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ON THE STRUCTURE OF HALFDIAGONAL-HALFTERMINAL-SYMMETRIC CATEGORIES WITH DIAGONAL INVERSIONS

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Dedicated to Hans-Jürgen Hoehnke on the occasion of his 75th birthday.

Abstract

The category of all binary relations between arbitrary sets turns out to be a certain symmetric monoidal category Rel with an additional structure characterized by a family $d = (d_A : A \to A \otimes A \mid A \in |\mathrm{Rel}|)$ of diagonal morphisms, a family $t = (t_A : A \to I \mid A \in |\mathrm{Rel}|)$ of terminal morphisms, and a family $\nabla = (\nabla_A : A \otimes A \to A \mid A \in |\mathrm{Rel}|)$ of diagonal inversions having certain properties. Using this properties in [11] was given a system of axioms which characterizes the abstract concept of a halfdiagonal-halfterminal-symmetric monoidal category with diagonal inversions ($hdht\nabla s$-category). Besides of certain identities this system of axioms contains two identical implications. In this paper is shown that there is an equivalent characterizing system of axioms for $hdht\nabla s$-categories consisting of identities only. Therefore, the class of all small $hdht\nabla$-symmetric categories (interpreted as hetrogeneous algebras of a certain type) forms a variety and hence there are free theories for relational structures.

Keywords: halfdiagonal-halfterminal-symmetric category, diagonal inversion, partial order relation, subidentity, equation.

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1. Defining conditions

Let $K^\bullet$ be any symmetric monoidal category in the sense of Eilenberg-Kelly ([2]) with the object class $|K|$, the morphism class K, the distinguished object I, the bifunctor $\otimes : K \times K \to K$, and the families a, r, l, s of isomorphisms of K such that the following axioms are valid for all objects and all morphisms of K. By $K[A, B]$ we denote the set of all morphisms $\rho \in K$ with the domain (source) $\operatorname{dom} \rho = A$ and the codomain (target) $\operatorname{codom} \rho = B$.

Bifunctor properties:

(F1) $\operatorname{dom}(\rho \otimes \rho') = \operatorname{dom} \rho \otimes \operatorname{dom} \rho'$,

(F2) $\operatorname{codom}(\rho \otimes \rho') = \operatorname{codom} \rho \otimes \operatorname{codom} \rho'$,

(F3) $1_{A \otimes B} = 1_A \otimes 1_B$,

(F4) $(\rho \otimes \rho')(\sigma \otimes \sigma') = \rho\sigma \otimes \rho'\sigma'$.

Conditions of monoidality:

(M1) $a_{A,B,C \otimes D} a_{A \otimes B,C,D} = (1_A \otimes a_{A,B,C}) a_{A,B \otimes C,D} (a_{A,B,C} \otimes 1_D)$,

(M2) $a_{A,I,B}(r_A \otimes 1_B) = 1_A \otimes l_B$,

(M3) $a_{A,B,C} s_{A \otimes B,C} a_{C,A,B} = (1_A \otimes s_{B,C}) a_{A,C,B} (s_{A,C} \otimes 1_B)$,

(M4) $s_{A,B} s_{B,A} = 1_{A \otimes B}$,

(M5) $s_{A,I} l_A = r_A$,

(M6) $a_{A,B,C}((\rho \otimes \sigma) \otimes \tau) = (\rho \otimes (\sigma \otimes \tau)) a_{A',B',C'}$,

(M7) $r_A \rho = (\rho \otimes 1_I) r_{A'}$,

(M8) $s_{A,B}(\sigma \otimes \rho) = (\rho \otimes \sigma) s_{A',B'}$.

Remark that the validity of an equation containing morphism compositions includes that they are defined on both sides.

An immediate consequence of the conditions above is the validity of

(M9) $\forall A,\ B \in |K|\ (a_{I,A,B}(l_A \otimes 1_B) = l_{A \otimes B})$,

(M10) $\forall A,\ B \in |K|\ (a_{A,B,I} r_{A \otimes B} = 1_A \otimes r_B)$,

(M11) $r_I = l_I$,

(M12) $s_{I,I} = 1_{I \otimes I}$,

(M13) $\forall A \in |K|\ (s_{I,A} r_A = l_A)$,

(M14) $\forall A \in |K|\ (l_A \rho = (1_I \otimes \rho) l_{A'})$.

Using the denotation

$$b_{A,B,C,D} := a_{A\otimes B,C,D}(a^{-1}_{A,B,C}(1_A \otimes s_{B,C}) a_{A,C,B} \otimes 1_D) a^{-1}_{A\otimes C,B,D}$$

one obtains the following properties for all objects $A, A', B, B', C, C', D, D'$ of K and all morphisms $\rho \in K[A, A']$, $\sigma \in K[B, B']$, $\lambda \in K[C, C']$, $\mu \in K[D, D']$:

(M15) $b_{A,B,C,D}((\rho \otimes \sigma) \otimes (\lambda \otimes \mu) = ((\rho \otimes \lambda) \otimes (\sigma \otimes \mu) b_{A',B',C'D'}$,

(M16) $b_{A,I,I,B} = 1_{A\otimes I} \otimes 1_{I\otimes B}$,

(M17) $b_{A,B,C,D} b_{A,C,B,D} = 1_{A\times B} \otimes 1_{C\otimes D}$,

(M18) $b_{A,B,C,D}(s_{A,C} \otimes s_{B,D}) = s_{A\otimes B,C\otimes D} b_{C,D,A,B}$.

Obviously, all morphisms $b_{A,B,C,D}$ are isomorphims in the category $K^\bullet$.

Definition 1.1 ([1]). A *diagonal-terminal-symmetric category* (shortly *dts*-category) $\underline{K} = (K^\bullet, d, t)$ is defined as a symmetric monoidal category endowed with morphism families

$$d = (d_A : A \to A \otimes A \mid A \in |K|) \ \text{ and } \ t = (t_A : A \to I \mid A \in |K|)$$

satisfying the following conditions for all objects $A, B, A' \in |K|$ and all morphisms $\rho \in K[A, A']$.

Diagonality:

(D1) $d_A(d_A \otimes 1_A) = d_A(1_A \otimes d_A) a_{A,A,A}$,

(D2) $d_A s_{A,A} = d_A$,

(D3) $d_{A\otimes B} = (d_A \otimes d_B) b_{A,A,B,B}$,

(D4) $d_A(\rho \otimes \rho) = \rho d_{A'}$.

Terminality:

(T1) $d_A(1_A \otimes t_A) r_A = 1_A$,

(T2) $t_I = 1_I$,

(T3) $\rho t_{A'} = t_A$.

Let A, A', B be arbitrary objects in K and let $\rho \in K[A, A']$ be any morphism in K. Then the properties

(D5) $d_A(d_A \otimes d_A) = d_A d_{A\otimes A}$,

(D6) $d_A(d_A \otimes d_A) = d_A(d_A \otimes d_A) b_{A,A,A,A}$,

(D7) $t_A d_I = d_A(t_A \otimes t_A)$,

(D9) $\rho d_{A'} d_{A'\otimes A'} = d_A(\rho d_{A'} \otimes d_A(\rho \otimes \rho)))$,

(T4) $d_A(t_A \otimes 1_A) l_A = 1_A$,

(T5) $d_{A\otimes B}((1_A \otimes t_B) r_A \otimes (t_A \otimes 1_B) l_B) = 1_{A\otimes B}$,

(T6) $t_{A\otimes B} = (t_A \otimes t_B) t_{I\otimes I}$,

(T7) $r_I = t_{I\otimes I}$,

(T8) $d_A t_{A\otimes A} = t_A$,

(T9) $\rho t_{A'} d_I = d_A(\rho t_{A'} \otimes t_A)$

are consequences of the conditions above ([1]).

The category *Set* of all total functions between arbitrary sets is a model of a *dts*-category by

$I := \{\emptyset\}, \quad A \otimes B := \{\langle a, b\rangle \mid a \in A \ \wedge \ b \in B\}$,

$\rho \in Set[A, B] :\Leftrightarrow \rho = \{(a, b) \mid a \in A \ \wedge \ b = \rho(a) \in B\}$,

$$\forall\, a \in A \ \exists!!\ b \in B \ (b = \rho(a)),$$

$\rho \in Set[A, B], \ \sigma \in Set[B, C] \ \Rightarrow \ \rho \circ \sigma := \{(a, c) \mid a \in A \ \wedge \ c = \sigma(\rho(a))\}$,

$$(a, c) \in \rho \circ \sigma \Leftrightarrow \exists\, b \in B \ ((a, b) \in \rho \ \wedge \ (b, c) \in \sigma),$$

$\rho \in Set[A, B], \rho' \in Set[A', B'] \Rightarrow \rho \otimes \rho' := \{(\langle a, a'\rangle, \langle \rho(a), \rho'(a')\rangle) \mid a \in A, a' \in A'\}$,

$a_{A,B,C} := \{(\langle a, \langle b, c\rangle\rangle, \langle\langle a, b\rangle, c\rangle) \mid a \in A, \ b \in B, \ c \in C\}$,

$s_{A,B} := \{(\langle a, b\rangle, \langle b, a\rangle) \mid a \in A, \ b \in B\}$,

$r_A := \{(\langle a, \emptyset\rangle, a) \mid a \in A\}$,

$l_A := \{(\langle \emptyset, a\rangle, a) \mid a \in A\}$,

$d_A := \{(a, \langle a, a\rangle) \mid a \in A\}$,

$t_A := \{(a, \emptyset) \mid a \in A\}$.

Remark that I is a *terminal object* in any *dts*-category $\underline{K}$ and $(A \otimes B;\, p_1^{A,B}, p_2^{A,B})$ forms a *categorical product* of the objects A, B in the category K, where $p_1^{A,B} := (1_A \otimes t_B)r_A$ and $p_2^{A,B} := (t_A \otimes 1_B)l_B$.

Moreover, $d_A(\rho \otimes \sigma) = \rho d_B$ is equivalent to $\rho = \sigma$ for all $A,\ B \in |K|$ and all $\rho, \sigma \in K[A, B]$ because of

$$\sigma = \sigma d_B p_2^{B,B} = d_A(\sigma t_B \otimes \sigma)l_B = d_A(t_A \otimes \sigma)l_B$$

$$= d_A(\rho t_B \otimes \sigma)l_B = d_A(\rho \otimes \sigma)p_2^{B,B} = \rho d_B p_2^{B,B} = \rho.$$

The morphisms $p_1^{A,B}$ and $p_2^{A,B}$ are called *canonical projections* in the category K.

Conditions (D9) and (T9) are equivalent to

$\rho d_{A'} = d_A(\rho d_{A'} \otimes d_A(\rho \otimes \rho))p_2^{A',A'}$ and $\rho t_{A'} = d_A(\rho t_{A'} \otimes t_A)p_2^{I,I}$, respectively.

Definition 1.2. Let $K^\bullet$ be again a symmetric monoidal category endowed with morhism families d and t as above. Then $\underline{K} = (K^\bullet, d, t)$ is called *halfdiagonal-terminal-symmetric category* (shortly *hdts-category*), if the conditions

(D1), (D2), (D3), (D5), (D7), (T1), (T2), (T3)

hold identically.

As above, the identities (T4), (T5), (T6), (T7), (T8), (T9) follow from the defining conditions in an *hdts*-category.

Definition 1.3. A *diagonal-halfterminal-symmetric category* (shortly *dhts*-category) ([3], [7], [10]) is defined as a sequence $\underline{K} := (K^\bullet; d, t, O, o)$ such that $K^\bullet$ is again a symmetric monoidal category, d and t are families as above, O is a distinguished zero-object of $K^\bullet$, $o : I \to O$ is a distinguished morphism of $K^\bullet$, and the following equations are fulfilled for all objects $A, B, A', B' \in |K|$ and all morphisms $\rho \in K[A, A']$, $\sigma \in K[B, B']$, $\lambda \in K[A, O]$, $\kappa \in K[O, A]$:

(D4), (T1), (T4), (T5), (T6), and

(o1) $t_A o = \lambda,$

(o2) $(1_A \otimes t_O)r_A = \kappa,$

(O1) $A \otimes O = O \otimes A = O.$

Remark that the conditions

(D1), (D2), (D3), (D5), (D6), (D7), (D9), (T2), (T7), (T8), (T9), and

(B1) $$b_{A,B,C,D}(1_{A\otimes C}\otimes t_{B\otimes D})r_{A\otimes C} = (1_A\otimes t_B)r_A\otimes(1_C\otimes t_D)r_C,$$

(B2) $$b_{A,B,C,D}(t_{A\otimes C}\otimes 1_{B\otimes D})l_{B\otimes D} = (t_A\otimes 1_B)l_B\otimes(t_C\otimes 1_D)l_D$$

are consequences of the other conditions ([3], [7], [10]).

Formulas (o1), (o2), and (O1) explain that the morphism sets $K[A,O]$ and $K[O,A]$ both consist of exactly one element $o_{A,O}$ and $o_{O,A}$, respectively, and O is a zero object in K. In any *dhts*-category there is a so-called *zero-morphism* $o_{A,B}$ to each pair of objects $A, B\in|K|$ with the properties

(o3) $$\forall\rho\in K[A,A'],\ \sigma\in K[B,B']\ (\rho o_{A,B} = o_{A',B}\ \wedge\ o_{A,B}\sigma = o_{A,B'}),$$

(o4) $$\forall\xi,\eta\in K\ (o_{A,B}\otimes\xi = o_{A,B} = \eta\otimes o_{A,B}),$$

(o5) $$o_{O,A} = (1_A\otimes t_O)r_A = (t_O\otimes 1_A)l_A.$$

The category *Par* of all partial functions between arbitrary sets is a model of a *dhts*-category by the same fixations as above and $O=\emptyset$ (the empty set) and $o: I\to O,\ o_{A,O}: A\to O,\ o_{O,A}: O\to A,\ o_{A,B}: A\to B$ as the empty functions. The morphisms are given by

$$\rho\in K[A,B]\quad:\Leftrightarrow\quad \rho=\{(a,\rho(a))\mid a\in D(\rho)\ \wedge\ \rho(a)\in B\},$$
$$\forall\ a\in D(\rho)\subseteq A\ \exists!!\ b\in B\ (b=\rho(a)).$$

The following fact is of importance for the consideration of *dhts*-categories.

Lemma 1.4. *Let* $\underline{K}$ *be a symmetric monoidal category endowed with morphism families* d *and* t *as above which fulfil conditions* (D4), (T1) *and* (T6). *Then conditions* (T4) *and* (T5) *are consequences of the validity of* (D2) *and* (D3) *in* $\underline{K}$.

Proof. Using (T1) and (D2) one obtains (T4) as follows:

$$1_A = d_A(1_A\otimes t_A)r_A = d_A s_{A,A}(1_A\otimes t_A)r_A = d_A(t_A\otimes 1_A)s_{I,A}r_A = d_A(t_A\otimes 1_A)l_A.$$

The calculation

$$d_{A\otimes B}((1_A \otimes t_B)r_A \otimes (t_A \otimes 1_B)l_B)$$

$$= (d_A \otimes d_B)b_{A,A,B,B}((1_A \otimes t_B)r_A \otimes (t_A \otimes 1_B)l_B) \qquad ((D3))$$

$$= (d_A(1_A \otimes t_A) \otimes d_B(t_B \otimes 1_B))b_{A,I,I,B}(r_A \otimes l_B) \qquad ((M15))$$

$$= (d_A(1_A \otimes t_A) \otimes d_B(t_B \otimes 1_B))(1_{A\otimes I} \otimes 1_{I\otimes B})(r_A \otimes l_B) \qquad ((M16))$$

$$= (d_A(1_A \otimes t_A) \otimes d_B(t_B \otimes 1_B))(r_A \otimes l_B) \qquad ((F3))$$

$$= (d_A(1_A \otimes t_A)r_A \otimes d_B(t_B \otimes 1_B)l_B) \qquad ((F4))$$

$$= 1_A \otimes 1_B \qquad ((T1),(T4))$$

shows the validity of (T5). ■

Let $\underline{K}$ be an arbitrary *dhts*-category. Then all morphisms $\rho \in K[A, A']$, $A, A' \in |K|$, fulfilling $\rho t_{A'} = t_A$, form a subcategory $\underline{M}^K$ of $\underline{K}$ which is even a *dts*-caregory. Denoting by $\underline{M}_K$ the smallest *dts*-subcategory of $\underline{M}^K$ containing all morphisms of the families $a,\ r,\ l,\ s,\ d,\ t$ one has

$$\underline{M}_K \subseteq \underline{\mathrm{Iso}}(K) \subseteq \underline{\mathrm{Cor}}(K) \subseteq \underline{M}^K,$$

where $\underline{\mathrm{Iso}}(K)$ ($\underline{\mathrm{Cor}}(K)$) is a *dts*-subcategory of $\underline{M}^K$ generated by all isomorphisms (coretractions) of K together with all terminal morphisms of K, since all coretractions and all terminal morphisms fulfil the condition (T3) (see [7], [10]).

The object $I \in |K|$ is a terminal object in the subcategories $\underline{M}_K$, $\underline{\mathrm{Iso}}(K)$, $\underline{\mathrm{Cor}}(K)$, and $\underline{M}^K$ but not in the whole category $\underline{K}$. Morphisms of the kind $p_1^{A,B} = (1_A \otimes t_B)r_A$ and $p_2^{A,B} = (t_A \otimes 1_B)l_B$ are called *canonincal projections* again and $(A \otimes B;\ p_1^{A,B}, p_2^{A,B})$ is a *categorical product* of A and B in $\underline{M}^K$, but in general not in the whole category.

Schreckenberger had proved ([7]) that

$$\rho \leq \sigma :\Leftrightarrow d_A(\rho \otimes \sigma) = \rho d_{A'} \qquad (\rho, \sigma \in K[A, A'])$$

defines a partial order relation which is stable under composition and $\otimes$-operation. Moreover, the following are equivaent:

(i) $$d_A(\rho \otimes \sigma) = \rho d_{A'},$$

(ii) $$d_A(\rho \otimes \sigma)p_2^{A',A'} = \rho,$$

(iii) $$d_A(\sigma \otimes \rho)p_1^{A',A'} = \rho.$$

Hoehnke had shown ([3]) the validity of the identical implication

$$\rho = d_A(\rho \otimes \sigma)p_2^{A',A'} \Rightarrow \rho = d_A(\rho \otimes \sigma)p_1^{A',A'}.$$

The relation $\leq$ in the *dhts*-category $\underline{Par}$ describes exactly the usual inclusion $\subseteq$.

Morphisms $e_A \in K[A, A]$ of any *dhts*-category $\underline{K}$ fulfilling $e_A \leq 1_A$ for any $A \in |K|$ are called *subidentities* ([7]). Especially, for each $\rho \in K[A, B]$, the morphism

$$\alpha(\rho) := d_A(\rho \otimes 1_A)p_2^{B,A} (= d_A(1_A \otimes \rho)p_1^{A,B})$$

is a subidentity of $A \in |K|$, since

$$\begin{aligned} d_A(d_A(\rho \otimes 1_A)p_2^{B,A} \otimes 1_A)p_2^{A,A} &= d_A(\rho \otimes d_A(1_A \otimes 1_A))a_{B,A,A}(p_2^{B,A} \otimes 1_A)p_2^{A,A} \\ &= d_A(\rho \otimes d_A)(1_B \otimes p_2^{A,A})p_2^{B,A} \\ &= d_A(\rho \otimes d_A p_2^{A,A})p_2^{B,A} = d_A(\rho \otimes 1_A)p_2^{B,A}. \end{aligned}$$

Important properties of subidentities are described in [7], [13], [15].

Definition 1.5. *A diagonal-halfterminal-symmetric category with diagonal inversion* ∇ (shortly *dht*∇*s-category*, [10]) is, by definition, a sequence $\underline{K} := (K^\bullet; d, t, \nabla, O, o)$ such that $(K^\bullet; d, t, O, o)$ is a *dhts*-category endowed with a morphism family $\nabla = (\nabla_A | \ A \in |K|)$ satisfying the following for all $A \in |K|$:

(∇1) $d_A \nabla_A = 1_A$,

(∇2) $\nabla_A d_A d_{A \otimes A} = d_{A \otimes A}(\nabla_A d_A \otimes 1_{A \otimes A})$.

The category Par is also a model of a *dht*∇*s*-category, where

$$\nabla_A := \{(\langle a, a \rangle, a) | \ a \in A\}, \ A \in |Par|.$$

The properties

(D8) $\nabla_A d_A = d_{A\otimes A}(\nabla_A \otimes \nabla_A)$,

(D9') $\rho d_{A'} = d_A(\rho d_{A'} \otimes d_A(\rho \otimes \rho))\nabla_{A'\otimes A'}$,

(T9') $\rho t_{A'} = d_A(\rho t_{A'} \otimes t_A)\nabla_I$,

(∇3) $a_{A,A,A}(\nabla_A \otimes 1_A)\nabla_A = (1_A \otimes \nabla_A)\nabla_A$,

(∇4) $s_{A,A}\nabla_A = \nabla_A$,

(∇5) $\nabla_{A\otimes B} = b_{A,B,A,B}(\nabla_A \otimes \nabla_B)$,

(∇6) $\nabla_A d_A = (d_A \otimes 1_A)a^{-1}_{A,A,A}(1_A \otimes \nabla_A)$,

(∇7) $\nabla_A d_A = (1_A \otimes d_A)a_{A,A,A}(\nabla_A \otimes 1_A)$,

(∇8) $\nabla_A d_A = (d_A \otimes d_A)\nabla_{A\otimes A}$,

(∇9) $\nabla_A \rho d_{A'} = d_{A\otimes A}(\nabla_A \rho \otimes (\rho \otimes \rho)\nabla_{A'})$,

(∇9') $\nabla_A \rho = d_{A\otimes A}(\nabla_A \rho \otimes (\rho \otimes \rho)\nabla_{A'})\nabla_{A'}$,

(∇10) $\nabla_{A\otimes A}\nabla_A = (\nabla_A \otimes \nabla_A)\nabla_A$,

(D∇) $\rho = d_A(\rho \otimes \rho)\nabla_{A'}$

follow from the axioms and the other properties of a $dht\nabla s$-category for all $A,\ A',\ B \in |K|$ and all $\rho \in K[A, A']$ (see [13]).

By the definition of the partial order relation, (T9) is equivalent to $\rho t_{A'} \leq t_A$, (∇2) is equivalent to $\nabla_A d_A \leq 1_{A^2}$, and ($\nabla$9) is equivalent to $\nabla_A \rho \leq (\rho \otimes \rho)\nabla_{A'}$ for $\rho \in K[A, A']$.

Moreover, one has the following important property in any $dht\nabla s$-category $\underline{K}$ ([11]):

(P∇) $\forall\, A, A' \in |K|\ \forall\, \rho, \sigma \in K[A, A']\ (d_A(\rho\otimes\sigma)p_2^{A',A'} = \rho \Leftrightarrow d_A(\rho\otimes\sigma)\nabla_{A'} = \rho)$.

In any $dht\nabla s$-category, conditions (D9), (T9), and (∇9) result in (D9'), (T9'), and (∇9'), respectively.

2. $hdht\nabla s$-categories

Definition 2.1 ([10]). A sequence $\underline{K} = (K^\bullet; d, t, \nabla, o)$ is called *halfdiagonal-halfterminal-symmetric monoidal category with diagonal inversion* ∇ (shortly *$hdht\nabla s$-category*), iff $K^\bullet$ is a symmetric monoidal category as above,

$(d_A : A \to A \otimes A \mid A \in |K|)$, $(t_A : A \to I \mid A \in |K|)$, $(\nabla_A : A \otimes A \to A \mid A \in |K|)$ are families of morphisms of K, and $o : I \to O$ $(I \neq O \in |K|)$ is a distinguished morphism of K such that for all objects and all morphisms of the underlying category K the conditions

(D1), (D2), (D3), (D5), (D7), (D8),

(T1), (T2), (T6), (T9'),

(∇1), (∇2), (∇3), (∇4), (∇5), (D∇),

(o1), (o2), (O1),

and

(*1) $$d_A(\rho \otimes \rho')\nabla_B d_B(\sigma \otimes \sigma')\nabla_C = d_A(d_A(\rho \otimes \rho')\nabla_B d_B(\sigma \otimes \sigma')\nabla_C \otimes d_A(\rho\sigma \otimes \rho'\sigma')\nabla_C)\nabla_C$$

are fulfilled.

The system of axioms given in this definition is free of contradictions, because the category $\underline{Rel}$ of all binary relations between sets is a model of it, i.e. $\underline{Rel}$ fulfils all the axioms of an $hdht\nabla s$-category, where $|Rel|$ is the class of all sets, the morphisms are characterized by

$$\rho \in Rel[A, A'] :\Leftrightarrow \rho = \{(a, a') \mid a \in D(\rho) \subseteq A \wedge a' \in W(\rho) \subseteq A' \wedge H(a, a')\},$$

where $H(x, y)$ is a sentence form in two variables, the distinguished objects are $I = \{\emptyset\}$ and $O = \emptyset$, the operation $\otimes$ for objects is given as in Set, the composition and the $\otimes$-operation of morphisms are described by

$$\rho \in Rel[A, B], \sigma \in Rel[B, C] \Rightarrow \rho \circ \sigma = \{(a, c) \mid \exists\, b \in B\ ((a, b) \in \rho \wedge (b, c) \in \sigma)\},$$

$$\rho \in Rel[A, B], \rho' \in Rel[A', B'] \Rightarrow \rho \otimes \rho' = \{(\langle a, a'\rangle, \langle b, b'\rangle) \mid (a, b) \in \rho \wedge (a', b') \in \rho'\},$$

and the morphisms of the families a, r, l, s, b, d, t, ∇, $(0_{A,B} \mid A, B \in |Rel|)$ are as in Par.

Lemma 2.2. *The relation $\leq$ defined by*

$$\rho \leq \sigma :\Leftrightarrow d_A(\rho \otimes \sigma)\nabla_B = \rho$$

is a partial order relation in any hdht∇-symmetric category which is compatible with compostion and $\otimes$-operation for morphisms. Moreover, the greatest *lower bound of two morphisms* $\lambda,\ \mu \in K[A,B]$ *with respect to the canonical order relation* $\leq$ *is given by*

$$d_A(\lambda \otimes \mu)\nabla_B = inf\{\lambda, \mu\}.$$

Proof. Condition $(D\nabla)$ shows the reflexivity of $\leq$. The relation is antisymmetric because of

$$\begin{aligned}
\rho \leq \sigma \ \wedge \ \sigma \leq \rho \quad \Rightarrow \quad \sigma &= d_A(\sigma \otimes \rho)\nabla_B \\
&= d_A s_{A,A}(\sigma \otimes \rho)\nabla_B && ((D2)) \\
&= d_A(\rho \otimes \sigma)s_{B,B}\nabla_B && ((M8)) \\
&= d_A(\rho \otimes \sigma)\nabla_B && ((\nabla 4)) \\
&= \rho.
\end{aligned}$$

The implication

$$\begin{aligned}
\rho \leq \sigma \wedge \sigma \leq \tau \Rightarrow \rho &= d_A(\rho \otimes \sigma)\nabla_B \\
&= d_A(\rho \otimes d_A(\sigma \otimes \tau)\nabla_B)\nabla_B \\
&= d_A(1_A \otimes d_A)(\rho \otimes (\sigma \otimes \tau))(1_B \otimes \nabla_B)\nabla_B \\
&= d_A(d_A \otimes 1_A)((\rho \otimes \sigma) \otimes \tau)a^{-1}_{B,B,B}(1_B \otimes \nabla_B)\nabla_B && ((M6), (D1)) \\
&= d_A(d_A(\rho \otimes \sigma) \otimes \tau)(\nabla_B \otimes 1_B)\nabla_B && ((\nabla 3)) \\
&= d_A(d_A(\rho \otimes \sigma)\nabla_B \otimes \tau)\nabla_B \\
&= d_A(\rho \otimes \tau)\nabla_B \\
&\Rightarrow \rho \leq \tau
\end{aligned}$$

yields the transitivity of the relation $\leq$.

Now suppose $\rho \leq \sigma,\ \ \lambda \leq \mu$, and cod $\rho =$ dom λ. Then $\rho\lambda \leq \sigma\mu$ follows via the definition of $\leq$ by condition $(*1)$:

$$\begin{aligned}
\rho\leq\sigma \ \wedge \ \lambda\leq\mu\Rightarrow \quad & d_A(\rho\otimes\sigma)\nabla_B = \rho \ \wedge \ d_B(\lambda\otimes\mu)\nabla_C = \lambda \\
\Rightarrow \rho\lambda = \ & d_A(\rho\otimes\sigma)\nabla_B d_B(\lambda\otimes\mu)\nabla_C \\
= \ & d_A(d_A(\rho\otimes\sigma)\nabla_B d_B(\lambda\otimes\mu)\nabla_C \otimes d_A(\rho\lambda\otimes\sigma\mu)\nabla_C)\nabla_C \\
= \ & d_A(\rho\lambda\otimes d_A(\rho\lambda\otimes\sigma\mu)\nabla_C)\nabla_C \\
= \ & d_A(d_A(\rho\lambda\otimes\rho\lambda)\otimes\sigma\mu)a^{-1}_{C,C,C}(1_C\otimes\nabla_C)\nabla_C \\
= \ & d_A(\rho\lambda\otimes\rho\lambda)\nabla_C\otimes\sigma\mu)\nabla_C \\
= \ & d_A(\rho\lambda\otimes\sigma\mu)\nabla_C \\
\Rightarrow \quad & \rho\lambda\leq\sigma\mu.
\end{aligned}$$

For morphisms $\rho\leq\sigma\in K[A,B]$ and $\rho'\leq\sigma'\in K[A',B']$ one obtains

$$\rho = d_A(\rho\otimes\sigma)\nabla_B \ \text{ and } \ \rho' = d_{A'}(\rho'\otimes\sigma')\nabla_{B'},$$

hence

$$\begin{aligned}
\rho\otimes\rho' &= d_A(\rho\otimes\sigma)\nabla_B\otimes d_{A'}(\rho'\otimes\sigma')\nabla_{B'} \\
&= (d_A\otimes d_{A'})((\rho\otimes\sigma)\otimes(\rho'\otimes\sigma'))(\nabla_B\otimes\nabla_{B'}) \\
&= d_{A\otimes A'}((\rho\otimes\rho')\otimes(\sigma\otimes\sigma'))b_{B,B',B,B'}(\nabla_B\otimes\nabla_{B'}) \quad ((D3),(M18)) \\
&= d_{A\otimes A'}((\rho\otimes\rho')\otimes(\sigma\otimes\sigma'))\nabla_{B\otimes B'} \quad ((\nabla 5)),
\end{aligned}$$

therefore $\rho\otimes\rho'\leq\sigma\otimes\sigma'$.

Now let λ and μ be morphisms from A into B. Then

$$\begin{aligned}
d_A(\lambda\otimes\mu)\nabla_B &= d_A(d_A(\lambda\otimes\lambda)\nabla_B\otimes\mu)\nabla_B && ((D\nabla)) \\
&= d_A(\lambda\otimes d_A(\lambda\otimes\mu)\nabla_B)\nabla_B && ((D1),(M6),(\nabla 3)) \\
&= d_A s_{A,A}(\lambda\otimes d_A(\lambda\otimes\mu)\nabla_B)\nabla_B && ((D2)) \\
&= d_A(d_A(\lambda\otimes\mu)\nabla_B\otimes\lambda)s_{B,B}\nabla_B && ((M8)) \\
&= d_A(d_A(\lambda\otimes\mu)\nabla_B\otimes\lambda)\nabla_B && ((\nabla 4)),
\end{aligned}$$

hence $d_A(\lambda \otimes \mu)\nabla_B \leq \lambda$. In the same manner one shows $d_A(\lambda \otimes \mu)\nabla_B \leq \mu$.

Further let be $\tau \leq \lambda$ and $\tau \leq \mu$. Then it follows

$$\tau = d_A(\tau \otimes \mu)\nabla_B = d_A(d_A(\tau \otimes \lambda)\nabla_B \otimes \mu)\nabla_B = d_A(\tau \otimes d_A(\lambda \otimes \mu)\nabla_B)\nabla_B,$$

therefore $\tau \leq d_A(\lambda \otimes \mu)\nabla_B$. Consequently, $d_A(\lambda \otimes \mu)\nabla_B$ is the greatest lower bound of λ and μ with respect to the partial order relation. ■

Lemma 2.3. *Any hdht∇s-category $\underline{K}$ has the following properties:*

$$\forall\ A \in |K| \qquad (\nabla_A d_a \leq 1_{A\otimes A}),$$

$$\forall\ A, A' \in |K|\ \forall\ \rho \in K[A, A'] \quad (\rho d_{A'} \leq d_A(\rho \otimes \rho)),$$

$$\forall\ A, A' \in |K|\ \forall\ \rho \in K[A, A'] \quad (\nabla_A \rho \leq (\rho \otimes \rho)\nabla_{A'}).$$

Proof. Composing the equation in condition (∇2) with $\nabla_{A',A'}$ and using (∇1) one obtains

$$\nabla_A d_A = \nabla_A d_A d_{A\otimes A}\nabla_{A'\otimes A'} = d_{A\otimes A}(\nabla_A d_A \otimes 1_{A\otimes A})\nabla_{A\otimes A},$$

hence $\nabla_A d_A \leq 1_{A\otimes A}$ by the definition of $\leq$.

Condition (D∇) gives rise to

$$\rho d_{A'} = (d_A(\rho \otimes \rho)\nabla_{A'})d_{A'} = (d_A(\rho \otimes \rho))(\nabla_{A'} d_{A'}) \leq d_A(\rho \otimes \rho) \qquad \text{and}$$

$$\nabla_A \rho = \nabla_A(d_A(\rho \otimes \rho)\nabla_{A'}) = (\nabla_A d_A)((\rho \otimes \rho)\nabla_{A'}) \leq (\rho \otimes \rho)\nabla_{A'},$$

respectively. ■

Corollary 2.4. *By the definition of the partial order relation,*

(D9') $\quad \rho d_{A'} = d_A(\rho d_{A'} \otimes d_A(\rho \otimes \rho))\nabla_{A'\otimes A'}$ *and*

(∇9') $\quad \nabla_A \rho = d_{A\otimes A}(\nabla_A \rho \otimes (\rho \otimes \rho)\nabla_{A'})\nabla_{A'}$

are identities in each hdht∇s-category $\underline{K}$. ■

Theorem 2.5. *Let $\underline{K}$ be an hdht∇s-category as defined above. Then the class*

$$F^K := \{\rho \in K \mid d_{\operatorname{dom}\rho}(\rho \otimes \rho) = \rho d_{\operatorname{cod}\rho}\}$$

of so-called functional morphisms forms an hdht∇s-subcategory $\underline{F}^K$ of $\underline{K}$ which is even a dht∇s-category.

The partial order relation in the dht∇-symmetric category $\underline{F}^K$ is the restriction of $\leq$ in the hdht∇-symmetric category $\underline{K}$.

Proof. The conditions (D5), (D7), and (D8) show that the class F^K contains all morphisms of the families d, t, and ∇, respectively.

Let $\rho \in K[A,B]$ be an isomorphism in $\underline{K}$. Then there is a $\rho^{-1} \in K[B,A]$ such that $\rho^{-1}d_A \leq d_B(\rho^{-1} \otimes \rho^{-1})$ and $\rho d_B \leq d_A(\rho \otimes \rho)$, hence $d_A(\rho \otimes \rho) \leq \rho d_B \leq d_A(\rho \otimes \rho)$, i.e. $\rho d_B = d_A(\rho \otimes \rho)$. Therefore, each isomorphism of $\underline{K}$ belongs to F^K, especially, all identities and all morphisms of the families $a,\ a^{-1},\ r,\ r^{-1},\ l,\ l^{-1},\ s,\ s^{-1},\ b,\ b^{-1}$ are in F^K. All zero morphisms $o_{A,B},\ A,B \in |K|,\ o = o_{I,O}$, are elements of F^K since $o_{A,B}d_B = o_{A,B\otimes B} = d_A(o_{A,B} \otimes o_{A,B})$.

Let $\rho \in K[A,B] \cap F^K$ and $\sigma \in K[B,C] \cap F^K$. Then

$$(\rho\sigma)d_C = \rho(\sigma d_C) = \rho(d_B(\sigma\otimes\sigma)) = (\rho d_B)(\sigma\otimes\sigma) = d_A(\rho\otimes\rho)(\sigma\otimes\sigma) = d_A(\rho\sigma\otimes\rho\sigma),$$

hence F^K is closed under composition.

If $\rho \in K[A,B]$ and $\rho' \in K[A',B']$ are morphisms of F^K, then $(\rho \otimes \rho') \in K[A \otimes A', B \otimes B']$ is in F^K too, since

$$\begin{aligned}(\rho \otimes \rho')d_{B\otimes B'} &= (\rho \otimes \rho')(d_B \otimes d_{B'})b_{B,B,B',B'}\\ &= (d_A(\rho \otimes \rho) \otimes d_{A'}(\rho' \otimes \rho')b_{B,B,B',B'}\\ &= (d_A \otimes d_{A'})b_{A,A,A',A'}((\rho \otimes \rho') \otimes (\rho \otimes \rho'))\\ &= d_{A\otimes A'}((\rho \otimes \rho') \otimes (\rho \otimes \rho')).\end{aligned}$$

With respect to the axioms of an $hdht\nabla s$-category, which are identities only, and because of the defining condition of $F^K \subseteq K$, one has a $dht\nabla s$-category $\underline{F}^K$.

The partial order relation $\leq$ in $\underline{K}$ is defined by $\rho \leq \sigma \Leftrightarrow \rho = d_A(\rho \otimes \sigma)\nabla_{A'}$ for morphisms $\rho, \sigma \in K[A,A']$. By property (P∇), this condition is equivalent to $\rho = d_A(\rho \otimes \sigma)p_2^{A',A'}$ for morphisms ρ, σ of F^K, hence $\rho \leq \sigma$ with respect to the partial order relation in the $dht\nabla s$-category $\underline{F}^K$. ∎

Proposition 2.6. *All morphisms $\rho \in K[A,B],\quad A,B \in |K|$, of an $hdht\nabla s$-category $\underline{K}$ fulfilling the condition $\rho t_B = t_A$ (so-called total morphisms) form a symmetric monoidal subcategory $T^{K\bullet}$ which contains all coretractions of $\underline{K}$ and all morphims $t_A,\quad A \in |K|$.*

Moreover, $\underline{T}^K := (T^{K\bullet}, d, t)$ is an hdts-category.

Proof. Obviously, all identity morphisms $1_A,\quad A \in |K|$, are in T^K. Because of

$$\rho t_B = t_A \ \wedge\ \sigma t_C = t_B \Rightarrow (\rho\sigma)t_c = \rho(\sigma t_C) = \rho t_B = t_A$$

and

$$\rho t_B = t_A \wedge \rho' t_{B'} = t_{A'} \Rightarrow (\rho\otimes\rho')t_{B\otimes B'} = (\rho\otimes\rho')(t_B\otimes t_{B'})t_{I\otimes I} = (t_A\otimes t_{A'})t_{I\otimes I} = t_{A\otimes A'}$$

the class T^K is closed under composition and $\otimes$-operation.

Let $\rho \in K[A,B]$ be a coretraction in $\underline{K}$. Then there is $\rho^* \in K[B,A]$ such that $\rho\rho^* = 1_A$. So, one has (see [6], p. 12)

$$\begin{aligned}
\rho t_B = 1_A\rho t_B &= d_A(1_A\otimes t_A)r_A\rho t_B && ((T1))\\
&= d_A(\rho t_B\otimes t_A)r_I && ((M7))\\
&= d_A(\rho\otimes\rho)(t_B\otimes\rho^* t_A)r_I && ((\rho\rho^* = 1_A))\\
&\geq \rho d_B(t_B\otimes 1_B)(1_I\otimes\rho^* t_A)l_I && ((2.3))\\
&= \rho d_b(t_B\otimes 1_B)l_B\rho^* t_A && ((M14))\\
&= \rho 1_B\rho^* t_A && ((T4))\\
&= t_A \geq \rho t_B,
\end{aligned}$$

therefore $\rho t_B = t_A$, hence $\rho\in T^K$.

Because of $t_A t_I = t_A 1_I = t_A$, $A\in|K|$, $d_A\nabla_A = 1_A$, $A\in|K|$, and each isomorphism is just a coretraction, all morphisms of the families a, a^{-1}, r, r^{-1}, l, l^{-1}, s, s^{-1}, b, b^{-1}, d, and t belong to T^K.

Since arbitrary suitable morphisms and objects of $\underline{K}$ fulfil the identities (D1), (D2), (D3), (D5), (D6), (D7), (T1), (T2), (T3), (T4), (T5), (T6), (T7), (T8), (T9), the sequence $(T^{K\bullet}, d, t)$ is an *hdts*-category. ■

Corollary 2.7. *Let $\underline{K}$ be any hdht∇s-category. Then all morphisms of the families* 1, a, r, l, s, b, d, t, ∇, *and* $(o_{A,B} \mid A,B\in|K|)$ *possess all properties of such morhisms in a dht∇s-category, especially the following identities are valid:*

(D8), (T4), (T5), (T7), (T8), (B1), (B2), (o3), (o4), (o5),

(∇6), ∇7), (∇8), (∇10),

(I1) $\nabla_I d_I = 1_{I\otimes I}$,

(I2) $t_{I\otimes I} = \nabla_I = l_I = r_I = d_I^{-1}$,

(I3) $d_I = r_I^{-1} = l_I^{-1}$,

(I4) $d_I\otimes d_I = d_{I\otimes I}$. ■

Lemma 2.8. *Let $\underline{K}$ be an hdht∇s-category. Then one has*

(T9) $\rho t_{A'} d_I = d_A(\rho t_{A'} \otimes t_A)$

for all objects $A, A' \in |K|$ and all morphisms $\rho \in K[A, A']$.

Moreover:

(i) $\forall\ A, A' \in |K|\ \forall\ \rho \in K[A, A']\ (\rho d_{A'} d_{A' \otimes A'} = d_A(\rho d_{A'} \otimes d_A(\rho \otimes \rho))$
$$\Rightarrow \rho d_{A'} = d_A(\rho d_{A'} \otimes d_A(\rho \otimes \rho)) \nabla_{A' \otimes A'}),$$

(ii) $\forall\ A, A' \in |K|\ \forall\ \rho \in K[A, A']\ (\nabla_A \rho d_{A'} = d_A(\nabla_A \rho \otimes (\rho \otimes \rho) \nabla_{A'})$
$$\Rightarrow \nabla_A \rho = d_A(\nabla_A \rho \otimes (\rho \otimes \rho) \nabla_{A'}) \nabla_{A'}),$$

(iii) $\forall\ A, A' \in |K|\ \forall\ \rho \in K[A, A']\ (\rho t_{A'} d_I = d_A(\rho t_{A'} \otimes t_A)$
$$\Leftrightarrow \rho t_{A'} = d_A(\rho t_{A'} \otimes t_A) \nabla_I).$$

Proof. Because of $\nabla_I d_I = 1_{I \otimes I}$ and $\nabla_I = r_I = l_I = t_{I \otimes I}$ the equation

$$d_A(\rho t_{A'} \otimes t_A) = d_A(\rho t_{A'} \otimes t_A) \nabla_I d_I = d_A(\rho t_{A'} \otimes t_A) r_I d_I$$
$$= d_A(1_A \otimes t_A) r_A \rho t_{A'} d_I = \rho t_{A'} d_I$$

is valid for each $\rho \in K[A, A']$ and all $A, A' \in |K|$, hence $\underline{K}$ fulfils condition (T9).

The condition (T9') is equivalent to (T9), since

$$d_A(\rho t_{A'} \otimes t_A) = \rho t_{A'} d_I \Rightarrow d_A(\rho t_{A'} \otimes t_A) \nabla_I = \rho t_{A'}$$

by $d_I \nabla_I = 1_I$ and

$$d_A(\rho t_{A'} \otimes t_A) \nabla_I = \rho t_{A'} \Rightarrow d_A(\rho t_{A'} \otimes t_A) = \rho t_{A'} d_I$$

by $\nabla_I d_I = 1_{I \otimes I}$, hence property (iii) is shown.

The implications (i) and (ii) are satisfied because of the general property

$$\xi d_B = d_A(\xi \otimes \eta) \Rightarrow \xi = \xi d_B \nabla_B = d_A(\xi \otimes \eta) \nabla_B.$$

■

Remark 2.9. The opposite of the implications (i) and (ii), respectively, is not true in general, since there are conterexamples in $\underline{Rel}$.

Remark 2.10. As in any $dht\nabla s$-category, the morphisms

$$p_1^{A,B} := (1_A \otimes t_B)r_A \in K[A \otimes B, A] \cap F^K,$$
$$p_2^{A,B} := (t_A \otimes 1_B)l_B \in K[A \otimes B, B] \cap F^K$$

of an arbitrary $hdht\nabla s$-category $\underline{K}$ are called *canonical projections* again and one has

$$\nabla_A = \inf\left\{p_1^{A,A}, p_2^{A,A}\right\} = d_A\left(p_1^{A,A} \otimes p_2^{A,A}\right)\nabla_A$$

for all $A \in |K|$.

Remark that $(A \otimes B; p_1^{A,B}, p_2^{A,B})$ is not a categorical product in the whole category $\underline{K}$, but in the subcategory T^K

The family $\nabla = (\nabla_A \mid A \in |K|)$ is uniquely determined by the family $d = (d_A \mid A \in |K|)$ and the conditions $(\nabla 1)$ and $(\nabla 2)$.

Lemma 2.11. *Let $\underline{K}$ be an arbitrary $hdht\nabla s$-category. Then there holds:*

(∗2) $\forall A, B, C \in |K|\ \forall \rho, \rho' \in K[A, B]\ \forall \sigma, \sigma' \in K[B, C]\ (d_A(\rho \otimes \rho')\nabla_B = \rho$
$\wedge\ d_B(\sigma \otimes \sigma')\nabla_C = \sigma \Rightarrow d_A(\rho\sigma \otimes \rho'\sigma')\nabla_C = \rho\sigma)$,

(∗3) $\forall A, B \in |K|\ \forall \rho, \sigma \in K[A, B]\ (d_A(\rho \otimes \sigma)\nabla_B = \rho\ \wedge\ d_A(\sigma \otimes \sigma) = \sigma d_B$
$\Rightarrow d_A(\rho \otimes \sigma)p_i^{B,B} = \rho\quad (i \in \{1, 2\}))$,

(∗4) $\forall A, B \in |K|\ \forall \rho \in K[A, B]\ (d_A(\rho \otimes \rho)p_i^{B,B} = \rho\quad (i \in \{1, 2\}))$,

(∗5) $\forall A, B \in |K|\ \forall \rho, \sigma \in K[A, B]\ (d_A(\rho \otimes \sigma)\nabla_B = \rho\ \wedge\ d_A(\sigma \otimes \sigma) = \sigma d_B$
$\Rightarrow d_A(\rho \otimes \rho) = \rho d_B)$,

(∗6) $\forall A \in |K|\ \forall \rho \in K[A, A]\ (d_A(1_A \otimes \rho)\nabla_A = \rho$
$\Rightarrow d_A(1_A \otimes \rho)p_1^{A,A} = d_A(1_A \otimes \rho)p_2^{A,A} = \rho)$.

Proof. Axiom (∗1) implies condition (∗2) because of $\rho \leq \rho' \quad \wedge \quad \sigma \leq \sigma' \Rightarrow \rho\sigma \leq \rho'\sigma'$. To show (∗3) not that $d_A(\rho \otimes \sigma)\nabla_B = \rho \Leftrightarrow \rho \leq \sigma$ and $d_A(\sigma \otimes \sigma) = \sigma d_B \Leftrightarrow \sigma \in F^K$. So one obtains

$$
\begin{aligned}
d_A(\rho\otimes\sigma)p_i^{B,B} &= d_A(d_A(\rho\otimes\sigma)\nabla_B\otimes\sigma)p_i^{B,B} && (\rho\le\sigma)\\
&= d_A(\rho\otimes d_A(\sigma\otimes\sigma))a_{B,B,B}(\nabla_B\otimes 1_B)p_i^{B,B} && (\sigma\in F_K)\\
&= d_A(\rho\otimes\sigma)(1_B\otimes d_B)a_{B,B,B}(\nabla_B\otimes 1_B)p_i^{B,B} && ((F4))\\
&= d_A(\rho\otimes\sigma)\nabla_B d_B p_i^{B,B} && ((\nabla 7))\\
&= d_A(\rho\otimes\sigma)\nabla_B=\rho
\end{aligned}
$$

with respect to the axioms of an $hdht\nabla s$-category.

The property $(*4)$ is a consequence of (D9') and (T9'):

$$
\begin{aligned}
&\rho=\rho d_B p_i^{B,B}\le d_A(\rho\otimes\rho)p_i^{B,B}\ \wedge\ \rho t_B\le t_A\\
&\quad\Rightarrow d_A(\rho\otimes\rho)p_1^{B,B}=d_A(\rho\otimes\rho t_B)r_B\le d_A(\rho\otimes t_A)r_B=d_A(1_A\otimes t_A)r_A\rho=\rho\\
&\quad\wedge\ \ d_A(\rho\otimes\rho)p_2^{B,B}=d_A(\rho t_B\otimes\rho)l_B\le d_A(t_A\otimes\rho)l_B=d_A(t_A\otimes 1_A)l_A\rho=\rho.
\end{aligned}
$$

(*5): Using the previous results and the assumption one obtains

$$
\begin{aligned}
d_A(\rho\otimes\rho)&=d_A(d_A(\rho\otimes\sigma)p_2^{B,B}\otimes d_A(\rho\otimes\sigma))p_2^{B,B})\\
&=d_A(d_A\otimes d_A)((\rho\otimes\sigma)\otimes(\rho\otimes\sigma)(p_2^{B,B}\otimes p_2^{B,B})\\
&=d_A d_{A\otimes A}((\rho\otimes\sigma)\otimes(\rho\otimes\sigma))(p_2^{B,B}\otimes p_2^{B,B})\\
&=d_A(d_A(\rho\otimes\rho)\otimes d_A(\sigma\otimes\sigma))b_{B,B,B,B}(p_2^{B,B}\otimes p_2^{B,B})\\
&=d_A(d_A(\rho\otimes\rho)\otimes\sigma d_B)p_2^{B\otimes B,B\otimes B}\\
&=d_A(\rho\otimes d_A(\rho\otimes\sigma))a_{B,B,B}(1_{B\otimes B}\otimes d_B)p_2^{B\otimes B,B\otimes B}\\
&=d_A(\rho\otimes d_A(\rho\otimes\sigma))a_{B,B,B}p_2^{B\otimes B,B}d_B\\
&=d_A(\rho\otimes d_A(\rho\otimes\sigma))(1_B\otimes p_2^{B,B})p_2^{B,B}d_B\\
&=d_A(\rho\otimes d_A(\rho\otimes\sigma)p_2^{B,B})p_2^{B,B}d_B\\
&=d_A(\rho\otimes\rho)p_2^{B,B}d_B=\rho d_B.
\end{aligned}
$$

The property (*6) arises from (*3) because of $1_A\in F^K$ for each $A\in|K|$. ■

Lemma 2.12. *Let $\underline{K}$ be a monoidal symmetric category endowed with morphisms families d, t, $(o_{A,B} \mid A, B \in |K|)$, and ∇ such that all axioms of an hdht∇s-category without (*1) are fulfilled. Moreover, let the condition (*2) be valid. Then $\underline{K}$ is an hdht∇s-category in the defined sense as above.*

Proof. It remains to show the condition (*1):

$$
\begin{aligned}
&d_A(d_A(\rho \otimes \rho')\nabla_B d_B(\sigma \otimes \sigma')\nabla_C \otimes d_A(\rho\sigma \otimes \rho'\sigma')\nabla_C)\nabla_C \\
&\quad = d_A(\rho\sigma \otimes d_A(\rho\sigma \otimes \rho'\sigma')\nabla_C)\nabla_C && ((*2)) \\
&\quad = d_A(1_A \otimes d_A)(\rho\sigma \otimes (\rho\sigma \otimes \rho'\sigma'))(1_C \otimes \nabla_C)\nabla_C && ((F4)) \\
&\quad = d_A(d_A \otimes 1_A)a^{-1}_{A,A,A}(\rho\sigma \otimes (\rho\sigma \otimes \rho'\sigma'))(1_C \otimes \nabla_C)\nabla_C && ((D3)) \\
&\quad = d_A(d_A \otimes 1_A)((\rho\sigma \otimes \rho\sigma) \otimes \rho'\sigma')a^{-1}_{C,C,C}(1_C \otimes \nabla_C)\nabla_C && ((M6)) \\
&\quad = d_A(d_A)(\rho\sigma \otimes \rho\sigma) \otimes \rho'\sigma'))(\nabla_C \otimes 1_C)\nabla_C && ((\nabla 3)) \\
&\quad = d_A(d_A(\rho\sigma \otimes \rho\sigma)\nabla_C \otimes \rho'\sigma'))\nabla_C && ((F4)) \\
&\quad = d_A(\rho\sigma \otimes \rho'\sigma')\nabla_C && ((D\nabla)) \\
&\quad = \rho\sigma && ((*2)) \\
&\quad = d_A(\rho \otimes \rho')\nabla_B d_B(\sigma \otimes \sigma')\nabla_C && ((*2))
\end{aligned}
$$

■

The results of the last both lemmata are important for the axiomization of *hdht*∇*s*- categories. The system of axioms for an *hdht*∇*s*-category given in [11] contains two identical implications, namely (21) ($\Leftrightarrow$ (*2)) and (20) ($\Leftrightarrow$ (*6)). The property (*6) is a consequence of the other properties and the conditions (*1) and (*2) are equivalent in a monoidal symmetric category $\underline{K}$ endowed with morphisms families d, t, $(o_{A,B} \mid A, B \in |K|)$, and ∇ such that

(D1), (D2), (D3), (D5), (D7), (D8), (T1), (T2), (T6), (T9'),
(∇1), (∇2), (∇3), (∇4), (∇5), (∇6), (∇7), (D∇),
(o1), (o2), (O1)

are fulfilled. Therefore, *hdht*∇*s*-categories are axiomatizable by identities only, hence all small *hdht*∇*s*-categories form a variety of many-sorted total algebras and there are free many-sorted algebras to each generating set with respect to this variety. Especially, there are free $hdht\nabla s$-theories, i.e. free algebraic theories for relational structures, by analogy with the existence of free algebraic theories for partial algebras ([3], [10]).

Lemma 2.13. *In any $hdht\nabla$-symmetric category the following conditions are fulfilled for arbitrary morphisms ρ, σ:*

(j) $$\rho\sigma = 1_A \quad \wedge \quad \sigma\rho \leq 1_B \quad \Rightarrow \quad d_A(\rho \otimes \rho) = \rho d_B$$

(jj) $$\rho\sigma \leq 1_A \quad \wedge \quad \sigma\rho = 1_B \quad \Rightarrow \quad \nabla_A\rho = (\rho \otimes \rho)\nabla_B$$

Proof. To show (j) we use at first the known property $\sigma d_A \leq d_B(\sigma \otimes \sigma)$. Further,

$$d_A(\rho \otimes \rho) = \rho\sigma d_A(\rho \otimes \rho) \leq \rho d_B(\sigma \otimes \sigma)(\rho \otimes \rho) \leq \rho d_B(1_B \otimes 1_B) = \rho d_B,$$

hence $d_A(\rho \otimes \rho) = \rho d_B$ by $\rho d_B \leq d_A(\rho \otimes \rho)$.

In a similar way one shows the statement (jj), namely because of $\nabla_B\sigma \leq (\sigma \otimes \sigma)\nabla_A$ and

$$(\rho \otimes \rho)\nabla_B = (\rho \otimes \rho)\nabla_B\sigma\rho \leq (\rho\sigma \otimes \rho\sigma)\nabla_A\rho \leq \nabla_A\rho \leq (\rho \otimes \rho)\nabla_B$$

one has $\nabla_A\rho = (\rho \otimes \rho)\nabla_B$. ∎

Definition 2.14. Morphisms $e \in K[A, A] \subseteq K$ with the property $e \leq 1_A$, i.e. $e = d_A(1_A \otimes e)\nabla_A$, are called *subidentities* in $\underline{K}$ (compare with ([7])).

Proposition 2.15 (cf. [7]). *For each morphism $\rho : A \to B$, $A, B \in |K|$, the morphism*

$$\alpha(\rho) := d_A(\rho \otimes 1_A)p_2^{B,A}$$

is a subidentity of A in $\underline{K}$ and there holds $\alpha(\rho)\rho = \rho$. Each subidentity e of $\underline{K}$ fulfils $d_A(e \otimes e) = ed_A$, therefore the subidentities of $\underline{K}$ are the subidentities of $\underline{F}^K$ and satisfy the following conditions for all suitable morphims and objects of K:

(e1) $e \leq 1_A \quad \Rightarrow \quad ee = e,$

(e2) $e_1, e_2 \leq 1_A \quad \Rightarrow \quad e_1e_2 = e_2e_1 = inf\{e_1, e_2\},$

(e3) $e_1 \leq e_2 \leq 1_A \quad \Leftrightarrow \quad e_1 = e_1e_2 \leq 1_A,$

(e4) $e \leq 1_A \quad \Leftrightarrow \quad \alpha(e) = e,$

(e5) $e \leq 1_A \quad \Rightarrow \quad ed_A = d_A(e \otimes e) = d_A(e \otimes 1_A),$

(e6) $e \leq 1_A \quad \Rightarrow \quad \nabla_A e = (e \otimes e)\nabla_A = (e \otimes 1_A)\nabla_A,$

(e7) $\rho, \sigma \in K[A, B] \quad \Rightarrow \alpha(\rho)\sigma = d_A(\rho \otimes \sigma)p_2^{B,B} \wedge \alpha(\sigma)\rho = d_A(\rho \otimes \sigma)p_1^{B,B},$

(e8) $\alpha(\rho)\sigma = \rho \quad \Rightarrow \quad \rho \leq \sigma,$

(e9) $e\rho = \rho \ \wedge \ e \leq 1_A \Leftrightarrow \quad \alpha(\rho) \leq e \leq 1_A,$

(e10) $cod\rho = dom\sigma \quad \Rightarrow \alpha(\rho\sigma) \leq \alpha(\rho),$

(e11) $e \leq 1_A \quad \Rightarrow \alpha(e\rho) \leq e,$

(e12) $e \leq 1_A \quad \Rightarrow \alpha(e\rho) = e\alpha(\rho),$

(e13) $\rho \leq \sigma \quad \Rightarrow \quad \alpha(\rho) \leq \alpha(\sigma),$

(e14) $cod\rho = dom\sigma \quad \Rightarrow \rho\alpha(\sigma) \leq \alpha(\rho\sigma)\rho,$

(e15) $cod\rho = dom\sigma \quad \Rightarrow \alpha(\rho\sigma) = \alpha(\rho\alpha(\sigma)),$

Proof. Because of $\rho t_B \leq t_A$ one obtains

$$\alpha(\rho) = d_A(\rho \otimes 1_A)p_2^{B,A} = d_A(\rho t_B \otimes 1_A)l_A \leq d_A(t_A \otimes 1_A)l_A = 1_A.$$

Using the definition of $\alpha(\rho)$, properties $(M14)$, $(M15)$, and $\alpha(\rho) \leq 1_A$ one receives $\alpha(\rho)\rho = \rho$ via

$$\alpha(\rho)\rho = d_A(\rho \otimes 1_A)p_2^{B,A}\rho = d_A(\rho \otimes \rho)p_2^{B,B} \geq \rho d_B p_2^{B,B} = \rho = 1_A\rho \geq \alpha(\rho)\rho.$$

Because of $e \leq 1_A$ the property $d_A(e \otimes e) = ed_A$ is a consequence of Lemma 2.11, (*5), and the subidentities of $\underline{K}$ are exactly the subidentities of $\underline{F}^K$, therefore, all subidentities have the properties (e1), (e2), (e3) and (e4) (cf. [7]).

To show property (e5) use the property (e4) $e \leq 1_A \Rightarrow e = \alpha(e) = d_A(e \otimes 1_A)p_2^{A,A}$:

$$d_A(e \otimes e) = d_A(e \otimes d_A(e \otimes 1_A)p_2^{A,A}) = d_A(d_A(e \otimes e) \otimes 1_A)a^{-1}_{A,A,A}(1_A \otimes p_2^{A,A})$$
$$= d_A(d_A(e \otimes e)p_1^{A,A} \otimes 1_A) = d_A(e \otimes 1_A).$$

The second part of the property (e6) is a consequence of (e2) and (e5) owing to $\nabla_A d_A \leq 1_{A\otimes A}$, $(e \otimes e) \leq 1_{A\otimes A}$, and $(e \otimes 1_A) \leq 1_{A\otimes A}$:

$$\begin{aligned} d_A(e \otimes e) = d_A(e \otimes 1_A) &\Rightarrow \nabla_A d_A(e \otimes e) = \nabla_A d_A(e \otimes 1_A) \\ &\Rightarrow (e \otimes e)\nabla_A d_A = (e \otimes 1_A)\nabla_A d_A && ((\text{e2})) \\ &\Rightarrow (e \otimes e)\nabla_A d_A \nabla_A = (e \otimes 1_A)\nabla_A d_A \nabla_A \\ &\Rightarrow (e \otimes e)\nabla_A = (e \otimes 1_A)\nabla_A. && ((\nabla 1)) \end{aligned}$$

Because of $(e \otimes e) \leq 1_{A\otimes A}$ and $\nabla_A d_A \leq 1_{A\otimes A}$ one has

$$\begin{aligned} (e \otimes e)\nabla_A &= (e \otimes e)\nabla_A d_A \nabla_A && (d_A \nabla_A = 1_A) \\ &= \nabla_A d_A (e \otimes e)\nabla_A && ((\text{e2})) \\ &= \nabla_A e. && ((D\nabla)) \end{aligned}$$

Property (e7) is an immediate consequence of $(M7)$, $(M14)$, $(M8)$, and $(M13)$.

To show (e8) take into consideration

$$\rho = \alpha(\rho)\sigma \leq 1_A\sigma = \sigma.$$

(e9): Assuming $e\rho = \rho$, $e \leq 1_A$ one gets

$$\alpha(\rho) = \alpha(e\rho) = d_A(e\rho \otimes 1_A)p_2^{B,A} = d_A(e\rho t_B \otimes 1_A)l_A \leq d_A(et_A \otimes 1_A)l_A = \alpha(e) = e.$$

Conversely, $\alpha(\rho) \leq e \leq 1_A$ yields

$$\rho = \alpha(\rho)\rho \leq e\rho \leq 1_A\rho = \rho.$$

Condition (e10) is true, since

$$\alpha(\rho\sigma) = d_A(\rho\sigma \otimes 1_A)p_2^{C,A} = d_A(\rho\sigma t_C \otimes 1_A)l_A \leq d_A(\rho t_B \otimes 1_A)l_A = \alpha(\rho).$$

Condition (e11) arises from $\alpha(e\rho) \leq \alpha(e) = e$.

Property (e12) is a consequence of (e5) as follows:

$$\begin{aligned}\alpha(e\rho) &= d_A(e\rho \otimes 1_A)p_2^{B,A} = d_A(e \otimes 1_A)(\rho \otimes 1_A)p_2^{B,A} \\ &= d_A(e \otimes e)(\rho \otimes 1_A)p_2^{B,A} = ed_A(\rho \otimes 1_A)p_2^{B,A} \\ &= e\alpha(\rho).\end{aligned}$$

To show (e13) use the definitions of $\leq$ and $\alpha(\rho)$ ($\rho : A \to B$, $\sigma : B \to C$):

$$\begin{aligned}\alpha(\rho) = d_A(\rho \otimes 1_A)p_2^{B,A} &= d_A(d_A(\rho \otimes \sigma)\nabla_B \otimes 1_A)p_2^{B,A} && (\rho \leq \sigma) \\ &\leq d_A(d_A(\rho \otimes \sigma)p_2^{B,B} \otimes 1_A)p_2^{B,A} && (\nabla_B \leq p_2^{B,B}) \\ &= d_A(d_A(\rho \otimes 1_A)p_2^{B,A}\sigma \otimes 1_A)p_2^{B,A} && ((M14)) \\ &= d_A(\alpha(\rho)\sigma \otimes 1_A)p_2^{B,A} \\ &\leq d_A(\sigma \otimes 1_A)p_2^{B,A} = \alpha(\sigma). && (\alpha(\rho)\sigma \leq \sigma)\end{aligned}$$

Assertion (e14) is true since

$$\rho\alpha(\sigma) = \rho d_B(\sigma \otimes 1_B)p_2^{C,B} \leq d_A(\rho\sigma \otimes \rho)p_2^{C,B} = \alpha(\rho\sigma)\rho.$$

Condition (e15) follows by (e10), (e13), and (e14):

Let ρ and σ be as above. Then one has

$$\alpha(\rho\sigma) = \alpha(\rho\alpha(\sigma)\sigma) \leq \alpha(\rho\alpha(\sigma)),$$

hence

$$\begin{aligned}\alpha(\rho\sigma) \leq \alpha(\rho\alpha(\sigma)) \quad &\leq \alpha(\alpha(\rho\sigma)\rho) \leq \alpha(\alpha(\rho\sigma)\alpha(\rho)) \\ &\leq \alpha(\alpha(\rho\sigma)1_A) = \alpha(\alpha(\rho\sigma)) \quad = \alpha(\rho\sigma).\end{aligned}$$

■

Remark that, as an easy example shows, in *Rel* the opposite implication to (e8) is not true: Let be given $A = \{a\}, B = \{b_1, b_2\}, \rho = \{(a, b_1)\}, \sigma = \{(a, b_1), (a, b_2)\}$. Then $\rho \leq \sigma$ and $\rho < \alpha(\rho)\sigma = \sigma$.

Furthermore, the equality in (e14) is not true in general. For this let be the sets A and B as above and let be $C = \{x\}$. For the relations σ as above and $\tau = \{(b_1, x)\}$ one obtains $\sigma\alpha(\tau) = \{(a, b_1)\}$ and $\sigma\tau = \{(a, x)\}$, hence $\alpha(\sigma\tau) = \{(a, a)\}$, consequently $\alpha(\sigma\tau)\sigma = \{(a, b_1), (a, b_2)\} = \sigma \neq \sigma\alpha(\tau)$.

References

[1] L. Budach and H.-J. Hoehnke, *Automaten und Funktoren*, Akademie-Verlag, Berlin 1975.

[2] S. Eilenberg and G.M. Kelly, *Closed categories*, p. 421–562 in: "*Proceedings of the Conference on Categorical Algebra (La Jolla, 1965)*", Springer-Verlag, New York 1966.

[3] H.-J. Hoehnke, *On partial algebras*, p. 373–412 in: Colloquia Mathematica Societatis János Bolyai, 29 ("*Universal Algebra, Esztergom (Hungary) 1977*"), North-Holland, Amsterdam 1981.

[4] G.M. Kelly, *On MacLane's condition for coherence of natural associativities, commutativities*, etc., J. Algebra **1** (1964), 397–402.

[5] S. Maclane, *Kategorien, Begriffssprache und mathematische Theorie*, Springer-Verlag, Berlin-New York 1972.

[6] J. Schreckenberger, *Über dht-symmetrische Kategorien*, Semesterarbeit Päd. Hochschule Köthen, Köthen 1978.

[7] J. Schreckenberger, *Über die Einbettung von dht-symmetrischen Kategorien in die Kategorie der partiellen Abbildungen zwischen Mengen*, AdW DDR Berlin, Zentralinstitut f. Mathematik und Mechanik, Preprint P-12/8 (1980).

[8] J. Schreckenberger, *Zur Theorie der dht-symmetrischen Kategorien*, Dissertation (B), eingereicht an der Math.-Naturwiss. Fak. d. Wiss. Rates d. Pädag. Hochschule "Karl Liebknecht", Potsdam 1984.

[9] J. Schreckenberger, *Zur Axiomatik von Kategorien partieller Morphismen*, Beiträge Algebra Geom. (Halle/Saale) **24** (1987), 83–98.

[10] H.-J. Vogel, *Eine kategorientheoretische Sprache zur Beschreibung von Birkhoff-Algebren*, AdW DDR, Institut f. Mathematik, Report R-MATH-06/84, Berlin (1984).

[11] H.-J. Vogel, *Relations as morphisms of a certain monoidal category*, p. 205–217, in: "*General Algebra and Applications in Discrete Mathematics*", Shaker Verlag, Aachen 1997.

[12] H.-J. Vogel, *On functors between dht∇-symmetric categories*, Discuss. Math. - Algebra & Stochastic Methods **18** (1998), 131–147.

[13] H.-J. Vogel, *On Properties of dht∇-symmetric categories*, Contributions to General Algebra **11** (1999), 211–223.

[14] H.-J. Vogel, *Halfdiagonal-halfterminal-symmetric monoidal categories with inversions*, p. 189–204 in: "*General Algebra and Discrete Mathematics*", Shaker Verlag, Aachen 1999.

[15] H.-J. Vogel, *On morphisms between prtial algebras*, p. 427–453 in: "*Algebras and Combinatorics. An International Congress, ICAC '97, Hong Kong*", Springer-Verlag, Singapore 1999.

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