\sigma}_{x_2x_1}[t] \in IdV &\Longrightarrow \hat{\sigma}_{x_2x_1}[\hat{\sigma}_{x_2x_1}[s]] \approx \hat{\sigma}_{x_2x_1}[\hat{\sigma}_{x_2x_1}[t]] \in IdV \\ &\Longrightarrow s \approx t \in IdV,\end{aligned}$$

showing that $\sigma_{x_2x_1}$ is in $\beta(\{V\})$.

Next, we show that none of the four remaining hypersubstitution representatives can be in $\beta(\{V\})$. Consider first the identity $x_1x_2 \approx (x_1x_2)x_1$, which does not hold in $V$. For $\sigma$ equal to either of $\sigma_{x_1}$ or $\sigma_{x_1x_2x_1}$, we do have $\hat{\sigma}[x_1x_2] \approx \hat{\sigma}[(x_1x_2)x_1] \in IdV$, so that the identity $x_1x_2 \approx (x_1x_2)x_1$ is in $ker_V\sigma$. A similar argument, using the identity $x_1x_2 \approx (x_2x_1)x_2$, works for the remaining two representative hypersubstitutions.

(vii): For $V = LN$ we have the set $\{\sigma_{x_1}, \sigma_{x_2}, \sigma_{x_1x_2}, \sigma_{x_2x_1}\}$ of representatives. The only representative which is both leftmost and regular is the identity hypersubstitution, and it is in $\beta(\{LN\})$. As in the previous cases, it is easy to verify that the other three representatives are not in $\beta(\{LN\})$.

(viii): This is the dual of the previous case. ∎

Now we determine the images under the mapping $\alpha$ for some sets of hypersubstitutions of type (2). For any term $t$ of type (2) we denote by $t^d$ the dual term, which is inductively defined by $x^d := x$ and $(f(t_1, t_2))^d =: f(t_2^d, t_1^d)$. A variety $V$ of semigroups is called dualizable if for every identity $s \approx t$ of $V$, the dual identity $s^d \approx t^d$ also holds in $V$.

**Theorem 5.2.** (i) $\alpha(\{\sigma_{x_2x_1}\}) = \{V \mid V \in \mathcal{L}(SEM)$ *and* $V$ *is dualizable* $\}$,

(ii) $\alpha(\{\sigma_{x_1^k}\}) = \{TR, LZ, Z, LZ \vee Z\}$, *for* $k \geq 2$,

(iii) $\alpha(\{\sigma_{x_2^k}\}) = \{TR, RZ, Z, RZ \vee Z\}$, *for* $k \geq 2$,

(iv) $\alpha(\{\sigma_{x_1}\}) = \{TR, LZ\}$,

(v) $\alpha(\{\sigma_{x_2}\}) = \{TR, RZ\}$.

***Proof.*** (i): The dual of an identity $s \approx t$ can be expressed as $\hat{\sigma}_{x_2x_1}[s] \approx \hat{\sigma}_{x_2x_1}[t]$, so $V$ dualizable means that $V \in \alpha(\{\sigma_{x_2x_1}\})$. If $V$ is not dualizable, then there is an identity $s \approx t \in IdV$ such that $\hat{\sigma}_{x_2x_1}[s] = s^d \approx t^d = \hat{\sigma}_{x_2x_1}[t] \notin IdV$, in which case $V \notin \alpha(\{\sigma_{x_2x_1}\})$.

(ii): If $V \in \alpha(\{\sigma_{x_1^k}\})$, then $V$ is a semigroup variety with $ker_V\sigma_{x_1^k} \subseteq IdV$. Since $\hat{\sigma}_{x_1^k}[x_1x_2] \approx \hat{\sigma}_{x_1^k}[x_1x_3] \in IdV$, this puts $x_1x_2 \approx x_1x_3 \in IdV$, and hence $V \subseteq LZ \vee Z = Mod\{(x_1x_2)x_3 \approx x_1(x_2x_3), x_1x_2 \approx x_1x_3\}$. But it is well-known that the subvariety lattice of $LZ \vee Z$ consists exactly of the varieties $TR, Z, LZ$, and $LZ \vee Z$.

We know that $TR$ satisfies $ker_{TR}\sigma_{x_1^k} \subseteq IdTR$. For $LZ$, we use the fact that if $\hat{\sigma}_{x_1}[s] \approx \hat{\sigma}_{x_1}[t]$, then the first variable of $s$ and of $t$ agree and so $s \approx t \in IdLZ$. Because $\sigma_{x_1} \sim_{LZ} \sigma_{x_1^k}$ for all $k \geq 2$, this gives $LZ \in \alpha(\{\sigma_{x_1^k}\})$. It can be shown similarly that $Z \in \alpha(\{\sigma_{x_1^k}\})$ for $k \geq 2$. Finally, by Lemma 3.6, we obtain $LZ \vee Z \in \alpha(\{\sigma_{x_1^k}\})$.

(iii): This is the dual of (ii).

(iv): Let $V$ be a variety of semigroups with $ker_V\sigma_{x_1} \subseteq IdV$. Then $\hat{\sigma}_{x_1}[x_1x_2] \approx \hat{\sigma}_{x_1}[x_1] \in IdV$, so $x_1x_2 \approx x_1 \in IdV$. This shows that either $V = TR$ or $V = LZ$, since $LZ$ is a minimal variety of semigroups. Conversely, for both $TR$ and $LZ$ we have that $ker_{TR}\sigma_{x_1} \subseteq IdTR$ and $ker_{LZ}\sigma_{x_1} \subseteq IdLZ$.

(v): This is the dual of (iv). ∎

For any fixed variety $V$ of semigroups, we can consider the subvariety lattice $\mathcal{L}(V)$ and the restriction of our mapping $\alpha$ to $\mathcal{L}(V)$. We will denote this restriction by $\alpha^*$. As in Theorem 5.1, we can use the relation $\sim_V$ to restrict our testing of hypersubstitutions. We will illustrate this now for the variety $B$ of all bands (idempotent semigroups), where the set $\{\sigma_{x_1}, \sigma_{x_2}, \sigma_{x_1x_2}, \sigma_{x_2x_1}, \sigma_{x_1x_2x_1}, \sigma_{x_2x_1x_2}\}$ is a full system of representatives of $Hyp$ with respect to the relation $\sim_B$.

**Theorem 5.3.** *Let $B$ be the variety of all bands. Then*

(i) $\alpha^*(\{\sigma_{x_1}\}) = \{TR, LZ\}$,

(ii) $\alpha^*(\{\sigma_{x_2}\}) = \{TR, RZ\}$,

(iii) $\alpha^*(\{\sigma_{x_1x_2}\}) = \mathcal{L}(B)$,

(iv) $\alpha^*(\{\sigma_{x_2x_1}\}) = \{V \mid V \in \mathcal{L}(V)$ *and* $V$ *is dualizable*$\}$,

(v) $\alpha^*(\{\sigma_{x_1x_2x_1}\}) = \{TR, LZ, LN, SL\}$,

(vi) $\alpha^*(\{\sigma_{x_2x_1x_2}\}) = \{TR, RZ, RN, SL\}$.

***Proof.*** (i), (ii) and (iv) follow from Theorem 5.2, and (iii) is clear.

(v): Let $V$ be a variety of bands with $ker_V\sigma_{x_1x_2x_1} \subseteq IdV$. Then from $(\hat{\sigma}_{x_1x_2x_1}[x_1x_2], \hat{\sigma}_{x_1x_2x_1}[(x_1x_2)x_1]) \in ker_V\sigma_{x_1x_2x_1}$, it follows that $x_1x_2 \approx x_1x_2x_1 \in IdV$. This forces $V \subseteq LN$, so $V$ must be one of $TR$, $LZ$, $LN$ or $SL$. Conversely, each of these four is in $\alpha^*(\{\sigma_{x_1x_2x_1}\})$, since $\sigma_{x_1x_2x_1} \sim_{TR} \sigma_{x_1}$, $\sigma_{x_1x_2x_1} \sim_{LZ} \sigma_{x_1}$, $\sigma_{x_1x_2x_1} \sim_{LN} \sigma_{x_1x_2}$ and $\sigma_{x_1x_2x_1} \sim_{SL} \sigma_{x_1x_2}$.

(vi): Can be proved in a similar way. ∎

In Theorem 5.1 we described the image $\beta(\{V\})$ for each of the seven non-trivial subvarieties of the variety $NB$ of normal bands. We can now extend this result to a complete description of $\beta(\{V\})$ for the remaining varieties of bands as well. Our previous Theorems give us a complete characterization of when a given representative hypersubstitution $\sigma$ is in $\beta(\{V\})$, for $V$ a variety of bands:

(i) $\sigma_{x_1x_2}$ is always in $\beta(\{V\})$,

(ii) $\sigma_{x_2x_1}$ is in $\beta(\{V\})$ iff $V$ is dualizable (see Theorem 5.3 (iv)),

(iii) $\sigma_{x_1}$ is in $\beta(\{V\})$ iff $V$ is equal to $TR$ or $LZ$ (see Theorem 5.3 (i)),

(iv) $\sigma_{x_2}$ is in $\beta(\{V\})$ iff $V$ is equal to $TR$ or $RZ$ (see Theorem 5.3 (ii)),

(v) $\sigma_{x_1x_2x_1}$ is in $\beta(\{V\})$ iff $V$ is a subvariety of $LN$ (see Theorem 5.3 (v)),

(vi) $\sigma_{x_2x_1x_2}$ is in $\beta(\{V\})$ iff $V$ is a subvariety of $RN$ (see Theorem 5.3 (vi)).

It follows from this that if $V$ is a variety of bands which is not a subvariety of $NB$, then $\beta(\{V\})$ is either $Out$, if $V$ is not dualizable, or $Out \cup \sigma_{x_2x_1} \circ_h Out$, if $V$ is dualizable.

## References


[1] K. Denecke, J. Hyndman and S.L. Wismath, *The Galois correspondence between subvariety lattices and monoids of hypersubstitutions*, Discuss. Math. - Gen. Algebra Appl. **20** (2000), 21–36.

[2] K. Denecke, J. Koppitz and St. Niwczyk, *Equational Theories generated by Hypersubstitutions of Type (n)*, Internat. J. Algebra Comput., in print.

[3] K. Denecke and S.L. Wismath, *The monoid of hypersubstitutions of type* (2), Contributions to General Algebra **10** (1998), 109–126.

[4] K. Denecke and S.L. Wismath, *Hyperidentities and Clones*, Gordon and Breach Sci. Publ., Amsterdam 2000.

[5] J. Płonka, *Proper and inner hypersubstitutions of varieties*, p. 106–116 in: "*Proccedings of the International Conference 'Summer School on Algebra and Ordered Sets', Olomouc 1994*," Palacký University, Olomouc 1994.

[6] J.H. Siekmann, *Universal Unification*, p. 1–42 in: Lecture Notes in Computer Science, no. **170** ("*International Conference on Automated Deductions (Napa, CA, 1984)*"), Springer-Verlag, Berlin 1984.

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