Abstract

The category of all binary relations between arbitrary sets turns out to be a certain symmetric monoidal category $\text{Rel}$ with an additional structure characterized by a family $d = \{d_A : A \to A \otimes A \mid A \in |\text{Rel}|\}$ of diagonal morphisms, a family $t = \{t_A : A \to I \mid A \in |\text{Rel}|\}$ of terminal morphisms, and a family $\nabla = \{\nabla_A : A \otimes A \to A \mid A \in |\text{Rel}|\}$ of diagonal inversions having certain properties. Using this properties in [11] was given a system of axioms which characterizes the abstract concept of a halfdiagonal-halfterminal-symmetric monoidal category with diagonal inversions ($\text{hdht}\nabla s$-category). Besides of certain identities this system of axioms contains two identical implications. In this paper is shown that there is an equivalent characterizing system of axioms for $\text{hdht}\nabla s$-categories consisting of identities only. Therefore, the class of all small $\text{hdht}\nabla s$-symmetric categories (interpreted as heterogeneous algebras of a certain type) forms a variety and hence there are free theories for relational structures.

Keywords: halfdiagonal-halfterminal-symmetric category, diagonal inversion, partial order relation, subidentity, equation.

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1. Defining conditions

Let $K$ be any symmetric monoidal category in the sense of Eilenberg-Kelly ([2]) with the object class $\mathcal{K}$, the morphism class $\mathcal{K}$, the distinguished object $I$, the bifunctor $\otimes : \mathcal{K} \times \mathcal{K} \to \mathcal{K}$, and the families $a, r, l, s$ of isomorphisms of $\mathcal{K}$ such that the following axioms are valid for all objects and all morphisms of $\mathcal{K}$. By $\mathcal{K}[A, B]$ we denote the set of all morphisms $\rho \in \mathcal{K}$ with the domain (source) $\text{dom}\rho = A$ and the codomain (target) $\text{codom}\rho = B$.

Bifunctor properties:

(F1) $\text{dom}(\rho \otimes \rho') = \text{dom}\rho \otimes \text{dom}\rho'$,
(F2) $\text{codom}(\rho \otimes \rho') = \text{codom}\rho \otimes \text{codom}\rho'$,
(F3) $1_{A \otimes B} = 1_A \otimes 1_B$,
(F4) $(\rho \otimes \rho') (\sigma \otimes \sigma') = \rho \sigma \otimes \rho' \sigma'$.

Conditions of monoidality:

(M1) $a_{A,B,C,D} a_{A \otimes B,C,D} = (1_A \otimes a_{A,B,C}) a_{A,B,C,D} (a_{A,B,C} \otimes 1_D)$,
(M2) $a_{A,B}(r_A \otimes 1_B) = 1_A \otimes l_B$,
(M3) $a_{A,B,C} s_{A \otimes B,C} a_{C,A,B} = (1_A \otimes s_{B,C}) a_{A,C,B} (s_{A,C} \otimes 1_B)$,
(M4) $s_{A,B}s_{B,A} = 1_{A \otimes B}$,
(M5) $s_{A,I} l_A = r_A$,
(M6) $a_{A,B,C} ((\rho \otimes \sigma) \otimes \tau) = (\rho \otimes (\sigma \otimes \tau)) a_{A',B',C'}$,
(M7) $r_A \rho = (\rho \otimes 1_I) r_A$,
(M8) $s_{A,B}(\sigma \otimes \rho) = (\rho \otimes \sigma) s_{A',B'}$.

Remark that the validity of an equation containing morphism compositions includes that they are defined on both sides.

An immediate consequence of the conditions above is the validity of

(M9) $\forall A, B \in \mathcal{K} \ (a_{I,A,B}(l_A \otimes 1_B) = l_{A \otimes B})$,
(M10) $\forall A, B \in \mathcal{K} \ (a_{A,B,I} r_{A \otimes B} = 1_A \otimes r_B)$,
(M11) $r_I = l_I$,
(M12) $s_{I,I} = 1_{I \otimes I}$.
(M13) $\forall A \in |K| \ (s_{I,A} r_A = l_A),$
(M14) $\forall A \in |K| \ (l_A \rho = (1_I \otimes \rho) l_{A'}).$

Using the denotation
\[
b_{A,B,C,D} := a_{A \otimes B,C,D}^{-1}(a_{A,B,C}^{-1}(1_A \otimes s_{B,C}) a_{A,C,B} \otimes 1_D) a_{A \otimes C,B,D}^{-1}
\]
one obtains the following properties for all objects $A, A', B, B', C, C', D, D'$ of $K$ and all morphisms $\rho \in K[A, A'], \ \sigma \in K[B, B'], \ \lambda \in K[C, C'], \ \mu \in K[D, D']$:

(M15) $b_{A,B,C,D}((\rho \otimes \sigma) \otimes (\lambda \otimes \mu)) = ((\rho \otimes \lambda) \otimes (\sigma \otimes \mu)) b_{A',B',C',D'}.$
(M16) $b_{A,I,I,B} = 1_{A \otimes I} \otimes 1_B.$
(M17) $b_{A,B,C,D} b_{A,C,B,D} = 1_{A \times B} \otimes 1_{C \otimes D},$
(M18) $b_{A,B,C,D} (s_{A,C} \otimes s_{B,D}) = s_{A \otimes B,C \otimes D} b_{C,D,A,B}.$

Obviously, all morphisms $b_{A,B,C,D}$ are isomorphims in the category $K^*.$

**Definition 1.1 ([1]).** A diagonal-terminal-symmetric category (shortly dts-category) $K = (K^*, d, t)$ is defined as a symmetric monoidal category endowed with morphism families
\[
d = (d_A : A \to A \otimes A \mid A \in |K|) \text{ and } t = (t_A : A \to I \mid A \in |K|)
\]
satisfying the following conditions for all objects $A, B, A' \in |K|$ and all morphisms $\rho \in K[A, A'].$

**Diagonality:**
(D1) $d_A(d_A \otimes 1_A) = d_A(1_A \otimes d_A)a_{A,A,A},$
(D2) $d_A s_{A,A} = d_A,$
(D3) $d_A \otimes B = (d_A \otimes d_B)b_{A,A,B,B},$
(D4) $d_A(\rho \otimes \rho) = \rho d_{A'}.$

**Terminality:**
(T1) $d_A(1_A \otimes t_A)r_A = 1_A,$
(T2) $t_I = 1_I,$
(T3) $\rho t_{A'} = t_A.$
Let $A$, $A'$, $B$ be arbitrary objects in $K$ and let $\rho \in K[A, A']$ be any morphism in $K$. Then the properties

(D5) $d_A(d_A \otimes d_A) = d_A d_{A \otimes A}$,

(D6) $d_A(d_A \otimes d_A) = d_A(d_A \otimes d_A)b_{A, A, A}$,

(D7) $t_A d_I = d_A(t_A \otimes t_A)$,

(D9) $\rho d_A d_{A' \otimes A'} = d_A(\rho d_A \otimes d_A(\rho \otimes \rho))$,

(T4) $d_A(t_A \otimes 1_A)l_A = 1_A$,

(T5) $d_{A \otimes B}((1_A \otimes t_B)r_A \otimes (t_A \otimes 1_B)l_B) = 1_{A \otimes B}$,

(T6) $t_{A \otimes B} = (t_A \otimes t_B)t_{I \otimes I}$,

(T7) $r_I = t_{I \otimes I}$,

(T8) $d_A t_{A \otimes A} = t_A$,

(T9) $\rho t_{A' \otimes I} = d_A(\rho t_{A'} \otimes t_A)$

are consequences of the conditions above ([1]).

The category $\text{Set}$ of all total functions between arbitrary sets is a model of a dts-category by

$I := \{\emptyset\}, \quad A \otimes B := \{(a, b) \mid a \in A \land b \in B\},$

$\rho \in \text{Set}[A, B] \Leftrightarrow \rho = \{(a, b) \mid a \in A \land b = \rho(a) \in B\},$

$\forall a \in A \exists! b \in B \ (b = \rho(a)),$

$\rho \in \text{Set}[A, B], \ \sigma \in \text{Set}[B, C] \Rightarrow \rho \circ \sigma := \{(a, c) \mid a \in A \land c = \sigma(\rho(a))\},$

$(a, c) \in \rho \circ \sigma \Leftrightarrow \exists b \in B \ ((a, b) \in \rho \land (b, c) \in \sigma),$

$\rho \in \text{Set}[A, B], \ \rho' \in \text{Set}[A', B'] \Rightarrow \rho \otimes \rho' := \{((a, a'), \langle \rho(a), \rho'(a') \rangle) \mid a \in A, a' \in A'\},$

$a_{A, B, C} := \{(\langle a, \langle b, c \rangle \rangle, \langle \langle a, b \rangle, c \rangle) \mid a \in A, \ b \in B, \ c \in C\},$

$s_{A, B} := \{\langle (a, b), (b, a) \rangle \mid a \in A, \ b \in B\},$

$r_A := \{(\langle a, \emptyset \rangle, a) \mid a \in A\},$

$l_A := \{((\emptyset, a), a) \mid a \in A\},$

$d_A := \{(a, \langle a, a \rangle) \mid a \in A\},$

$t_A := \{(a, \emptyset) \mid a \in A\}. $
Remark that $I$ is a terminal object in any dts-category $K$ and $(A \otimes B; p^A_{1,B}, p^A_{2,B})$ forms a categorical product of the objects $A, B$ in the category $K$, where $p^A_{1,B} := (1_A \otimes t_B)r_A$ and $p^A_{2,B} := (t_A \otimes 1_B)l_B$.

Moreover, $d_A(\rho \otimes \sigma) = \rho d_B$ is equivalent to $\rho = \sigma$ for all $A, B \in |K|$ and all $\rho, \sigma \in K[A, B]$ because of

$$
\begin{align*}
\sigma &= \sigma d_B p^B_{2,B} = d_A(\sigma t_B \otimes \sigma)l_B = d_A(t_A \otimes \sigma)l_B = d_A(\rho t_B \otimes \sigma)l_B = d_A(\rho \otimes \sigma)p^B_{2,B} = \rho d_B p^B_{2,B} = \rho.
\end{align*}
$$

The morphisms $p^A_{1,B}$ and $p^A_{2,B}$ are called canonical projections in the category $K$.

Conditions (D9) and (T9) are equivalent to

$$
\begin{align*}
\rho d_A &= d_A(\rho d_A \otimes d_A(\rho \otimes \rho)) p^A_{2,A} \quad \text{and} \quad \rho t_A &= d_A(\rho t_A \otimes t_A) p^A_{1,A},
\end{align*}
$$

respectively.

**Definition 1.2.** Let $K^\bullet$ be again a symmetric monoidal category endowed with morphism families $d$ and $t$ as above. Then $K = (K^\bullet; d, t)$ is called 
halfdiagonal-terminal-symmetric category (shortly hdts-category), if the conditions

(D1), (D2), (D3), (D5), (D7), (T1), (T2), (T3)

hold identically.

As above, the identities (T4), (T5), (T6), (T7), (T8), (T9) follow from the defining conditions in an hdts-category.

**Definition 1.3.** A diagonal-halfterminal-symmetric category (shortly dhts-category) ([3], [7], [10]) is defined as a sequence $K := (K^\bullet; d, t, O, o)$ such that $K^\bullet$ is again a symmetric monoidal category, $d$ and $t$ are families as above, $O$ is a distinguished zero-object of $K^\bullet$, $o : I \rightarrow O$ is a distinguished morphism of $K^\bullet$, and the following equations are fulfilled for all objects $A, B, A', B' \in |K|$ and all morphisms $\rho \in K[A, A'], \sigma \in K[B, B']$, $\lambda \in K[A, O]$, $\kappa \in K[O, A]$:

(D4), (T1), (T4), (T5), (T6), and

$$
\begin{align*}
(o1) & \quad t_A o = \lambda, \\
(o2) & \quad (1_A \otimes t_O)r_A = \kappa, \\
(O1) & \quad A \otimes O = O \otimes A = O.
\end{align*}
$$
Remark that the conditions
(D1), (D2), (D3), (D5), (D6), (D7), (D9), (T2), (T7), (T8), (T9),
and
\[(B1)\] \[b_{A,B,C,D}(1_{A\otimes C} \otimes t_{B \otimes D})r_{A \otimes C} = (1_{A} \otimes t_{B})r_{A} \otimes (1_{C} \otimes t_{D})r_{C},\]
\[(B2)\] \[b_{A,B,C,D}(t_{A \otimes C} \otimes 1_{B \otimes D})l_{B \otimes D} = (t_{A} \otimes 1_{B})l_{B} \otimes (t_{C} \otimes 1_{D})l_{D}\]
are consequences of the other conditions ([3], [7], [10]).

Formulas (o1), (o2), and (O1) explain that the morphism sets
\[K[A, O]\] and \[K[O, A]\] both consist of exactly one element \(o_{A,O}\) and \(o_{O,A}\), respectively,
and \(O\) is a zero object in \(K\). In any \(dhts\)-category there is a so-called
zero-morphism \(o_{A,B}\) to each pair of objects \(A, B \in |K|\) with the properties
\[(o3)\] \[\forall \rho \in K[A, A'], \sigma \in K[B, B'] (\rho o_{A,B} = o_{A',B} \land o_{A,B} \sigma = o_{A,B'}),\]
\[(o4)\] \[\forall \xi, \eta \in K (o_{A,B} \otimes \xi = o_{A,B} = \eta \otimes o_{A,B}),\]
\[(o5)\] \[o_{O,A} = (1_{A} \otimes t_{O})r_{A} = (t_{O} \otimes 1_{A})l_{A}.\]

The following fact is of importance for the consideration of \(dhts\)-categories.

**Lemma 1.4.** Let \(K\) be a symmetric monoidal category endowed with mor-
phism families \(d\) and \(t\) as above which fulfil conditions (D4), (T1) and (T6).
Then conditions (T4) and (T5) are consequences of the validity of (D2) and
(D3) in \(K\).

**Proof:** Using (T1) and (D2) one obtains (T4) as follows:
\[1_{A} = d_{A}(1_{A} \otimes t_{A})r_{A} = d_{A}s_{A,A}(1_{A} \otimes t_{A})r_{A} = d_{A}(t_{A} \otimes 1_{A})s_{I,A}r_{A} = d_{A}(t_{A} \otimes 1_{A})l_{A}.\]
The calculation
\[ d_{A \otimes B}((1_A \otimes t_B)r_A \otimes (t_A \otimes 1_B)l_B) \]
\[ = (d_A \otimes d_B)b_{A,A,B,B}((1_A \otimes t_B)r_A \otimes (t_A \otimes 1_B)l_B) \]  \quad ((D3))
\[ = (d_A(1_A \otimes t_A) \otimes d_B(t_B \otimes 1_B))b_{A,I,I,B}(r_A \otimes l_B) \]  \quad ((M15))
\[ = (d_A(1_A \otimes t_A) \otimes d_B(t_B \otimes 1_B))(1_{A \otimes I} \otimes 1_{I \otimes B})(r_A \otimes l_B) \]  \quad ((M16))
\[ = (d_A(1_A \otimes t_A)r_A \otimes d_B(t_B \otimes 1_B)l_B) \]  \quad ((F3))
\[ = (d_A(1_A \otimes t_A)r_A \otimes d_B(t_B \otimes 1_B)l_B)(1_A \otimes 1_B) \]  \quad ((T1), (T4))
\[ = 1_A \otimes 1_B \]
shows the validity of (T5).

Let \( K \) be an arbitrary \textit{dhts}-category. Then all morphisms \( \rho \in K[A, A'] \), \( A, A' \in |K| \), fulfilling \( pt_{A'} = t_A \), form a subcategory \( M^K \) of \( K \) which is even a \textit{dts}-category. Denoting by \( M^K \) the smallest \textit{dts}-subcategory of \( M^K \) containing all morphisms of the families \( a, r, l, s, d, t \) one has
\[ M^K \subseteq \text{Iso}(K) \subseteq \text{Cor}(K) \subseteq M^K, \]
where \( \text{Iso}(K) \) (\( \text{Cor}(K) \)) is a \textit{dts}-subcategory of \( M^K \) generated by all isomorphisms (coretractions) of \( K \) together with all terminal morphisms of \( K \), since all coretractions and all terminal morphisms fulfil the condition (T3) (see [7], [10]).

The object \( I \in |K| \) is a terminal object in the subcategories \( M^K, \text{Iso}(K), \text{Cor}(K) \), and \( M^K \) but not in the whole category \( K \). Morphisms of the kind \( p^A_{1,B} = (1_A \otimes l_B)r_A \) and \( p^A_{2,B} = (t_A \otimes 1_B)l_B \) are called \textit{canonical projections} again and \( A \otimes B; \ p^A_{1,B},p^A_{2,B} \) is a \textit{categorical product} of \( A \) and \( B \) in \( M^K \), but in general not in the whole category.

Schreckenberger had proved ([7]) that
\[ \rho \leq \sigma :\Leftrightarrow d_A(\rho \otimes \sigma) = \rho d_{A'} \quad (\rho, \sigma \in K[A, A']) \]
defines a partial order relation which is stable under composition and \( \otimes \)-operation. Moreover, the following are equivalent:
(i) \( d_A(\rho \otimes \sigma) = \rho d_{A'} \),
(ii) \( d_A(\rho \otimes \sigma)p_{A''A''} = \rho, \)
(iii) \( d_A(\sigma \otimes \rho)p_{A'1} = \rho. \)

Hoehnke had shown ([3]) the validity of the identical implication
\[
\rho = d_A(\rho \otimes \sigma)p_{A''A''} \Rightarrow \rho = d_A(\rho \otimes \sigma)p_{A'1}.
\]
The relation \( \leq \) in the \( \text{dhts-category} \) \( \text{Par} \) describes exactly the usual inclusion \( \subseteq \).

Morphisms \( e_A \in K[A,A] \) of any \( \text{dhts-category} \) \( K \) fulfilling \( e_A \leq 1_A \) for any \( A \in |K| \) are called \textit{subidentities} ([7]). Especially, for each \( \rho \in K[A,B] \), the morphism
\[
\alpha(\rho) := d_A(\rho \otimes 1_A)p_{B,A} = d_A(1_A \otimes \rho)p_{A,B}
\]
is a subidentity of \( A \in |K| \), since
\[
d_A(d_A(\rho \otimes 1_A)p_{B,A} = d_A(\rho \otimes d_A(1_A \otimes 1_A))a_{B,A}p_{B,A} = d_A(\rho \otimes d_A)p_{B,A}.
\]
Important properties of subidentities are described in [7], [13], [15].

**Definition 1.5.** A diagonal-halfterminal-symmetric category with diagonal inversion \( \nabla \) (shortly \( \text{dht}\nabla\text{s-category}, [10] \)) is, by definition, a sequence \( K := (K^*,d,t,\nabla,O,o) \) such that \( (K^*,d,t,O,o) \) is a \( \text{dhts-category} \) endowed with a morphism family \( \nabla = (\nabla_A| A \in |K|) \) satisfying the following for all \( A \in |K| \):
\[
(\nabla 1) \quad d_A \nabla_A = 1_A,
(\nabla 2) \quad \nabla_A d_A d_A = d_A(\nabla_A d_A \otimes 1_A).\]

The category \( \text{Par} \) is also a model of a \( \text{dht}\nabla\text{s-category}, \) where
\[
\nabla_A := \{((a,a),a)| a \in A\}, \quad A \in |\text{Par}|.
\]
The properties

\[(D8) \quad \nabla_A d_A = d_A \otimes_A (\nabla_A \otimes \nabla_A),\]

\[(D9') \quad \rho d_{A'} = d_A (\rho d_{A'} \otimes d_A (\rho \otimes \rho)) \nabla_{A' \otimes A'},\]

\[(T9') \quad \rho t_{A'} = d_A (\rho t_{A'} \otimes t_A) \nabla_I,\]

\[(\nabla 3) \quad a_{A,A,A} (\nabla_A \otimes 1_A) \nabla_A = (1_A \otimes \nabla_A) \nabla_A,\]

\[(\nabla 4) \quad s_{A,A} \nabla_A = \nabla_A,\]

\[(\nabla 5) \quad \nabla_{A \otimes B} = b_{A,B,A,B} (\nabla_A \otimes \nabla_B),\]

\[(\nabla 6) \quad \nabla_A d_A = (d_A \otimes 1_A) a_{A,A,A}^{-1} (1_A \otimes \nabla_A),\]

\[(\nabla 7) \quad \nabla_A d_A = (1_A \otimes d_A) a_{A,A,A} (\nabla_A \otimes 1_A),\]

\[(\nabla 8) \quad \nabla_A d_A = (d_A \otimes d_A) \nabla_{A \otimes A},\]

\[(\nabla 9) \quad \nabla_A \rho d_{A'} = d_A (\nabla_A \rho \otimes (\rho \otimes \rho)) \nabla_{A'},\]

\[(\nabla 9') \quad \nabla_A \rho = d_A \nabla_A (\nabla_A \rho \otimes (\rho \otimes \rho)) \nabla_{A'},\]

\[(\nabla 10) \quad \nabla_{A \otimes A} \nabla_A = (\nabla_A \otimes \nabla_A) \nabla_A,\]

\[(D\nabla) \quad \rho = d_A (\rho \otimes \rho) \nabla_{A'}\]

follow from the axioms and the other properties of a \(dht\nabla s\)-category for all \(A, A', B \in |K|\) and all \(\rho \in K[A, A']\) (see [13]).

By the definition of the partial order relation, \((T9)\) is equivalent to \(\rho t_{A'} \leq t_A\), \((\nabla 2)\) is equivalent to \(\nabla_A d_A \leq 1_A\), and \((\nabla 9)\) is equivalent to \(\nabla_A \rho \leq (\rho \otimes \rho) \nabla_{A'}\) for \(\rho \in K[A, A']\).

Moreover, one has the following important property in any \(dht\nabla s\)-category \(K\) ([11]):

\[(P \nabla) \quad \forall A, A' \in |K| \forall \rho, \sigma \in K[A, A'] \quad (d_A (\rho \otimes \sigma) p_{A',A'}^{\rho,\sigma} = \rho \iff d_A (\rho \otimes \sigma) \nabla_{A'} = \rho).\]

In any \(dht\nabla s\)-category, conditions \((D9), (T9), (\nabla 9)\) result in \((D9'), (T9'), (\nabla 9')\), respectively.

### 2. \(hdht\nabla s\)-categories

**Definition 2.1** ([10]). A sequence \(K = (K^*; d, t, \nabla, o)\) is called halfdiagonal-halfterminal-symmetric monoidal category with diagonal inversion \(\nabla\) (shortly \(hdht\nabla s\)-category), iff \(K^*\) is a symmetric monoidal category as above,
are families of morphisms of $K$, and $o : I \to O$ ($I \neq O \in |K|$) is a distinguished morphism of $K$ such that for all objects and all morphisms of the underlying category $K$ the conditions

\begin{align*}
(D1), & \ (D2), \ (D3), \ (D5), \ (D7), \ (D8), \\
(T1), & \ (T2), \ (T6), \ (T9'), \\
(\nabla1), & \ (\nabla2), \ (\nabla3), \ (\nabla4), \ (\nabla5), \ (D\nabla), \\
(o1), & \ (o2), \ (O1), \\
\end{align*}

and

\begin{equation}
(*)_1 \ d_A(\rho \otimes \rho') \nabla_B d_B(\sigma \otimes \sigma') \nabla_C = d_A(d_A(\rho \otimes \rho') \nabla_B d_B(\sigma \otimes \sigma') \nabla_C \otimes d_A(\rho \sigma \otimes \rho' \sigma') \nabla_C) \nabla_C
\end{equation}

are fulfilled.

The system of axioms given in this definition is free of contradictions, because the category $Rel$ of all binary relations between sets is a model of it, i.e. $Rel$ fulfills all the axioms of an $hdht\nabla$-category, where $|Rel|$ is the class of all sets, the morphisms are characterized by

$\rho \in \text{Rel}[A, A'] \iff \rho = \{(a, a') \mid a \in D(\rho) \subseteq A \land a' \in W(\rho) \subseteq A' \land H(a, a')\}$,

where $H(x, y)$ is a sentence form in two variables, the distinguished objects are $I = \{\emptyset\}$ and $O = \emptyset$, the operation $\otimes$ for objects is given as in $\text{Set}$, the composition and the $\otimes$-operation of morphisms are described by

$\rho \in \text{Rel}[A, B], \sigma \in \text{Rel}[B, C] \Rightarrow \rho \circ \sigma = \{(a, c) \mid \exists b \in B \ (\{(a, b) \in \rho \land (b, c) \in \sigma\})\}$,

$\rho \in \text{Rel}[A, B], \rho' \in \text{Rel}[A', B'] \Rightarrow \rho \otimes \rho' = \{\{(a, a'), (b, b')\} \mid (a, b) \in \rho \land (a', b') \in \rho'\}$,

and the morphisms of the families $a, r, l, s, b, d, t, \nabla, \ (0_{A,B} \mid A, B \in |Rel|)$ are as in $\text{Par}$.

**Lemma 2.2.** The relation $\leq$ defined by

$\rho \leq \sigma \iff d_A(\rho \otimes \sigma) \nabla_B = \rho$

is a partial order relation in any $hdht\nabla$-symmetric category which is compatible with composition and $\otimes$-operation for morphisms. Moreover, the greatest
lower bound of two morphisms $\lambda, \mu \in K[A, B]$ with respect to the canonical order relation $\leq$ is given by

$$d_A(\lambda \otimes \mu) \nabla_B = \inf \{ \lambda, \mu \}.$$

**Proof.** Condition $(D\nabla)$ shows the reflexivity of $\leq$. The relation is antisymmetric because of

$$\rho \leq \sigma \land \sigma \leq \rho \Rightarrow \sigma = d_A(\sigma \otimes \rho) \nabla_B$$

$$= d_A s_{A, A}(\sigma \otimes \rho) \nabla_B \quad ((D2))$$

$$= d_A(\rho \otimes \sigma) s_{B, B} \nabla_B \quad ((M8))$$

$$= d_A(\rho \otimes \sigma) \nabla_B \quad ((\nabla4))$$

$$= \rho.$$

The implication

$$\rho \leq \sigma \land \sigma \leq \tau \Rightarrow \rho = d_A(\rho \otimes \sigma) \nabla_B$$

$$= d_A(\rho \otimes d_A(\sigma \otimes \tau) \nabla_B) \nabla_B$$

$$= d_A(1_A \otimes d_A(\rho \otimes (\sigma \otimes \tau))) (1_B \otimes \nabla_B) \nabla_B$$

$$= d_A(d_A \otimes 1_A)(\rho \otimes (\sigma \otimes \tau)) a_{B, B, B}^{-1}(1_B \otimes \nabla_B) \nabla_B \quad ((M6), (D1))$$

$$= d_A(d_A(\rho \otimes \sigma) \otimes \tau)(\nabla_B \otimes 1_B) \nabla_B \quad ((\nabla3))$$

$$= d_A(d_A(\rho \otimes \sigma) \nabla_B \otimes \tau) \nabla_B$$

$$= d_A(\rho \otimes \sigma) \nabla_B$$

$$\Rightarrow \rho \leq \tau$$

yields the transitivity of the relation $\leq$.

Now suppose $\rho \leq \sigma$, $\lambda \leq \mu$, and $\text{cod} \rho = \text{dom} \lambda$. Then $\rho \lambda \leq \sigma \mu$ follows via the definition of $\leq$ by condition $(\ast1)$:
\( \rho \leq \sigma \land \lambda \leq \mu \Rightarrow d_A(\rho \otimes \sigma)\nabla_B = \rho \land d_B(\lambda \otimes \mu)\nabla_C = \lambda \)

\hspace{1cm} \Rightarrow \rho\lambda = d_A(\rho \otimes \sigma)\nabla_B dB(\lambda \otimes \mu)\nabla_C

\hspace{1cm} = d_A(d_A(\rho \otimes \sigma)\nabla_B dB(\lambda \otimes \mu)\nabla_C) \cdots (M6), (∇3)

\hspace{1cm} = d_A(d_A(\lambda \otimes \mu)\nabla_B \otimes \lambda)s_B,B \nabla_B ((D2))

\hspace{1cm} = d_A(d_A(\lambda \otimes \mu)\nabla_B \otimes \lambda)\nabla_B ((∇4)),

For morphisms \( \rho \leq \sigma \in K[A, B] \) and \( \rho' \leq \sigma' \in K[A', B'] \) one obtains

\[ \rho = d_A(\rho \otimes \sigma)\nabla_B \quad \text{and} \quad \rho' = d_{A'}(\rho' \otimes \sigma')\nabla_{B'}, \]

hence

\[ \rho \otimes \rho' = d_A(\rho \otimes \sigma)\nabla_B \otimes d_{A'}(\rho' \otimes \sigma')\nabla_{B'} \]

\[ = (d_A \otimes d_{A'})((\rho \otimes \sigma) \otimes (\rho' \otimes \sigma'))(\nabla_B \otimes \nabla_{B'}) \]

\[ = d_{A \otimes A'}((\rho \otimes \rho') \otimes (\sigma \otimes \sigma'))b_{B,B',B,B'}(\nabla_B \otimes \nabla_{B'}) \quad ((D3), (M18)) \]

\[ = d_{A \otimes A'}((\rho \otimes \rho') \otimes (\sigma \otimes \sigma'))\nabla_{B \otimes B'} \quad ((∇5)), \]

therefore \( \rho \otimes \rho' \leq \sigma \otimes \sigma' \).

Now let \( \lambda \) and \( \mu \) be morphisms from \( A \) into \( B \). Then

\[ d_A(\lambda \otimes \mu)\nabla_B = d_A(d_A(\lambda \otimes \lambda)\nabla_B \otimes \mu)\nabla_B \quad ((D\nabla)) \]

\[ = d_A(\lambda \otimes d_A(\lambda \otimes \mu)\nabla_B)\nabla_B \quad ((D1), (M6), (∇3)) \]

\[ = d_A s_{A,A}(\lambda \otimes d_A(\lambda \otimes \mu)\nabla_B)\nabla_B \quad ((D2)) \]

\[ = d_A(d_A(\lambda \otimes \mu)\nabla_B \otimes \lambda)s_{B,B}\nabla_B \quad ((M8)) \]

\[ = d_A(d_A(\lambda \otimes \mu)\nabla_B \otimes \lambda)\nabla_B \quad ((∇4)), \]
hence \( d_A(\lambda \otimes \mu) \nabla B \leq \lambda \). In the same manner one shows \( d_A(\lambda \otimes \mu) \nabla B \leq \mu \).

Further let be \( \tau \leq \lambda \) and \( \tau \leq \mu \). Then it follows
\[
\tau = d_A(\tau \otimes \mu) \nabla B = d_A(d_A(\tau \otimes \lambda) \nabla B \otimes \mu) \nabla B = d_A(\tau \otimes d_A(\lambda \otimes \mu) \nabla B) \nabla B,
\]
therefore \( \tau \leq d_A(\lambda \otimes \mu) \nabla B \). Consequently, \( d_A(\lambda \otimes \mu) \nabla B \) is the greatest lower bound of \( \lambda \) and \( \mu \) with respect to the partial order relation.

**Lemma 2.3.** Any hdht\( \nabla s \)-category \( K \) has the following properties:
\[
\forall \ A \in |K| \quad (\nabla d_A \leq 1_{A \otimes A}),
\]
\[
\forall \ A, A' \in |K| \forall \ \rho \in K[A, A'] \quad (\rho d_{A'} \leq d_A(\rho \otimes \rho)),
\]
\[
\forall \ A, A' \in |K| \forall \ \rho \in K[A, A'] \quad (\nabla \rho \leq (\rho \otimes \rho) \nabla A').
\]

**Proof.** Composing the equation in condition (\( \nabla 2 \)) with \( \nabla_{A', A'} \) and using (\( \nabla 1 \)) one obtains
\[
\nabla d_A = \nabla d_A d_{A \otimes A} \nabla_{A' \otimes A'} = d_{A \otimes A}(\nabla d_A \otimes 1_{A \otimes A}) \nabla_{A \otimes A},
\]
hence \( \nabla d_A \leq 1_{A \otimes A} \) by the definition of \( \leq \).

Condition (\( D \nabla \)) gives rise to
\[
\rho d_{A'} = (d_A(\rho \otimes \rho) \nabla A')d_{A'} = (d_A(\rho \otimes \rho))(\nabla d_{A'}) \leq d_A(\rho \otimes \rho)
\]
and
\[
\nabla \rho = \nabla(d_A(\rho \otimes \rho) \nabla A') = (\nabla d_A)((\rho \otimes \rho) \nabla A') \leq (\rho \otimes \rho) \nabla A',
\]
respectively.

**Corollary 2.4.** By the definition of the partial order relation,
\[
(\text{D9}') \quad \rho d_{A'} = d_A(\rho d_{A'} \otimes d_A(\rho \otimes \rho)) \nabla_{A' \otimes A'} \quad \text{and}
\]
\[
(\text{V9'}) \quad \nabla \rho = d_{A \otimes A}(\nabla \rho \otimes (\rho \otimes \rho) \nabla A') \nabla A'
\]
are identities in each hdht\( \nabla s \)-subcategory \( K \).

**Theorem 2.5.** Let \( K \) be an hdht\( \nabla s \)-category as defined above. Then the class
\[
F^K := \{ \rho \in K \mid d_{\text{dom}}(\rho \otimes \rho) = \rho d_{\text{cod}} \rho \}
\]
of so-called functional morphisms forms an hdht\( \nabla s \)-subcategory \( F^K \) of \( K \), which is even a dhht\( \nabla s \)-category.

The partial order relation in the dhht\( \nabla s \)-symmetric category \( F^K \) is the restriction of \( \leq \) in the hdht\( \nabla s \)-symmetric category \( K \).
Proof. The conditions (D5), (D7), and (D8) show that the class $F^K$ contains all morphisms of the families $d$, $t$, and $\nabla$, respectively.

Let $\rho \in K[A, B]$ be an isomorphism in $K$. Then there is a $\rho^{-1} \in K[B, A]$ such that $\rho^{-1}d_A \leq d_B(\rho^{-1} \otimes \rho^{-1})$ and $\rho d_B \leq d_A(\rho \otimes \rho)$, hence $d_A(\rho \otimes \rho) \leq \rho d_B \leq d_A(\rho \otimes \rho)$, i.e. $\rho d_B = d_A(\rho \otimes \rho)$. Therefore, each isomorphism of $K$ belongs to $F^K$, especially, all identities and all morphisms of the families $a$, $a^{-1}$, $r$, $r^{-1}$, $l$, $l^{-1}$, $s$, $s^{-1}$, $b$, $b^{-1}$ are in $F^K$. All zero morphisms $o_{A,B}$, $A, B \in |K|$, $\alpha = o_{I,O}$, are elements of $F^K$ since $o_{A,B}d_B = o_{A,B}d_B = d_A(o_{A,B} \otimes o_{A,B})$.

Let $\rho \in K[A, B] \cap F^K$ and $\sigma \in K[B, C] \cap F^K$. Then

$$(\rho \sigma)d_C = \rho(\sigma d_C) = \rho(d_B(\sigma \otimes \sigma)) = (\rho d_B)(\sigma \otimes \sigma) = d_A(\rho \otimes \rho)(\sigma \otimes \sigma) = d_A(\rho \sigma \otimes \rho \sigma),$$

hence $F^K$ is closed under composition.

If $\rho \in K[A, B]$ and $\rho' \in K[A', B']$ are morphisms of $F^K$, then $(\rho \otimes \rho') \in K[A \otimes A', B \otimes B']$ is in $F^K$ too, since

$$(\rho \otimes \rho')d_{B \otimes B'} = (\rho \otimes \rho')(d_B \otimes d_B')b_{B, B', B'}'$$

$$= (d_A(\rho \otimes \rho) \otimes d_{A'}(\rho' \otimes \rho'))b_{B, B', B'}'$$

$$= (d_A \otimes d_{A'})(b_{A, A', A'}((\rho \otimes \rho') \otimes (\rho \otimes \rho'))$$

$$= d_{A \otimes A'}((\rho \otimes \rho') \otimes (\rho \otimes \rho')).$$

With respect to the axioms of an $hdht\nabla s$-category, which are identities only, and because of the defining condition of $F^K \subseteq K$, one has a $dht\nabla s$-category $F^K$.

The partial order relation $\leq$ in $K$ is defined by $\rho \leq \sigma \iff \rho = d_A(\rho \otimes \sigma)\nabla A'$ for morphisms $\rho, \sigma \in K[A, A']$. By property (P\nabla), this condition is equivalent to $\rho = d_A(\rho \otimes \sigma)p_{A, A'}^2$ for morphisms $\rho, \sigma$ of $F^K$, hence $\rho \leq \sigma$ with respect to the partial order relation in the $dht\nabla s$-category $F^K$.

Proposition 2.6. All morphisms $\rho \in K[A, B]$, $A, B \in |K|$, of an $hdht\nabla s$-category $K$ fulfilling the condition $\rho t_B = t_A$ (so-called total morphisms) form a symmetric monoidal subcategory $T^K \bullet$ which contains all coretractions of $K$ and all morphisms $t_A$, $A \in |K|$.

Moreover, $\prod^K := (T^K \bullet, d, t)$ is an $hdts$-category.

Proof. Obviously, all identity morphisms $1_A$, $A \in |K|$, are in $T^K$.

Because of

$$\rho t_B = t_A \land \sigma t_C = t_B \Rightarrow (\rho \sigma)t_C = \rho(\sigma t_C) = \rho t_B = t_A$$
and

\[ \rho t_B = t_A \wedge \rho' t_B = t_A' \Rightarrow (\rho \otimes \rho')(t_B \otimes t_B') t_I \otimes I = (t_A \otimes t_A') t_I \otimes I = t_A \otimes A' \]

the class \( T^K \) is closed under composition and \( \otimes \)-operation.

Let \( \rho \in K[A, B] \) be a coretraction in \( K \). Then there is \( \rho^* \in K[B, A] \)
such that \( \rho \rho^* = 1_A \). So, one has (see [6], p. 12)

\[ \rho t_B = t_A \rho t_B = \Delta_A (t_A \otimes t_A') r_I \rho t_B (T_1) \]

\[ \geq \rho d_B (t_B \otimes 1_B)(1_I \otimes \rho^* t_A) l_I (T_2) \]

\[ = \rho d_B (t_B \otimes 1_B) l_B \rho^* t_A (T_3) \]

\[ = \rho 1_B \rho^* t_A (T_4) \]

\[ = t_A \geq \rho t_B, \]

therefore \( \rho t_B = t_A \), hence \( \rho \in T^K \).

Because of \( t_A t_I = t_A 1_I = t_A, \ A \in |K|, \ d_A \Delta_A = 1_A, \ A \in |K|, \)
and each isomorphism is just a coretraction, all morphisms of the families \( a, a^{-1}, r, r^{-1}, l, l^{-1}, s, s^{-1}, b, b^{-1}, d, \) and \( t \) belong to \( T^K \).

Since arbitrary suitable morphisms and objects of \( K \) fulfil the identities (D1), (D2), (D3), (D5), (D6), (D7), (T1), (T2), (T3), (T4), (T5), (T6),
(T7), (T8), (T9), the sequence \( (T^K^*, d, t) \) is an \text{hdts}-category.

\textbf{Corollary 2.7.} Let \( K \) be any \text{hdht}\text{\textbackslash}s\text{-category}. Then all morphisms of the families \( 1, a, r, l, s, b, d, t, \Delta, \) and \( (o_{AB} \mid A, B \in |K|) \) possess all properties of such morphisms in a \text{dht}\text{\textbackslash}s\text{-category}, especially the following identities are valid:

(\text{D8}), (T4), (T5), (T7), (T8), (B1), (B2), (o3), (o4), (o5),
(\Delta6), (\Delta7), (\Delta8), (\Delta10),
(I1) \quad \Delta_I d_I = 1_I \otimes I,
(I2) \quad t_I \otimes I = \Delta_I = l_I = r_I = d_I^{-1},
(I3) \quad d_I = r_I^{-1} = l_I^{-1},
(I4) \quad d_I \otimes d_I = d_I \otimes I.
Lemma 2.8. Let $K$ be an $hdht\nabla s$-category. Then one has
\[ (T9) \quad \rho A A' d I = d A (\rho A A' \otimes t A) \]
for all objects $A, A' \in |K|$ and all morphisms $\rho \in K[A, A']$.

Moreover:

(i) \quad $\forall A, A' \in |K| \forall \rho \in K[A, A'] (\rho d A A' = d A (\rho d A A' \otimes (\rho \otimes \rho)))$
\[ \Rightarrow \rho d A A' = d A (\rho d A A' \otimes (\rho \otimes \rho)) \nabla A A', \]
(ii) \quad $\forall A, A' \in |K| \forall \rho \in K[A, A'] (\nabla A A' \rho d A A' = d A (\nabla A A' \rho \otimes (\rho \otimes \rho)) \nabla A A')$
\[ \Rightarrow \nabla A A' \rho = d A (\nabla A A' \rho \otimes (\rho \otimes \rho) \nabla A A'), \]
(iii) \quad $\forall A, A' \in |K| \forall \rho \in K[A, A'] (\rho t A A' d I = d A (\rho t A A' \otimes t A))$
\[ \Leftrightarrow \rho A A' = d A (\rho A A' \otimes t A) \nabla I. \]

Proof. Because of $\nabla I d I = 1_I \otimes I$ and $\nabla I = r I = l I = t I \otimes I$ the equation
\[ d A (\rho A A' \otimes t A) = d A (\rho A A' \otimes t A) \nabla I d I = d A (\rho A A' \otimes t A) r I d I \]
\[ = d A (1_A \otimes t A) r A \rho A A' d I = \rho A A' d I \]
is valid for each $\rho \in K[A, A']$ and all $A, A' \in |K|$, hence $K$ fulfils condition (T9).

The condition (T9') is equivalent to (T9), since
\[ d A (\rho A A' \otimes t A) = \rho A A' d I \Rightarrow d A (\rho A A' \otimes t A) \nabla I = \rho A A' \]
by $d I \nabla I = 1_I$ and
\[ d A (\rho A A' \otimes t A) \nabla I = \rho A A' \Rightarrow d A (\rho A A' \otimes t A) = \rho A A' d I \]
by $\nabla I d I = 1_I \otimes I$, hence property (iii) is shown.

The implications (i) and (ii) are satisfied because of the general property
\[ \xi d B = d A (\xi \otimes \eta) \Rightarrow \xi = \xi d B \nabla B = d A (\xi \otimes \eta) \nabla B. \]

Remark 2.9. The opposite of the implications (i) and (ii), respectively, is not true in general, since there are examples in $\text{Rel}$. 

Remark 2.10. As in any dht∇s-category, the morphisms

\[ p_{1}^{A,B} := (1_{A} \otimes t_{B})r_{A} \in K[A \otimes B, A] \cap F^{K}, \]
\[ p_{2}^{A,B} := (t_{A} \otimes 1_{B})l_{B} \in K[A \otimes B, B] \cap F^{K} \]

of an arbitrary hdht∇s-category \( K \) are called canonical projections again and one has

\[ \nabla_{A} = \inf \{ p_{1}^{A,A}, p_{2}^{A,A} \} = d_{A} \left( p_{1}^{A,A} \otimes p_{2}^{A,A} \right) \nabla_{A} \]

for all \( A \in |K| \).

Remark that \((A \otimes B; p_{1}^{A,B}, p_{2}^{A,B})\) is not a categorical product in the whole category \( K \), but in the subcategory \( T^{K} \).

The family \( \nabla = (\nabla_{A} \mid A \in |K|) \) is uniquely determined by the family \( d = (d_{A} \mid A \in |K|) \) and the conditions \((\nabla 1)\) and \((\nabla 2)\).

Lemma 2.11. Let \( K \) be an arbitrary hdht∇s-category. Then there holds:

\[ \forall A, B, C \in |K| \quad \forall \rho, \rho' \in K[A, B] \quad \forall \sigma, \sigma' \in K[B, C] \quad (d_{A}(\rho \otimes \rho')\nabla_{B} = \rho \wedge d_{B}(\sigma \otimes \sigma')\nabla_{C} = \sigma) \Rightarrow d_{A}(\rho \otimes \rho')p_{i}^{B,B} = \rho \quad (i \in \{1, 2\}), \]

\[ \forall A, B \in |K| \quad \forall \rho, \sigma \in K[A, B] \quad (d_{A}(\rho \otimes \sigma)\nabla_{B} = \rho \wedge d_{A}(\sigma \otimes \sigma) = \sigma d_{B}) \Rightarrow d_{A}(\rho \otimes \sigma)p_{i}^{B,B} = \rho \quad (i \in \{1, 2\}), \]

\[ \forall A, B \in |K| \quad \forall \rho, \sigma \in K[A, B] \quad (d_{A}(\rho \otimes \sigma)\nabla_{B} = \rho \wedge d_{A}(\sigma \otimes \sigma) = \sigma d_{B}) \Rightarrow d_{A}(\rho \otimes \rho') = \rho d_{B}, \]

\[ \forall A \in |K| \quad \forall \rho \in K[A, A] \quad (d_{A}(1_{A} \otimes \rho)\nabla_{A} = \rho) \Rightarrow d_{A}(1_{A} \otimes \rho)p_{1}^{A,A} = d_{A}(1_{A} \otimes \rho)p_{2}^{A,A} = \rho. \]

**Proof.** Axiom \((\ast 1)\) implies condition \((\ast 2)\) because of \( \rho \leq \rho' \wedge \sigma \leq \sigma' \Rightarrow \rho \sigma \leq \rho' \sigma' \). To show \((\ast 3)\) not that \( d_{A}(\rho \otimes \sigma)\nabla_{B} = \rho \iff \rho \leq \sigma \) and \( d_{A}(\sigma \otimes \sigma) = \sigma d_{B} \iff \sigma \in F^{K} \). So one obtains
\[ d_A(\rho \otimes \sigma)p_{1,B}^{B,B} = d_A(d_A(\rho \otimes \sigma)\nabla_B \otimes \sigma)p_{1,B}^{B,B} \quad (\rho \leq \sigma) \]
\[ = d_A(\rho \otimes d_A(\sigma \otimes \sigma))a_{B,B,B}(\nabla_B \otimes 1_B)p_{1,B}^{B,B} \quad (\sigma \in F_K) \]
\[ = d_A(\rho \otimes \sigma)(1_B \otimes d_B)a_{B,B,B}(\nabla_B \otimes 1_B)p_{1,B}^{B,B} \quad ((F4)) \]
\[ = d_A(\rho \otimes \sigma)\nabla_B d_Bp_{1,B}^{B,B} \quad ((\nabla7)) \]
\[ = d_A(\rho \otimes \sigma)\nabla_B = \rho \]

with respect to the axioms of an \(hdht\nabla_s\)-category.

The property \((*4)\) is a consequence of \((D9')\) and \((T9')\):

\[ \rho = \rho d_Bp_{1,B}^{B,B} \leq d_A(\rho \otimes \rho)p_{1,B}^{B,B} \land \rho t_B \leq t_A \]
\[ \Rightarrow d_A(\rho \otimes \rho)p_{1,B}^{B,B} = d_A(\rho \otimes \rho t_B)r_B \leq d_A(\rho \otimes t_A)r_B = d_A(1_A \otimes t_A)r_A = \rho \]
\[ \land \quad d_A(\rho \otimes \rho)p_{2,B}^{B,B} = d_A(\rho t_B \otimes \rho)t_B \leq d_A(t_A \otimes \rho)t_B = d_A(t_A \otimes 1_A)t_A = \rho. \]

\((*5)\): Using the previous results and the assumption one obtains

\[ d_A(\rho \otimes \rho) = d_A(d_A(\rho \otimes \sigma)p_{2,B}^{B,B} \otimes d_A(\rho \otimes \sigma))p_{2,B}^{B,B} \]
\[ = d_A(d_A(\rho \otimes \rho) \otimes (\rho \otimes \sigma))p_{2,B}^{B,B} \otimes p_{2,B}^{B,B} \]
\[ = d_A(d_A(\rho \otimes \sigma) \otimes (\rho \otimes \sigma))p_{2,B}^{B,B} \otimes p_{2,B}^{B,B} \]
\[ = d_A(d_A(\rho \otimes \rho) \otimes d_A(\sigma \otimes \sigma))b_{B,B,B,B}(p_{2,B}^{B,B} \otimes p_{2,B}^{B,B}) \]
\[ = d_A(d_A(\rho \otimes \rho) \otimes \sigma d_B)p_{2,B,B,B}^{B,B} \]
\[ = d_A(\rho \otimes d_A(\rho \otimes \sigma))a_{B,B,B}(1_B \otimes d_B)p_{2,B,B,B}^{B,B} \]
\[ = d_A(\rho \otimes d_A(\rho \otimes \sigma))a_{B,B,B}(p_{2,B,B,B}^{B,B} \otimes d_B) \]
\[ = d_A(\rho \otimes d_A(\rho \otimes \sigma))(1_B \otimes p_{2,B,B,B}^{B,B} \otimes d_B) \]
\[ = d_A(\rho \otimes d_A(\rho \otimes \sigma))(p_{2,B,B,B}^{B,B} \otimes d_B) \]
\[ = d_A(\rho \otimes \rho)p_{2,B,B,B}^{B,B} \otimes d_B = \rho d_B. \]

The property \((*6)\) arises from \((*3)\) because of \(1_A \in F_K\) for each \(A \in |K|\).
Lemma 2.12. Let $K$ be a monoidal symmetric category endowed with morphisms families $d, t, (o_{A,B} | A, B \in |K|)$, and $\nabla$ such that all axioms of an $hdht\nabla s$-category without (*1) are fulfilled. Moreover, let the condition (*2) be valid. Then $K$ is an $hdht\nabla s$-category in the defined sense as above.

Proof. It remains to show the condition (*1):

$$d_A(d_A(\rho \otimes \rho') \nabla_B d_B(\sigma \otimes \sigma') \nabla_C \otimes d_A(\rho \otimes \rho' \sigma') \nabla_C) \nabla_C$$

$$= d_A(\rho \sigma \otimes d_A(\rho \sigma \otimes \rho' \sigma') \nabla_C) \nabla_C$$

$$= d_A(1_A \otimes d_A)(\rho \sigma \otimes (\rho \sigma \otimes \rho' \sigma'))(1_C \otimes \nabla_C) \nabla_C$$

$$= d_A(d_A \otimes 1_A)a_{A,A,A}^{-1}(\rho \sigma \otimes (\rho \sigma \otimes \rho' \sigma'))(1_C \otimes \nabla_C) \nabla_C$$

$$= d_A(d_A \otimes 1_A)((\rho \sigma \otimes \rho) \otimes \rho' \sigma')a_{C,C,C}^{-1}(1_C \otimes \nabla_C) \nabla_C$$

$$= d_A(d_A)(\rho \sigma \otimes \rho) \nabla_C \otimes 1_C) \nabla_C$$

$$= d_A(d_A)(\rho \sigma \otimes \rho \sigma) -(\rho \sigma \otimes \rho) \otimes 1_C) \nabla_C$$

$$= d_A(\rho \otimes \rho') \nabla_B d_B(\sigma \otimes \sigma') \nabla_C$$

The results of the last both lemmata are important for the axiomization of $hdht\nabla s$-categories. The system of axioms for an $hdht\nabla s$-category given in [11] contains two identical implications, namely (21) ($\Leftrightarrow$ (*2)) and (20) ($\Leftrightarrow$ (*6)). The property (*6) is a consequence of the other properties and the conditions (*1) and (*2) are equivalent in a monoidal symmetric category $K$ endowed with morphisms families $d, t, (o_{A,B} | A, B \in |K|)$, and $\nabla$ such that

(D1), (D2), (D3), (D5), (D7), (D8), (T1), (T2), (T6), (T9'),

(\nabla1), (\nabla2), (\nabla3), (\nabla4), (\nabla5), (\nabla6), (\nabla7), (D\nabla),

(o1), (o2), (O1)

are fulfilled. Therefore, $hdht\nabla s$-categories are axiomatizable by identities only, hence all small $hdht\nabla s$-categories form a variety of many-sorted total
algebras and there are free many-sorted algebras to each generating set with respect to this variety. Especially, there are free $hdht\nabla s$-theories, i.e. free algebraic theories for relational structures, by analogy with the existence of free algebraic theories for partial algebras ([3], [10]).

**Lemma 2.13.** In any $hdht\nabla$-symmetric category the following conditions are fulfilled for arbitrary morphisms $\rho, \sigma$:

(j) \[ \rho\sigma = 1_A \land \sigma \rho \leq 1_B \Rightarrow d_A(\rho \otimes \rho) = \rho d_B \]

(jj) \[ \rho\sigma \leq 1_A \land \sigma \rho = 1_B \Rightarrow \nabla_A \rho = (\rho \otimes \rho) \nabla_B \]

**Proof.** To show (j) we use at first the known property $\sigma d_A \leq d_B(\sigma \otimes \sigma)$. Further,

\[ d_A(\rho \otimes \rho) = \rho \sigma d_A(\rho \otimes \rho) \leq \rho d_B(\sigma \otimes \sigma)(\rho \otimes \rho) \leq \rho d_B(1_B \otimes 1_B) = \rho d_B, \]

hence $d_A(\rho \otimes \rho) = \rho d_B$ by $\rho d_B \leq d_A(\rho \otimes \rho)$.

In a similar way one shows the statement (jj), namely because of $\nabla_B\sigma \leq (\sigma \otimes \sigma)\nabla_A$ and

\[ (\rho \otimes \rho)\nabla_B = (\rho \otimes \rho)\nabla_B\sigma \rho \leq (\rho \sigma \otimes \rho\sigma)\nabla_A \rho \leq \nabla_A \rho \leq (\rho \otimes \rho)\nabla_B \]

one has $\nabla_A \rho = (\rho \otimes \rho)\nabla_B$.

**Definition 2.14.** Morphisms $e \in K[A, A] \subseteq K$ with the property $e \leq 1_A$, i.e. $e = d_A(1_A \otimes e)\nabla_A$, are called *subidentities* in $K$ (compare with ([7])).

**Proposition 2.15** (cf. [7]). For each morphism $\rho : A \to B$, $A, B \in |K|$, the morphism

\[ \alpha(\rho) := d_A(\rho \otimes 1_A)p^{B,A}_2 \]

is a subidentity of $A$ in $K$ and there holds $\alpha(\rho)\rho = \rho$. Each subidentity $e$ of $K$ fulfils $d_A(e \otimes e) = cd_A$, therefore the subidentities of $K$ are the subidentities of $F^K$ and satisfy the following conditions for all suitable morphisms and objects of $K$: 

\[ \text{(j)} \quad \rho\sigma = 1_A \land \sigma \rho \leq 1_B \Rightarrow d_A(\rho \otimes \rho) = \rho d_B \]

\[ \text{(jj)} \quad \rho\sigma \leq 1_A \land \sigma \rho = 1_B \Rightarrow \nabla_A \rho = (\rho \otimes \rho) \nabla_B \]
\[(e_1)\quad e \leq 1_A \implies ee = e,\]
\[(e_2)\quad e_1, e_2 \leq 1_A \implies e_1e_2 = e_2e_1 = \inf\{e_1, e_2\},\]
\[(e_3)\quad e_1 \leq e_2 \leq 1_A \iff e_1 = e_1e_2 \leq 1_A,\]
\[(e_4)\quad e \leq 1_A \iff \alpha(e) = e,\]
\[(e_5)\quad e \leq 1_A \implies ed_A = d_A(e \otimes e) = d_A(e \otimes 1_A),\]
\[(e_6)\quad e \leq 1_A \implies \nabla_A e = (e \otimes e)\nabla_A = (e \otimes 1_A)\nabla_A,\]
\[(e_7)\quad \rho, \sigma \in K[A, B] \implies \alpha(\rho)\sigma = d_A(\rho \otimes \sigma)p_2^{B,B} \land \alpha(\sigma)\rho = d_A(\rho \otimes \sigma)p_1^{B,B},\]
\[(e_8)\quad \alpha(\rho)\sigma = \rho \implies \rho \leq \sigma,\]
\[(e_9)\quad e \rho = \rho \land e \leq 1_A \implies \alpha(\rho) \leq e \leq 1_A,\]
\[(e_{10})\quad \text{cod} \rho = \text{dom} \sigma \implies \alpha(\rho \sigma) \leq \alpha(\rho),\]
\[(e_{11})\quad e \leq 1_A \implies \alpha(e \rho) \leq e,\]
\[(e_{12})\quad e \leq 1_A \implies \alpha(e \rho) = e\alpha(\rho),\]
\[(e_{13})\quad \rho \leq \sigma \implies \alpha(\rho) \leq \alpha(\sigma),\]
\[(e_{14})\quad \text{cod} \rho = \text{dom} \sigma \implies \rho \alpha(\sigma) \leq \alpha(\rho \sigma) \rho,\]
\[(e_{15})\quad \text{cod} \rho = \text{dom} \sigma \implies \alpha(\rho \sigma) = \alpha(\rho \alpha(\sigma)).\]

**Proof.** Because of \(pt_B \leq t_A\) one obtains
\[
\alpha(\rho) = d_A(\rho \otimes 1_A)p_2^{B,A} = d_A(pt_B \otimes 1_A)l_A \leq d_A(t_A \otimes 1_A)l_A = 1_A.
\]

Using the definition of \(\alpha(\rho)\), properties \((M14), (M15)\), and \(\alpha(\rho) \leq 1_A\) one receives \(\alpha(\rho)\rho = \rho\) via
\[
\alpha(\rho)\rho = d_A(\rho \otimes 1_A)p_2^{B,A} \rho = d_A(\rho \otimes \rho)p_2^{B,B} \geq \rho d_B p_2^{B,B} = \rho = 1_A \rho \geq \alpha(\rho)\rho.
\]

Because of \(e \leq 1_A\) the property \(d_A(e \otimes e) = ed_A\) is a consequence of Lemma 2.11, \((*5)\), and the subidentities of \(K\) are exactly the subidentities of \(F^K\), therefore, all subidentities have the properties \((e_1), (e_2), (e_3)\) and \((e_4)\) (cf. \[7\]).
To show property (e5) use the property (e4) $e \leq 1_A \Rightarrow e = \alpha(e) = d_A(e \otimes 1_A)p_{2,A}^A$:

$$d_A(e \otimes e) = d_A(e \otimes d_A(e \otimes 1_A)p_{2,A}^A) = d_A(d_A(e \otimes e) \otimes 1_A)d_{A,A,A}(1_A \otimes p_{2,A}^A)$$

$$= d_A(d_A(e \otimes e)p_{1,A}^A \otimes 1_A) = d_A(e \otimes 1_A).$$

The second part of the property (e6) is a consequence of (e2) and (e5) owing to

$$\nabla_A d_A \leq 1_A \otimes A \otimes 1_A \otimes A,$$

and

$$\nabla_A d_A \leq 1_A \otimes A \otimes 1_A \otimes A.$$ 

To show (e8) take into consideration

$$\rho = \alpha(\rho) \sigma \leq 1_A \sigma = \sigma.$$ 

(e9): Assuming $e \rho = \rho$, $e \leq 1_A$ one gets

$$\alpha(\rho) = \alpha(e \rho) = d_A(e \rho \otimes 1_A)p_{2,A}^B = d_A(e \rho t_B \otimes 1_A)l_A \leq d_A(e t_A \otimes 1_A)l_A = \alpha(e) = e.$$ 

Conversely, $\alpha(\rho) \leq e \leq 1_A$ yields

$$\rho = \alpha(\rho) \rho \leq e \rho \leq 1_A \rho = \rho.$$ 

Condition (e10) is true, since

$$\alpha(\rho \sigma) = d_A(\rho \sigma \otimes 1_A)p_{2,A}^C = d_A(\rho \sigma t_C \otimes 1_A)l_A \leq d_A(\rho t_B \otimes 1_A)l_A = \alpha(\rho).$$
Condition (e11) arises from $\alpha(e\rho) \leq \alpha(e) = e$.

Property (e12) is a consequence of (e5) as follows:

$$
\alpha(e\rho) = d_A(e\rho \otimes 1_A)p_2^{B,A} = d_A(e \otimes 1_A)(\rho \otimes 1_A)p_2^{B,A} \\
= d_A(e \otimes e)(\rho \otimes 1_A)p_2^{B,A} = ed_A(\rho \otimes 1_A)p_2^{B,A} \\
= e\alpha(\rho).
$$

To show (e13) use the definitions of $\leq$ and $\alpha(\rho)$ ($\rho : A \rightarrow B$, $\sigma : B \rightarrow C$):

$$
\alpha(\rho) = d_A(\rho \otimes 1_A)p_2^{B,A} = d_A(d_A(\rho \otimes \sigma)\nabla_B \otimes 1_A)p_2^{B,A} \quad (\rho \leq \sigma) \\
\leq d_A(d_A(\rho \otimes \sigma)p_2^{B,B} \otimes 1_A)p_2^{B,A} \quad (\nabla_B \leq p_2^{B,B}) \\
= d_A(d_A(\rho \otimes 1_A)p_2^{B,A} \sigma \otimes 1_A)p_2^{B,A} \quad ((M14)) \\
= d_A(\alpha(\rho)\sigma \otimes 1_A)p_2^{B,A} \\
\leq d_A(\sigma \otimes 1_A)p_2^{B,A} = \alpha(\sigma). \quad (\alpha(\rho)\sigma \leq \sigma)
$$

Assertion (e14) is true since

$$
\rho\alpha(\sigma) = \rho d_B(\sigma \otimes 1_B)p_2^{C,B} \leq d_A(\rho\sigma \otimes \rho)p_2^{C,B} = \alpha(\rho\sigma)\rho.
$$

Condition (e15) follows by (e10), (e13), and (e14):

Let $\rho$ and $\sigma$ be as above. Then one has

$$
\alpha(\rho\sigma) = \alpha(\rho\alpha(\sigma)\sigma) \leq \alpha(\rho\alpha(\sigma)),
$$

hence

$$
\alpha(\rho\sigma) \leq \alpha(\rho\alpha(\sigma)) \leq \alpha(\alpha(\rho\sigma)\rho) \leq \alpha(\alpha(\rho\sigma)\alpha(\rho)) \\
\leq \alpha(\alpha(\rho\sigma)1_A) = \alpha(\rho\sigma) = \alpha(\rho\sigma).
$$

Remark that, as an easy example shows, in Rel the opposite implication to (e8) is not true: Let be given $A = \{a\}$, $B = \{b_1, b_2\}$, $\rho = \{(a, b_1)\}$, $\sigma = \{(a, b_1), (a, b_2)\}$. Then $\rho \leq \sigma$ and $\rho < \alpha(\rho)\sigma = \sigma$.

Furthermore, the equality in (e14) is not true in general. For this let be the sets $A$ and $B$ as above and let be $C = \{x\}$. For the relations $\sigma$ as above and $\tau = \{(b_1, x)\}$ one obtains $\sigma\alpha(\tau) = \{(a, b_1)\}$ and $\sigma\tau = \{(a, x)\}$, hence $\alpha(\sigma\tau) = \{(a, a)\}$, consequently $\alpha(\sigma\tau)\sigma = \{(a, b_1), (a, b_2)\} = \sigma \neq \sigma\alpha(\tau)$.
References


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