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SOME MODIFICATIONS OF CONGRUENCE PERMUTABILITY AND DUALLY CONGRUENCE REGULAR VARIETIES *

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Abstract

It is well known that every congruence regular variety is n-permutable (in the sense of [9]) for some $n \geq 2$. For the explicit proof see e.g. [2]. The connections between this n and Mal'cev type characterizations of congruence regularity were studied by G.D. Barbour and J.G. Raftery [1]. The concept of local congruence regularity was introduced in [3]. A common generalization of congruence regularity and local congruence regularity was given in [6] under the name "dual congruence regularity with respect to a unary term g". The natural problem arises what modification of n-permutability is satisfied by dually congruence regular varieties. The aim of this paper is to find out such a modification, to characterize varieties satisfying it by a Mal'cev type condition and to show connections with normally presented varieties (see e.g. [5], [8], [11]). The latter concept was introduced already

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by J. Płonka under a different term; the names "normal identity" and "normal variety" were firstly used by E. Graczyńska in [8].

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1. Preliminaries

At first we recall some basic concepts.

An algebra $\mathfrak{A} = (A, F)$ is *congruence regular* if for every two congruences $\Theta, \Phi \in Con(\mathfrak{A})$,

$$[a]\Theta = [a]\Phi \text{ for some } a \in A \text{ implies } \Theta = \Phi.$$

Suppose that 0 is a constant term of $\mathfrak{A}$. We say that $\mathfrak{A}$ is *locally congruence regular* (at 0) if for each $\Theta, \Phi \in Con(\mathfrak{A})$,

$$[a]\Theta = [a]\Phi \text{ for some } a \in A \text{ implies } [0]\Theta = [0]\Phi.$$

A variety $\mathcal{V}$ is *congruence regular* or *locally congruence regular* if each $\mathfrak{A} \in \mathcal{V}$ has this property. For examples of congruence regular or locally congruence regular algebras see e.g. [3].

The afore mentioned concepts were generalized in [6] as follows:

Let $g(x)$ be a unary term function of an algebra $\mathfrak{A}$. We say that $\mathfrak{A}$ is *dually congruence regular with respect to* g if for every $\Theta, \Phi \in Con(\mathfrak{A})$,

$$[b]\Theta = [b]\Phi \text{ for some } b \in A \text{ implies } [g(a)]\Theta = [g(a)]\Phi \text{ for every } a \in A.$$

A variety $\mathcal{V}$ is *dually congruence regular with respect to a term* $g(x)$ *of* $\mathcal{V}$ if each $\mathfrak{A} \in \mathcal{V}$ has this property.

Let us note that if $g(x) = x$ then dual congruence regularity with respect to g is congruence regularity; if $g(x) = 0$ then it turns out to be local congruence regularity. Hence, this concept is a common generalization of congruence regularity as well as local congruence regularity. The following characterization was given in [6]:

Proposition 1. *Let $g(x)$ be a unary term of a variety $\mathcal{V}$. The following are equivalent:*

(1) *$\mathcal{V}$ is dually congruence regular with respect to g;*

(2) *there exist – for some k – ternary terms $t_1, \ldots, t_k$ and 5-ary terms $p_1, \ldots, p_k$ such that $t_i(x, g(x), z) = z, i = 1, \ldots, k$, and*

$$g(x) = p_1(t_1(x, y, z), z, x, y, z),$$

$$p_j(z, t_j(x, y, z), x, y, z) = p_{j+1}(t_{j+1}(x, y, z), z, x, y, z) \text{ for } j = 1, \ldots, k-1,$$

$$y = p_k(z, t_k(x, y, z), x, y, z);$$

(3) *there exist – for some k – ternary terms $t_1, \ldots, t_k$ such that*

$$t_1(x, y, z) = \cdots = t_k(x, y, z) = z \text{ if and only if } y = g(x).$$

2. Locally n-permutable varieties

We introduce the following concept: Let $\mathfrak{A} = (A, F)$ be an algebra and $g(x)$ a unary term function of $\mathfrak{A}$. We say that $\mathfrak{A}$ is *locally n-permutable at g* if for each $\Theta, \Phi \in Con(\mathfrak{A})$ the following holds:

$$(g(a), g(b)) \in \Theta \cdot \Phi \cdot \Theta \cdots \text{ if and only if } (g(a), g(b)) \in \Phi \cdot \Theta \cdot \Phi \cdots$$

for each $a, b \in A$, where the relational products have n factors on both sides. A variety $\mathcal{V}$ with a unary term $g(x)$ is *locally n-permutable at g* if each $\mathfrak{A} \in \mathcal{V}$ has this property. If $n = 2$, we will call $\mathfrak{A}$ to be *locally permutable at g*.

To illuminate these concepts, denote by $g(A)$ the set $\{g(a); a \in A\}$. Then the algebra $\mathfrak{A} = (A, F)$ is locally n-permutable at g if

$$g(A)^2 \cap \Theta \cdot \Phi \cdot \Theta \cdots = g(A)^2 \cap \Phi \cdot \Theta \cdot \Phi \cdots$$

(with n factors on both sides) and $\mathfrak{A}$ is locally permutable at g if

$$g(A)^2 \cap \Theta \cdot \Phi = g(A)^2 \cap \Phi \cdot \Theta.$$

Lemma 1. *Let $g(x)$ be a unary term of a variety $\mathcal{V}$. If there exist ternary terms $h_1, \ldots, h_{n-1}$ such that*

$$g(x) = h_1(g(x), y, y),$$

$$h_i(x, x, y) = h_{i+1}(x, y, y) \text{ for } i = 1, \ldots, n-2,$$

$$g(y) = h_{n-1}(x, x, g(y)),$$

then $\mathcal{V}$ is locally n-permutable at g.

Proof. Suppose $(g(a), g(b)) \in \Theta \cdot \Phi \cdot \Theta \cdots$ (n factors). Then there exist $c_1, \dots, c_{n-1}$ with
$$g(a)\Theta c_1 \Phi c_2 \Theta c_3 \cdots c_{n-1} \Lambda g(b),$$
where $\Lambda = \Theta$ for n odd and $\Lambda = \Phi$ for n even. Then

$$\begin{aligned} g(a) &= h_1(g(a), c_1, c_1)\Phi h_1(g(a), c_1, c_2)\Theta h_1(c_1, c_1, c_2) \\ &= h_2(c_1, c_2, c_2)\Theta h_2(c_1, c_2, c_3)\Phi h_2(c_2, c_2, c_3) \\ &= h_3(c_2, c_3, c_3)\Phi \cdots \\ &\vdots \\ &= h_{n-1}(c_{n-2}, c_{n-1}, c_{n-1})\Lambda h_{n-1}(c_{n-2}, c_{n-1}, g(b))\Lambda^* \\ &\quad h_{n-1}(c_{n-1}, c_{n-1}, g(b)) = g(b), \end{aligned}$$

where $\Lambda^* = \Theta$ if $\Lambda = \Phi$ and conversely. Hence, $(g(a), g(b)) \in \Phi \cdot \Theta \cdot \Phi \cdots$ (n factors). The converse inclusion can be shown similarly. ■

We say that the unary term $g(x)$ of the variety $\mathcal{V}$ is *idempotent* if $g(g(x)) = g(x)$ is an identity of $\mathcal{V}$.

We are ready to state the following variant of n-permutability which will be involved in dually congruence regular varieties:

Theorem 1. *Let $\mathcal{V}$ be a variety and $g(x)$ an idempotent unary term of $\mathcal{V}$. If $\mathcal{V}$ is dually congruence regular with respect to g, then $\mathcal{V}$ is locally $(k+1)$-permutable at g, where k is the number of terms in* (2) *of Proposition* 1.

Proof. For the terms p_i, t_i of (2) in Proposition 1, let us set

$$h_i(x, y, z) = p_i(t_i(x, g(y), z), t_i(y, g(z), z), x, g(z), z) \text{ for } i = 1, \dots, k.$$

One can easily verify

$$\begin{aligned} h_1(g(x), y, y) &= p_1(t_1(g(x), g(y), y), t_1(y, g(y), y), g(x), g(y), y) \\ &= p_1(t_1(g(x), g(y), y), y, g(x), g(y), y) \\ &= g(g(x)) = g(x), \end{aligned}$$

$$\begin{aligned}
h_i(x,x,y) &= p_i(t_i(x,g(x),y),t_i(x,g(y),y),x,g(y),y)\\
&= p_i(y,t_i(x,g(y),y),x,g(y),y)\\
&= p_{i+1}(t_{i+1}(x,g(y),y),y,x,g(y),y)\\
&= h_{i+1}(x,y,y) \text{ for } i=1,\dots,k-1, \quad \text{and}\\
h_k(x,x,g(y)) &= p_k(t_k(x,g(x),g(y)),t_k(x,g(g(y)),g(y)),x,g(g(y)),y)\\
&= p_k(g(y),t_k(x,g(y),g(y)),x,g(y),g(y))\\
&= g(y).
\end{aligned}$$

By Lemma 1, $\mathcal{V}$ is locally $(k+1)$-permutable at g. ■

Unfortunately, we are not able to convert the assertion of Lemma 1. On the other hand, we can give a Mal'cev type condition characterizing locally n-permutable varieties:

Theorem 2. *Let $g(x)$ be a unary term of a variety $\mathcal{V}$. The following are equivalent:*

(1) *$\mathcal{V}$ is locally n-permutable at g;*

(2) *there exist $(n+1)$-ary terms $q_1,\dots,q_{n-1}$ such that:*
for n odd

$$\begin{aligned}
&g(x_0) = q_1(x_0,x_2,x_2,x_4,x_4,\dots,x_{n-1},x_{n-1},x_n),\\
&q_i(x_0,g(x_0),x_3,x_3,\dots,x_{n-2},x_{n-2},g(x_n),x_n) =\\
&q_{i+1}(x_0,g(x_0),x_3,x_3,\dots,x_{n-2},x_{n-2},g(x_n),x_n) \textit{ for } i \textit{ odd},\\
&q_i(x_0,x_2,x_2,\dots,x_{n-1},x_{n-1},x_n) =\\
&q_{i+1}(x_0,x_2,x_2,\dots,x_{n-1},x_{n-1},x_n) \textit{ for } i \textit{ even},\\
&g(x_n) = q_{n-1}(x_0,x_2,x_2,\dots,x_{n-1},x_{n-1},x_n);
\end{aligned}$$

for n even

$$g(x_0) = q_1(x_0, x_2, x_2, \dots, x_{n-2}, x_{n-2}, g(x_n), x_n),$$

$$q_i(x_0, g(x_0), x_3, x_3, \dots, x_{n-1}, x_{n-1}, x_n) =$$

$$q_{i+1}(x_0, g(x_0), x_3, x_3, \dots, x_{n-1}, x_{n-1}, x_n) \text{ for } i \text{ odd},$$

$$q_i(x_0, x_2, x_2, \dots, x_{n-2}, x_{n-2}, g(x_n), x_n) =$$

$$q_{i+1}(x_0, x_2, x_2, \dots, x_{n-2}, x_{n-2}, g(x_n), x_n) \text{ for } i \text{ even},$$

$$g(x_n) = q_{n-1}(x_0, g(x_0), x_3, x_3, \dots, x_{n-1}, x_{n-1}, x_n).$$

Proof. (1)⇒(2): Take $\mathfrak{A} = F_{\mathcal{V}}(x_0, x_1, \dots, x_n)$, the free algebra of $\mathcal{V}$ generated by $x_0, x_1, \dots, x_n$. For n even, we set

$$\Theta = \Theta((g(x_0), x_1), (x_2, x_3), \dots, (x_{n-2}, x_{n-1})),$$

$$\Phi = \Theta((x_1, x_2), (x_3, x_4), \dots, (x_{n-1}, g(x_n))),$$

the congruences on $\mathfrak{A}$ generated by the pairs in brackets. Evidently,

$$g(x_0)\Theta x_1 \Phi x_2 \Theta x_3 \cdots x_{n-2} \Theta x_{n-1} \Phi g(x_n).$$

By (1) then also

$$g(x_0)\Phi q_1 \Theta q_2 \Phi q_3 \cdots q_{n-2} \Phi q_{n-1} \Theta g(x_n)$$

for some $q_1, \dots, q_{n-1} \in \mathfrak{A}$. Every q_i is an $(n+1)$-ary term, i.e. $q_i = q_i(x_0, x_1, \dots, x_n)$. We consider the algebras $\mathfrak{A}/\Theta$ and $\mathfrak{A}/\Phi$ of $\mathcal{V}$ and by a standard method similar to that of [9], we easily obtain the identities of (2) for n even. A similar argument works for n odd.

(2)⇒(1): Let $\mathfrak{A} \in \mathcal{V}$ and $\Theta, \Phi \in Con(\mathfrak{A})$. Suppose that n is even and that

$$g(a)\Theta c_1 \Phi c_2 \Theta c_3 \cdots c_{n-1} \Phi g(b)$$

for some $c_1, \dots, c_{n-1}$ of $\mathfrak{A}$. Applying the terms $q_1, \dots, q_{n-1}$ and the identities of (2), we obtain:

$$\begin{aligned} g(a) &= q_1(a, c_2, c_2, \dots, g(b), b) \Phi q_1(a, c_1, c_2, \dots, c_{n-1}, b) \Theta q_1(a, g(a), c_3, c_3, \dots, b) \\ &= q_2(a, g(a), c_3, c_3, \dots, b) \Theta q_2(a, c_1, c_2, \dots, c_{n-1}, b) \Phi q_2(a, c_2, c_2, \dots, g(b), b) \\ &= q_3(a, c_2, c_2, \dots, g(b), b) \Phi q_3(a, c_1, c_2, \dots, c_{n-1}, b) \Theta \cdots \\ &\quad \cdots g(b). \end{aligned}$$

Hence $(g(a), g(b)) \in \Theta \cdot \Phi \cdot \Theta \cdots$ implies $(g(a), g(b)) \in \Phi \cdot \Theta \cdot \Phi \cdots$ (with n factors). In this way we obtain (1). For n odd (1) can be shown analogously. ∎

Corollary 1. *Let $g(x)$ be an idempotent unary term of a variety $\mathcal{V}$. Then $\mathcal{V}$ is locally permutable at g if and only if there exists a ternary term $t(x, y, z)$ such that the identities*

$$g(x) = t(x, z, z) \text{ and } g(z) = t(x, x, z)$$

hold in $\mathcal{V}$.

Proof. Consider the terms of Theorem 2. For $n = 2$ we have $g(x) = q_1(x, g(z), z)$ and $g(z) = q_1(x, g(x), z)$. We can set $t(x, y, z) = q_1(x, g(y), z)$ to obtain the assertion. Conversely, put $q_1(x, y, z) = t(g(x), y, g(z))$, then we obtain $q_1(x, g(z), z) = t(g(x), g(z), g(z)) = g(g(x)) = g(x)$ and $q_1(x, g(x), z)$ $= t(g(x), g(x), g(z)) = g(g(z)) = g(z)$. ∎

3. Examples

Example 1. Let $\mathcal{P}$ be the variety of all pseudocomplemented $\wedge$-semilattices. Denote by x^* the pseudocomplement of x and put

$$a + b = ((a^* \wedge b)^* \wedge (a \wedge b^*)^*)^*.$$

As it was shown in [7], the term $+$ is associative, commutative and satisfies the identities $x + x = 0$ and $x + 0 = x^{**}$. Therefore, $\mathcal{P}$ is locally permutable at $g(x) = x^{**}$ and the ternary term $t(x, y, z)$ of Corollary 1 can be chosen as: $t(x, y, z) = x + y + z$.

Remark 1. Applying Theorem 2, a variety $\mathcal{V}$ is locally 3-permutable at g if there exist 4-ary terms q_1, q_2 satisfying the following identities:

$$g(x) = q_1(x, y, y, z),$$

$$q_1(x, g(x), g(z), z) = q_2(x, g(x), g(z), z),$$

$$g(z) = q_2(x, y, y, z).$$

Example 2. A grupoid $(G, \cdot)$ is called a *semi-implication algebra* (see [4]), whenever it satisfies the identities

$$z \cdot ((x \cdot y) \cdot x) = z \cdot x, \qquad ((x \cdot y) \cdot x) \cdot z = x \cdot z,$$
$$(x \cdot y) \cdot y = (y \cdot x) \cdot x, \qquad x \cdot (y \cdot z) = y \cdot (x \cdot z).$$

As it was proved in Theorem 4 of [4], it also satisfies $x \cdot x = y \cdot y$. Thus the term $x \cdot x$ has a constant value denoted by 1. Moreover, it also satisfies $1 \cdot (1 \cdot x) = 1 \cdot x$, $x \cdot 1 = 1$ and $(1 \cdot x) \cdot y = x \cdot (1 \cdot y) = x \cdot y$. Applying Theorem 2 (and the foregoing Remark 1), we can easily show that the variety of all semi-implication algebras is locally 3-permutable at $g(x) = 1 \cdot x$. For this, take $q_1(x, y, z, v) = (z \cdot y) \cdot x$ and $q_2(x, y, z, v) = (y \cdot z) \cdot v$. Then

$$\begin{aligned} q_1(x, y, y, z) &= (y \cdot y) \cdot x = 1 \cdot x = g(x), \\ q_1(x, g(x), g(z), z) &= (g(z) \cdot g(x)) \cdot x = ((1 \cdot z) \cdot (1 \cdot x)) \cdot x = (z \cdot x) \cdot x \\ &= (x \cdot z) \cdot z = ((1 \cdot x) \cdot (1 \cdot z)) \cdot z = (g(x) \cdot g(z)) \cdot z \\ &= q_2(x, g(x), g(z), z), \quad \text{and} \\ q_2(x, y, y, z) &= (y \cdot y) \cdot z = 1 \cdot z = g(z). \end{aligned}$$

Let us note that this variety is not n-permutable for every $n \geq 1$.

4. Normal varieties

One can be interested in the question how frequent dually congruence regular or locally n-permutable varieties are among those which are often treated. The following construction will show that they are relatively common among the so-called normal varieties. For this, we firstly recall some concepts of [5], [8] and [11].

An n-ary ($n \geq 1$) term of a variety $\mathcal{V}$ is called *proper* if it is not a single variable. An identity $p(x_1, \ldots, x_n) = q(x_1, \ldots, x_n)$ is called *normal* if either both p, q are proper terms or both are equal to the same variable. For a variety $\mathcal{V}$, denote by $Id\mathcal{V}$ or $Id_N\mathcal{V}$ the set of all or of all normal identities of $\mathcal{V}$, respectively. A variety $\mathcal{V}$ is called *normal* if $Id\mathcal{V} = Id_N\mathcal{V}$. The class of all models of $Id_N\mathcal{V}$ is denoted by $N(\mathcal{V})$ and in [11] it is called a *nilpotent shift of* $\mathcal{V}$. One can easily see that $\mathcal{V}$ is normal if and only if $\mathcal{V} = N(\mathcal{V})$ since $Id_N\mathcal{V}$ is a closed set of identities (see [8]).

If $\mathcal{V} \neq N(\mathcal{V})$, then there exists a proper unary term $g(x)$ such that the identity $g(x) = x$ holds in $\mathcal{V}$. If $\mathcal{V}$ satisfies also $w(x) = x$ for a unary term w, then $\mathcal{V}$ satisfies $w(x) = g(x)$. Thus $g(x)$ is unique "up to identity". Hence, in [5] $g(x)$ is called an *assigned term* of $\mathcal{V}$. Clearly, $g(x)$ is idempotent in $N(\mathcal{V})$. It was shown in [11] that if $\mathcal{V} \neq N(\mathcal{V})$ and $g(x)$ is an assigned term of $\mathcal{V}$, then $Id\mathcal{V}$ is generated by $Id_N\mathcal{V} \cup \{g(x) = x\}$.

Remark 2. Let $\mathcal{V}$ be a normal variety. Then $\mathcal{V}$ does not have any property which can be characterized by an idempotent Mal'cev condition. Hence, no normal variety can be congruence regular or n-permutable or congruence modular since for each of those properties there exists an idempotent Mal'cev condition.

On the other hand, we can prove

Theorem 3. *Let $\mathcal{V}$ be a variety which is permutable or n-permutable or congruence regular. Let $g(x)$ be an assigned term of $\mathcal{V}$. Then the variety $N(\mathcal{V})$ is locally permutable at g or locally n-permutable at g or dually congruence regular with respect to g, respectively.*

Proof. By [10], $\mathcal{V}$ is permutable if and only if there exists a ternary term t satisfying the identities $x = t(x, z, z)$ and $z = t(x, x, z)$. Of course, $g(x) = t(x, x, x)$ is an assigned term of $\mathcal{V}$. Hence, $g(x) = t(x, z, z)$ and $g(z) = t(x, x, z)$ are also identities of $\mathcal{V}$, however they are normal and thus satisfied by $N(\mathcal{V})$. By Corollary 1, $N(\mathcal{V})$ is locally permutable at g.

Analogously, comparing the identities for n-permutability of [9] with those of Theorem 2 we conclude the second assertion.

For the third assertion one only needs to compare the identities of [2] or [1] with those of Proposition 1. ■

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