ON THE STRUCTURE OF
HALFDIAGONAL-HALFTERMINAL-SYMMETRIC
CATEGORIES WITH DIAGONAL INVERSIONS

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Dedicated to Hans-Jürgen Hoehnke on the occasion of his 75th birthday.

Abstract

The category of all binary relations between arbitrary sets turns out to be a certain symmetric monoidal category $\text{Rel}$ with an additional structure characterized by a family $d = \{d_A : A \to A \otimes A \mid A \in |\text{Rel}|\}$ of diagonal morphisms, a family $t = \{t_A : A \to I \mid A \in |\text{Rel}|\}$ of terminal morphisms, and a family $\nabla = \{\nabla_A : A \otimes A \to A \mid A \in |\text{Rel}|\}$ of diagonal inversions having certain properties. Using this properties in [11] was given a system of axioms which characterizes the abstract concept of a halfdiagonal-halfterminal-symmetric monoidal category with diagonal inversions ($\text{hdht}\nabla$-category). Besides of certain identities this system of axioms contains two identical implications. In this paper is shown that there is an equivalent characterizing system of axioms for $\text{hdht}\nabla$-categories consisting of identities only. Therefore, the class of all small $\text{hdht}\nabla$-symmetric categories (interpreted as heterogeneous algebras of a certain type) forms a variety and hence there are free theories for relational structures.

Keywords: halfdiagonal-halfterminal-symmetric category, diagonal inversion, partial order relation, subidentity, equation.

2000 AMS Subject Classification: 18D10, 18B10, 18D20, 08A05, 08A02.
1. Defining conditions

Let $K^\bullet$ be any symmetric monoidal category in the sense of Eilenberg-Kelly ([2]) with the object class $|K|$, the morphism class $K$, the distinguished object $I$, the bifunctor $\otimes : K \times K \to K$, and the families $a$, $r$, $l$, $s$ of isomorphisms of $K$ such that the following axioms are valid for all objects and all morphisms of $K$. By $K[A, B]$ we denote the set of all morphisms $\rho \in K$ with the domain (source) $\text{dom} \rho = A$ and the codomain (target) $\text{codom} \rho = B$.

Bifunctor properties:

(F1) $\text{dom}(\rho \otimes \rho') = \text{dom} \rho \otimes \text{dom} \rho'$,

(F2) $\text{codom}(\rho \otimes \rho') = \text{codom} \rho \otimes \text{codom} \rho'$,

(F3) $1_{A \otimes B} = 1_A \otimes 1_B$,

(F4) $(\rho \otimes \rho')(\sigma \otimes \sigma') = \rho \sigma \otimes \rho' \sigma'$.

Conditions of monoidality:

(M1) $a_{A,B,C,D}a_{A \otimes B,C,D} = (1_A \otimes a_{A,B,C})a_{A,B \otimes C,D}(a_{A,B,C} \otimes 1_D)$,

(M2) $a_{A,B}(r_A \otimes 1_B) = 1_A \otimes l_B$,

(M3) $a_{A,B,C}a_{A \otimes B,C}a_{C,A,B} = (1_A \otimes s_{B,C})a_{A,C,B}(s_{A,C} \otimes 1_B)$,

(M4) $s_{A,B}s_{B,A} = 1_{A \otimes B}$,

(M5) $s_{A,I}l_A = r_A$,

(M6) $a_{A,B,C}((\rho \otimes \sigma) \otimes \tau) = (\rho \otimes (\sigma \otimes \tau))a_{A',B',C'}$,

(M7) $r_A \rho = (\rho \otimes 1_I)r_{A'}$,

(M8) $s_{A,B}(\sigma \otimes \rho) = (\rho \otimes \sigma)s_{A',B'}$.

Remark that the validity of an equation containing morphism compositions includes that they are defined on both sides.

An immediate consequence of the conditions above is the validity of

(M9) $\forall A, B \in |K| \ (a_{I,A,B}(l_A \otimes 1_B) = l_{A \otimes B})$,

(M10) $\forall A, B \in |K| \ (a_{A,B,I}r_{A \otimes B} = 1_A \otimes r_B)$,

(M11) $r_I = l_I$,

(M12) $s_{I,I} = 1_{I \otimes I}$. 
(M13) \( \forall A \in |K| \) \((s_I,Ar_A = l_A)\),
(M14) \( \forall A \in |K| \) \((l_A \rho = (1_I \otimes \rho)l_{A'})\).

Using the denotation
\[
b_{A,B,C,D} := a_{A\otimes B,C,D}(a_{A,B,C}^{-1}(1_A \otimes s_{B,C})a_{A,C,B} \otimes 1_D)a_{A\otimes C,B,D}^{-1}
\]
one obtains the following properties for all objects \(A, A', B, B', C, C', D, D'\) of \(K\) and all morphisms \(\rho \in K[A, A'], \sigma \in K[B, B'], \lambda \in K[C, C'], \mu \in K[D, D']\):

(M15) \(b_{A,B,C,D}((\rho \otimes \sigma) \otimes (\lambda \otimes \mu)) = ((\rho \otimes \lambda) \otimes (\sigma \otimes \mu))b_{A',B',C',D'}\),

(M16) \(b_{A,I,I,B} = 1_{A\otimes I} \otimes 1_{I\otimes B}\),

(M17) \(b_{A,B,C,D}b_{A,C,B,D} = 1_{A\otimes B} \otimes 1_{C\otimes D}\),

(M18) \(b_{A,B,C,D}(s_{A,C} \otimes s_{B,D}) = s_{A\otimes B,C\otimes D}b_{C,D,A,B}\).

Obviously, all morphisms \(b_{A,B,C,D}\) are isomorphisms in the category \(K^\bullet\).

**Definition 1.1 ([1]).** A diagonal-terminal-symmetric category (shortly dts-category) \(K = (K^\bullet, d, t)\) is defined as a symmetric monoidal category endowed with morphism families

\[
d = (d_A : A \rightarrow A \otimes A \mid A \in |K|) \quad \text{and} \quad t = (t_A : A \rightarrow I \mid A \in |K|)
\]
satisfying the following conditions for all objects \(A, B, A' \in |K|\) and all morphisms \(\rho \in K[A, A']\).

**Diagonality:**

(D1) \(d_A(d_A \otimes 1_A) = d_A(1_A \otimes d_A)a_{A,A,A}\),
(D2) \(d_A s_{A,A} = d_A\),
(D3) \(d_A \otimes B = (d_A \otimes d_B)b_{A,A,B,B}\),
(D4) \(d_A(\rho \otimes \rho) = \rho d_{A'}\).

**Terminality:**

(T1) \(d_A(1_A \otimes t_A)r_A = 1_A\),
(T2) \(t_I = 1_I\),
(T3) \(\rho t_{A'} = t_A\).
Let $A, A', B$ be arbitrary objects in $K$ and let $\rho \in K[A, A']$ be any morphism in $K$. Then the properties

\[(D5) \quad d_A(d_A \otimes d_A) = d_A d_{A \otimes A},\]
\[(D6) \quad d_A(d_A \otimes d_A) = d_A(d_A \otimes d_A)b_{A, A, A, A},\]
\[(D7) \quad t_A d_I = d_A(t_A \otimes t_A),\]
\[(D9) \quad \rho d_A d_{A' \otimes A'} = d_A(\rho d_A \otimes d_A(\rho \otimes \rho)),\]
\[(T4) \quad d_A(t_A \otimes 1_A)l_A = 1_A,\]
\[(T5) \quad d_{A \otimes B}((1_A \otimes t_B)r_A \otimes (t_A \otimes 1_B)l_B) = 1_{A \otimes B},\]
\[(T6) \quad t_{A \otimes B} = (t_A \otimes t_B)t_{I \otimes I},\]
\[(T7) \quad r_I = t_{I \otimes I},\]
\[(T8) \quad d_A t_{A \otimes A} = t_A,\]
\[(T9) \quad \rho t_A d_I = d_A(\rho t_{A'} \otimes t_A)\]

are consequences of the conditions above ([1]).

The category $\text{Set}$ of all total functions between arbitrary sets is a model of a $\mathrm{dts}$-category by

$I := \{\emptyset\}, \quad A \otimes B := \{(a, b) \mid a \in A \land b \in B\},$
\[\rho \in \text{Set}[A, B] :\Leftrightarrow \rho = \{(a, b) \mid a \in A \land b = \rho(a) \in B\},\]
\[\forall a \in A \exists! b \in B (b = \rho(a)),\]
\[\rho \in \text{Set}[A, B], \quad \sigma \in \text{Set}[B, C] \Rightarrow \rho \circ \sigma = \{(a, c) \mid a \in A \land c = \sigma(\rho(a))\},\]
\[(a, c) \in \rho \circ \sigma \Leftrightarrow \exists b \in B ((a, b) \in \rho \land (b, c) \in \sigma),\]
\[\rho \in \text{Set}[A, B], \quad \rho' \in \text{Set}[A', B'] \Rightarrow \rho \otimes \rho' = \{((a, a'), \langle \rho(a), \rho'(a') \rangle) \mid a \in A, a' \in A'\},\]
\[a_{A, B, C} := \{(\langle a, b, c \rangle), \langle a, b, c \rangle) \mid a \in A, b \in B, c \in C\},\]
\[s_{A, B} := \{(\langle a, b, a \rangle) \mid a \in A, b \in B\},\]
\[r_A := \{(a, \emptyset, a) \mid a \in A\},\]
\[l_A := \{(\emptyset, a, a) \mid a \in A\},\]
\[d_A := \{(a, \langle a, a \rangle) \mid a \in A\},\]
\[t_A := \{(a, \emptyset) \mid a \in A\}.\]
Remark that $I$ is a terminal object in any dts-category $K$ and $(A \otimes B; p_{1}^{A,B}, p_{2}^{A,B})$ forms a categorical product of the objects $A, B$ in the category $K$, where $p_{1}^{A,B} := (1_{A} \otimes t_{B})r_{A}$ and $p_{2}^{A,B} := (t_{A} \otimes 1_{B})l_{B}$.

Moreover, $d_{A}(\rho \otimes \sigma) = \rho d_{B}$ is equivalent to $\rho = \sigma$ for all $A, B \in |K|$ and all $\rho, \sigma \in K[A, B]$ because of

\[ \sigma = \sigma d_{B}p_{2}^{B,B} = d_{A}(\sigma t_{B} \otimes \sigma)l_{B} = d_{A}(t_{A} \otimes \sigma)l_{B} = d_{A}(\rho t_{B} \otimes \sigma)l_{B} = \rho d_{B}p_{2}^{B,B} = \rho. \]

The morphisms $p_{1}^{A,B}$ and $p_{2}^{A,B}$ are called canonical projections in the category $K$.

Conditions (D9) and (T9) are equivalent to
\[ \rho d_{A'} = d_{A}(\rho d_{A'} \otimes d_{A}(\rho \otimes \rho))p_{2}^{A',A'} \quad \text{and} \quad \rho t_{A'} = d_{A}(\rho t_{A'} \otimes t_{A})p_{2}^{I,I}, \]
respectively.

**Definition 1.2.** Let $K^{•}$ be again a symmetric monoidal category endowed with morphism families $d$ and $t$ as above. Then $K = (K^{•}, d, t)$ is called halfdiagonal-terminal-symmetric category (shortly hdts-category), if the conditions (D1), (D2), (D3), (D5), (D7), (T1), (T2), (T3) hold identically.

As above, the identities (T4), (T5), (T6), (T7), (T8), (T9) follow from the defining conditions in an hdts-category.

**Definition 1.3.** A diagonal-halfterminal-symmetric category (shortly dhts-category) ([3], [7], [10]) is defined as a sequence $K := (K^{•}; d, t, O, o)$ such that $K^{•}$ is again a symmetric monoidal category, $d$ and $t$ are families as above, $O$ is a distinguished zero-object of $K^{•}, o : I \to O$ is a distinguished morphism of $K^{•}$, and the following equations are fulfilled for all objects $A, B, A', B' \in |K|$ and all morphisms $\rho \in K[A, A'], \ sigma \in K[B, B'], \ lambda \in K[A, O], \ k \in K[O, A]$

(D4), (T1), (T4), (T5), (T6), and

\[ (o1) \quad t_{A}o = \lambda, \]
\[ (o2) \quad (1_{A} \otimes t_{O})r_{A} = \kappa, \]
\[ (O1) \quad A \otimes O = O \otimes A = O. \]
Remark that the conditions (D1), (D2), (D3), (D5), (D6), (D7), (D9), (T2), (T7), (T8), (T9), and (B1) are consequences of the other conditions ([3], [7], [10]).

Formulas (o1), (o2), and (O1) explain that the morphism sets \( K[A, O] \) and \( K[O, A] \) both consist of exactly one element \( o_{A,O} \) and \( o_{O,A} \), respectively, and \( O \) is a zero object in \( K \). In any dhts-category there is a so-called zero-morphism \( o_{A,B} \) to each pair of objects \( A, B \in |K| \) with the properties

\[
\begin{align*}
(\text{o3}) \quad & \forall \rho \in K[A, A'], \sigma \in K[B, B'] \ (\rho o_{A,B} = o_{A',B'} \land o_{A,B} \sigma = o_{A,B'}), \\
(\text{o4}) \quad & \forall \xi, \eta \in K \ (o_{A,B} \otimes \xi = o_{A,B} = \eta \otimes o_{A,B}), \\
(\text{o5}) \quad & o_{O,A} = (1_A \otimes o_{O}) r_A = (o_{O} \otimes 1_A) l_A.
\end{align*}
\]

The category Par of all partial functions between arbitrary sets is a model of a dhts-category by the same fixations as above and \( O = \emptyset \) (the empty set) and \( o : I \to O, o_{A,O} : A \to O, o_{O,A} : O \to A, o_{A,B} : A \to B \) as the empty functions. The morphisms are given by

\[
\rho \in K[A, B] \iff \rho = \{(a, \rho(a)) \mid a \in D(\rho) \land \rho(a) \in B\}, \\
\forall a \in D(\rho) \subseteq A \exists! b \in B \ (b = \rho(a)).
\]

The following fact is of importance for the consideration of dhts-categories.

**Lemma 1.4.** Let \( K \) be a symmetric monoidal category endowed with morphism families \( d \) and \( t \) as above which fulfil conditions (D4), (T1) and (T6). Then conditions (T4) and (T5) are consequences of the validity of (D2) and (D3) in \( K \).

**Proof:** Using (T1) and (D2) one obtains (T4) as follows:

\[
1_A = d_A(1_A \otimes t_A)r_A = d_A s_{A,A}(1_A \otimes t_A)r_A = d_A(t_A \otimes 1_A)s_{I,A}r_A = d_A(t_A \otimes 1_A)l_A.
\]
The calculation
\[ d_{A \otimes B}((1_A \otimes t_B)r_A \otimes (t_A \otimes 1_B)l_B) \]
\[ = (d_A \otimes d_B)b_{A,A,B,B}((1_A \otimes t_B)r_A \otimes (t_A \otimes 1_B)l_B) \quad ((D3)) \]
\[ = (d_A(1_A \otimes t_A) \otimes d_B(t_B \otimes 1_B)b_{A,I,B,I}(r_A \otimes l_B) \quad ((M15)) \]
\[ = (d_A(1_A \otimes t_A) \otimes d_B(t_B \otimes 1_B))(1_{A \otimes I} \otimes 1_{I \otimes B})(r_A \otimes l_B) \quad ((M16)) \]
\[ = (d_A(1_A \otimes t_A)r_A \otimes d_B(t_B \otimes 1_B)l_B) \quad ((F3)) \]
\[ = 1_A \otimes 1_B \quad ((T1),(T4)) \]
shows the validity of (T5).

Let \( K \) be an arbitrary dhts-category. Then all morphisms \( \rho \in K[A,A'] \), \( A, A' \in \mathcal{K} \), fulfilling \( pt_A' = t_A \), form a subcategory \( M^K \) of \( K \) which is even a dts-category. Denoting by \( M^K \) the smallest dts-subcategory of \( M^K \) containing all morphisms of the families \( a, r, l, s, d, t \) one has
\[ M^K \subseteq \text{Iso}(K) \subseteq \text{Cor}(K) \subseteq M^K, \]
where \( \text{Iso}(K) \) (\( \text{Cor}(K) \)) is a dts-subcategory of \( M^K \) generated by all isomorphisms (coretractions) of \( K \) together with all terminal morphisms of \( K \), since all coretractions and all terminal morphisms fulfil the condition (T3) (see [7], [10]).

The object \( I \in \mathcal{K} \) is a terminal object in the subcategories \( M^K \), \( \text{Iso}(K) \), \( \text{Cor}(K) \), and \( M^K \) but not in the whole category \( K \). Morphisms of the kind \( p^1_A = (1_A \otimes l_B)r_A \) and \( p^2_A = (t_A \otimes 1_B)l_B \) are called canonical projections again and \( (A \otimes B; p^1_{A,B}, p^2_{A,B}) \) is a categorical product of \( A \) and \( B \) in \( M^K \), but in general not in the whole category.

Schreckenberger had proved ([7]) that
\[ \rho \leq \sigma : \Leftrightarrow d_A(\rho \otimes \sigma) = \rho d_{A'} \quad (\rho, \sigma \in K[A,A']) \]
defines a partial order relation which is stable under composition and \( \otimes \)-operation. Moreover, the following are equivalent:
(i) \( d_A(\rho \otimes \sigma) = \rho d_{A'} \).

(ii) \( d_A(\rho \otimes \sigma)p_{A',A'}^2 = \rho \).

(iii) \( d_A(\sigma \otimes \rho)p_{A',A'}^1 = \rho \).

Hoehnke had shown ([3]) the validity of the identical implication

\[ \rho = d_A(\rho \otimes \sigma)p_{A',A'}^2 \Rightarrow \rho = d_A(\rho \otimes \sigma)p_{A',A'}^1. \]

The relation \( \leq \) in the dhts-category \( \text{Par} \) describes exactly the usual inclusion \( \subseteq \).

Morphisms \( e_A \in K[A,A] \) of any dhts-category \( K \) fulfilling \( e_A \leq 1_A \) for any \( A \in |K| \) are called subidentities ([7]). Especially, for each \( \rho \in K[A,B] \), the morphism

\[ \alpha(\rho) := d_A(\rho \otimes 1_A)p_{B,A}^2 = d_A(1_A \otimes \rho)p_{A,B}^1 \]

is a subidentity of \( A \in |K| \), since

\[
d_A(d_A(\rho \otimes 1_A)p_{B,A}^2 \otimes 1_A)p_{B,A}^{A,A} = d_A(\rho \otimes d_A(1_A \otimes 1_A))a_{B,A,A}(p_{B,A}^{B,A} \otimes 1_A)p_{B,A}^2
\]

\[
= d_A(\rho \otimes d_A(1_B \otimes p_{B,A}^{A,A}))p_{B,A}^2
\]

\[
= d_A(\rho \otimes d_A)p_{B,A}^2 = d_A(\rho \otimes 1_A)p_{B,A}^2.
\]

Important properties of subidentities are described in [7], [13], [15].

**Definition 1.5.** A diagonal-halfterminal-symmetric category with diagonal inversion \( \nabla \) (shortly dht\( \nabla \)-category, [10]) is, by definition, a sequence \( K := (K^*,d,t,\nabla,O,o) \) such that \( (K^*,d,t,O,o) \) is a dhts-category endowed with a morphism family \( \nabla = (\nabla_A| A \in |K|) \) satisfying the following for all \( A \in |K| \):

\[
(\nabla 1) \quad \nabla_A = 1_A,
\]

\[
(\nabla 2) \quad \nabla_A d_A d_{A \otimes A} = d_{A \otimes A}(\nabla_A d_A \otimes 1_{A \otimes A}).
\]

The category \( \text{Par} \) is also a model of a dht\( \nabla \)-category, where

\[ \nabla_A := \{(a,a)| a \in A\}, \quad A \in |\text{Par}|. \]
The properties

\((D8)\) \(\nabla d_A = d_{A \otimes A}(\nabla_A \otimes \nabla_A)\),

\((D9')\) \(\rho d_A' = d_A(\rho d_A' \otimes d_A(\rho \otimes \rho))\),

\((T9')\) \(\rho t_A' = d_A(\rho t_A' \otimes t_A)\),

\((\nabla 3)\) \(a_{A,A,A}(\nabla_A \otimes 1_A)\nabla_A = (1_A \otimes \nabla_A)\nabla_A\),

\((\nabla 4)\) \(s_{A,A} \nabla_A = \nabla_A\),

\((\nabla 5)\) \(\nabla_{A \otimes B} = b_{A,B,A,B}(\nabla_A \otimes \nabla_B)\),

\((\nabla 6)\) \(\nabla d_A = (d_A \otimes 1_A) a_{A,A,A}^{-1}(1_A \otimes \nabla_A)\),

\((\nabla 7)\) \(\nabla d_A = (1_A \otimes d_A) a_{A,A,A}(\nabla_A \otimes 1_A)\),

\((\nabla 8)\) \(\nabla d_A = (d_A \otimes d_A)\nabla_{A \otimes A}\),

\((\nabla 9)\) \(\nabla_A \rho d_A' = d_{A \otimes A}(\nabla_A \rho \otimes (\rho \otimes \rho))\),

\((\nabla 9')\) \(\nabla_A \rho = d_{A \otimes A}(\nabla_A \rho \otimes (\rho \otimes \rho))\),

\((\nabla 10)\) \(\nabla_{A \otimes A} \nabla_A = (\nabla_A \otimes \nabla_A)\nabla_A\),

\((D\nabla)\) \(\rho = d_A(\rho \otimes \rho)\nabla_A\)

follow from the axioms and the other properties of a dht\(\nabla\)-s-category for all \(A, A', B \in |K|\) and all \(\rho \in K[A,A']\) (see [13]).

By the definition of the partial order relation, \((T9)\) is equivalent to \(\rho t_A' \leq t_A\), \((\nabla 2)\) is equivalent to \(\nabla_A d_A \leq 1_{A^2}\), and \((\nabla 9)\) is equivalent to \(\nabla_A \rho \leq (\rho \otimes \rho)\nabla_A'\) for \(\rho \in K[A,A']\).

Moreover, one has the following important property in any dht\(\nabla\)-category \(K\) ([11]):

\((P\nabla)\) \(\forall A, A' \in |K| \forall \rho, \sigma \in K[A,A'] (d_A(\rho \otimes \sigma)p_{A,A}' = \rho \leftrightarrow d_A(\rho \otimes \sigma)\nabla_A' = \rho)\).

In any dht\(\nabla\)-category, conditions \((D9)\), \((T9)\), and \((\nabla 9)\) result in \((D9')\), \((T9')\), and \((\nabla 9')\), respectively.

### 2. hdht\(\nabla\)-categories

**Definition 2.1** ([10]). A sequence \(K = (K^\bullet; d, t, \nabla, o)\) is called halfdiagonal-halfterminal-symmetric monoidal category with diagonal inversion \(\nabla\) (shortly hdht\(\nabla\)-category), iff \(K^\bullet\) is a symmetric monoidal category as above,
are families of morphisms of $K$, and $o : I \to O$ ($I \neq O \in |K|$) is a distinguished morphism of $K$ such that for all objects and all morphisms of the underlying category $K$ the conditions

(D1), (D2), (D3), (D5), (D7), (D8),

(T1), (T2), (T6), (T9'),

(ν1), (ν2), (ν3), (ν4), (ν5), (Dν),

(o1), (o2), (O1),

and

(*1) $d_A(\rho \otimes \rho')ν_Bd_B(\sigma \otimes \sigma')ν_C$

$= d_A(d_A(\rho \otimes \rho')ν_Bd_B(\sigma \otimes \sigma')ν_C \otimes d_A(\rhoσ \otimes \rho'σ')ν_C)ν_C$

are fulfilled.

The system of axioms given in this definition is free of contradictions, because the category $\text{Rel}$ of all binary relations between sets is a model of it, i.e. $\text{Rel}$ fulfils all the axioms of an $\text{hdht}ν$-category, where $|\text{Rel}|$ is the class of all sets, the morphisms are characterized by

$\rho \in \text{Rel}[A, A'] :⇔ \rho = \{(a, a') \mid a \in D(\rho) \subseteq A \land a' \in W(\rho) \subseteq A' \land H(a, a')\}$,

where $H(x, y)$ is a sentence form in two variables, the distinguished objects are $I = \{\emptyset\}$ and $O = \emptyset$, the operation $\otimes$ for objects is given as in $\text{Set}$, the composition and the $\otimes$-operation of morphisms are described by

$\rho \in \text{Rel}[A, B], \sigma \in \text{Rel}[B, C] \Rightarrow \rho \circ \sigma = \{(a, c) \mid \exists b \in B (\{(a, b) \in \rho \land (b, c) \in \sigma\}\}$,

$\rho \in \text{Rel}[A, B], \rho' \in \text{Rel}[A', B'] \Rightarrow \rho \otimes \rho' = \{((a, a'), (b, b')) \mid (a, b) \in \rho \land (a', b') \in \rho'\}$,

and the morphisms of the families $a, r, l, s, b, d, t, ν, (0_{A,B} \mid A, B \in |\text{Rel}|)$ are as in $\text{Par}$.

Lemma 2.2. The relation $\leq$ defined by

$\rho \leq \sigma :⇔ d_A(\rho \otimes \sigma)ν_B = \rho$

is a partial order relation in any $\text{hdht}ν$-symmetric category which is compatible with composition and $\otimes$-operation for morphisms. Moreover, the greatest
lower bound of two morphisms $\lambda, \mu \in K[A, B]$ with respect to the canonical order relation $\leq$ is given by

$$d_A(\lambda \otimes \mu) \nabla_B = \inf \{\lambda, \mu\}.$$ 

**Proof.** Condition $(D\nabla)$ shows the reflexivity of $\leq$. The relation is antisymmetric because of

$$\rho \leq \sigma \land \sigma \leq \rho \Rightarrow \sigma = d_A(\sigma \otimes \rho) \nabla_B$$

$$= d_A s_A A(\sigma \otimes \rho) \nabla_B \quad ((D2))$$

$$= d_A(\rho \otimes \sigma) s_B B \nabla_B \quad ((M8))$$

$$= d_A(\rho \otimes \sigma) \nabla_B \quad ((\nabla4))$$

$$= \rho.$$

The implication

$$\rho \leq \sigma \land \sigma \leq \tau \Rightarrow \rho = d_A(\rho \otimes \sigma) \nabla_B$$

$$= d_A(\rho \otimes d_A(\sigma \otimes \tau) \nabla_B) \nabla_B$$

$$= d_A(1_A \otimes d_A)(\rho \otimes (\sigma \otimes \tau))(1_B \otimes \nabla_B) \nabla_B$$

$$= d_A(d_A \otimes 1_A)((\rho \otimes \sigma) \otimes \tau)a_{B,B,B}^{-1}(1_B \otimes \nabla_B) \nabla_B \quad ((M6), (D1))$$

$$= d_A(d_A(\rho \otimes \sigma) \otimes \tau)(\nabla_B \otimes 1_B) \nabla_B \quad ((\nabla3))$$

$$= d_A(d_A(\rho \otimes \sigma) \nabla_B \otimes \tau) \nabla_B$$

$$= d_A(\rho \otimes \tau) \nabla_B$$

$$\Rightarrow \rho \leq \tau$$

yields the transitivity of the relation $\leq$.

Now suppose $\rho \leq \sigma, \lambda \leq \mu$, and $\text{cod} \rho = \text{dom} \lambda$. Then $\rho \lambda \leq \sigma \mu$ follows via the definition of $\leq$ by condition $(\ast 1)$:
\(\rho \leq \sigma \land \lambda \leq \mu \Rightarrow \) 
\[d_A(\rho \otimes \sigma)\nabla_B = \rho \land d_B(\lambda \otimes \mu)\nabla_C = \lambda\]

\[\Rightarrow \rho \lambda = d_A(\rho \otimes \sigma)\nabla_B d_B(\lambda \otimes \mu)\nabla_C\]

\[= d_A(d_A(\rho \otimes \sigma)\nabla_B d_B(\lambda \otimes \mu)\nabla_C \otimes d_A(\rho \lambda \otimes \mu)\nabla_C)\nabla_C\]

\[= d_A(d_A(\rho \lambda \otimes \rho \lambda) \otimes \sigma \mu)\nabla_C^{-1}(1_C \otimes \nabla_C)\nabla_C\]

\[= d_A(\rho \lambda \otimes \sigma \mu)\nabla_C\]

\[\Rightarrow \rho \lambda \leq \sigma \mu.\]

For morphisms \(\rho \leq \sigma \in K[A, B]\) and \(\rho' \leq \sigma' \in K[A', B']\) one obtains

\[\rho = d_A(\rho \otimes \sigma)\nabla_B\] and \(\rho' = d_A(\rho' \otimes \sigma')\nabla_{B'},\]

hence

\[\rho \otimes \rho' = d_A(\rho \otimes \sigma)\nabla_B \otimes d_A(\rho' \otimes \sigma')\nabla_{B'}\]

\[= (d_A \otimes d_A')(\rho \otimes \sigma) \otimes (\rho' \otimes \sigma')(\nabla_B \otimes \nabla_{B'})\]

\[= d_A(d_A(\rho' \otimes \sigma'(\sigma \otimes \sigma'))b_{B', B'}(\nabla_B \otimes \nabla_{B'}) (D3), (M18))\]

\[= d_A(d_A(\rho' \otimes (\sigma \otimes \sigma'))\nabla_{B \otimes B'} (\nabla5)),\]

therefore \(\rho \otimes \rho' \leq \sigma \otimes \sigma'\).

Now let \(\lambda\) and \(\mu\) be morphisms from \(A\) into \(B\). Then

\[d_A(\lambda \otimes \mu)\nabla_B = d_A(d_A(\lambda \otimes \lambda)\nabla_B \otimes \mu)\nabla_B (D4)\]

\[= d_A(\lambda \otimes d_A(\lambda \otimes \mu)\nabla_B)\nabla_B (D1), (M6), (\nabla3)\]

\[= d_A s_A, A(\lambda \otimes d_A(\lambda \otimes \mu)\nabla_B)\nabla_B (D2)\]

\[= d_A(d_A(\lambda \otimes \mu)\nabla_B \otimes \lambda) s_{B, B'} \nabla_B (M8)\]

\[= d_A(d_A(\lambda \otimes \mu)\nabla_B \otimes \lambda)\nabla_B (\nabla4)\]
hence $d_A(\lambda \otimes \mu)\nabla B \leq \lambda$. In the same manner one shows $d_A(\lambda \otimes \mu)\nabla B \leq \mu$.

Further let be $\tau \leq \lambda$ and $\tau \leq \mu$. Then it follows
\[
\tau = d_A(\tau \otimes \mu)\nabla B = d_A(d_A(\tau \otimes \lambda)\nabla B \otimes \mu)\nabla B = d_A(\tau \otimes d_A(\lambda \otimes \mu)\nabla B)\nabla B,
\]
therefore $\tau \leq d_A(\lambda \otimes \mu)\nabla B$. Consequently, $d_A(\lambda \otimes \mu)\nabla B$ is the greatest lower bound of $\lambda$ and $\mu$ with respect to the partial order relation.

**Lemma 2.3.** Any $hdht\nabla s$-category $K$ has the following properties:
\[
\forall A \in |K| \quad (\nabla A d_a \leq 1_{A \otimes A}),
\]
\[
\forall A, A' \in |K| \forall \rho \in K[A, A'] \quad (\rho d_A \leq d_A(\rho \otimes \rho)),
\]
\[
\forall A, A' \in |K| \forall \rho \in K[A, A'] \quad (\nabla A \rho \leq (\rho \otimes \rho)\nabla A').
\]

**Proof.** Composing the equation in condition (D2) with $\nabla A, A'$ and using (\nabla 1) one obtains
\[
\nabla A d_A = \nabla A d_A d_A A \nabla A' A' = d_A A (\nabla A d_A \otimes 1_{A \otimes A'}) \nabla A \otimes A,
\]
hence $\nabla A d_A \leq 1_{A \otimes A}$ by the definition of $\leq$.

Condition (D\nabla) gives rise to
\[
\rho d_A = (d_A(\rho \otimes \rho)\nabla A') d_A = (d_A(\rho \otimes \rho))(\nabla A' d_A) \leq d_A(\rho \otimes \rho) \quad \text{and}
\]
\[
\nabla A \rho = \nabla A (d_A(\rho \otimes \rho)\nabla A') = (\nabla A d_A)((\rho \otimes \rho)\nabla A') \leq (\rho \otimes \rho)\nabla A',
\]
respectively.

**Corollary 2.4.** By the definition of the partial order relation,
\[
(D9') \quad \rho d_{A'} = d_A(\rho d_{A'} \otimes d_A(\rho \otimes \rho))\nabla A' \otimes A' \quad \text{and}
\]
\[
(\nabla 9') \quad \nabla A \rho = d_A(\nabla A \rho \otimes (\rho \otimes \rho))\nabla A'
\]
are identities in each $hdht\nabla s$-category $K$.

**Theorem 2.5.** Let $K$ be an $hdht\nabla s$-category as defined above. Then the class
\[
F^K := \{\rho \in K \mid d_{\text{dom}, \rho}(\rho \otimes \rho) = \rho d_{\text{cod}, \rho}\}
\]
of so-called functional morphisms forms an $hdht\nabla s$-subcategory $F^K$ of $K$ which is even a $dht\nabla s$-category.

The partial order relation in the $dht\nabla s$-symmetric category $F^K$ is the restriction of $\leq$ in the $hdht\nabla s$-symmetric category $K$. 
Proof. The conditions (D5), (D7), and (D8) show that the class $F^K$ contains all morphisms of the families $d, t, \nabla$, respectively.

Let $\rho \in K[A, B]$ be an isomorphism in $K$. Then there is a $\rho^{-1} \in K[B, A]$ such that $\rho^{-1}d_A \leq d_B(\rho^{-1} \otimes \rho^{-1})$ and $\rho d_B \leq d_A(\rho \otimes \rho)^{-1}$, hence $d_A(\rho \otimes \rho)^{-1} \leq \rho d_B \leq d_A(\rho \otimes \rho)$, i.e. $pd_B = d_A(\rho \otimes \rho)$. Therefore, each isomorphism of $K$ belongs to $F^K$, especially, all identities and all morphisms of the families $a, a^{-1}, r, r^{-1}, l, l^{-1}, s, s^{-1}, b, b^{-1}$ are in $F^K$. All zero morphisms $o_{A,B}, A, B \in |K|$, $o = o_{I,O}$, are elements of $F^K$ since $o_A \cdot o_Bd_B = o_{A,B} = d_A(o_{A,B} \otimes o_{A,B})$.

Let $\rho \in K[A, B] \cap F^K$ and $\sigma \in K[B, C] \cap F^K$. Then

\[(\rho \sigma)d_C = \rho(\sigma d_C) = \rho(d_B(\sigma \otimes \sigma)) = (\rho d_B)(\sigma \otimes \sigma) = d_A(\rho \otimes \rho)(\sigma \otimes \sigma) = d_A(\rho \sigma \otimes \rho \sigma),\]

hence $F^K$ is closed under composition.

If $\rho \in K[A, B]$ and $\rho' \in K[A', B']$ are morphisms of $F^K$, then $(\rho \otimes \rho') \in K[A \otimes A', B \otimes B']$ is in $F^K$ too, since

\[d_{B \otimes B'} = (\rho \otimes \rho')(d_B \otimes d_B')b_{B, B', B'} \]
\[= (d_A(\rho \otimes \rho) \otimes d_{A'}(\rho' \otimes \rho'))b_{B, B', B'} \]
\[= (d_A \otimes d_{A'})(b_{A, A', A'}((\rho \otimes \rho' \otimes (\rho \otimes \rho'))) \]
\[= d_{A \otimes A'}((\rho \otimes \rho' \otimes (\rho \otimes \rho')).\]

With respect to the axioms of an hdht$\nabla$s-category, which are identities only, and because of the defining condition of $F^K \subseteq K$, one has a $\nabla$ht$\nabla$s-category $E^K$.

The partial order relation $\leq$ in $K$ is defined by $\rho \leq \sigma \iff \rho = d_A(\rho \otimes \sigma)\nabla_{A'}$ for morphisms $\rho, \sigma \in K[A, A']$. By property (P$\nabla$), this condition is equivalent to $\rho = d_A(\rho \otimes \sigma)p_{A, A'}^{\rho}$ for morphisms $\rho, \sigma$ of $F^K$, hence $\rho \leq \sigma$ with respect to the partial order relation in the $\nabla$ht$\nabla$s-category $E^K$.

Proposition 2.6. All morphisms $\rho \in K[A, B], A, B \in |K|$, of an hdht$\nabla$s-category $K$ fulfilling the condition $\rho d_B = t_A$ (so-called total morphisms) form a symmetric monoidal subcategory $T^K\bullet$ which contains all coretractions of $K$ and all morphisms $t_A, A \in |K|$.

Moreover, $T^K := (T^K\bullet, d, t)$ is an hdts-category.

Proof. Obviously, all identity morphisms $1_A, A \in |K|$, are in $T^K$.

Because of

\[\rho t_B = t_A \land \sigma t_C = t_B \Rightarrow (\rho \sigma)t_C = \rho(\sigma t_C) = \rho t_B = t_A,\]
and
\[ \rho t_B = t_A \land \rho' t_B = t_A' \Rightarrow (\rho \otimes \rho') t_B \otimes B = (\rho \otimes \rho') (t_B \otimes t_B') t_I \otimes I = (t_A \otimes t_A') t_I \otimes I = t_A \otimes A' \]

the class \( T^K \) is closed under composition and \( \otimes \)-operation.

Let \( \rho \in K[A, B] \) be a coretraction in \( K \). Then there is \( \rho^* \in K[B, A] \) such that \( \rho \rho^* = 1_A \). So, one has (see [6], p. 12)

\[ \rho t_B = 1_A \rho t_B = d_A (1_A \otimes t_A) r_A \rho t_B \]

\[ = d_A (\rho t_B \otimes t_A) r_I \]

\[ = d_A (\rho \otimes \rho) (t_B \otimes \rho^* t_A) r_I \]

\[ \geq \rho d_B (t_B \otimes 1_B) (1_I \otimes \rho^* t_A) l_I \]

\[ = \rho d_B (t_B \otimes 1_B) l_B \rho^* t_A \]

\[ \geq \rho t_B \geq \rho t_B \]

therefore \( \rho t_B = t_A \), hence \( \rho \in T^K \).

Because of \( t_A l_I = t_A 1_I = t_A \), \( A \in |K| \), \( d_A \nabla A = 1_A \), \( A \in |K| \), and each isomorphism is just a coretraction, all morphisms of the families \( a, a^{-1}, r, r^{-1}, l, l^{-1}, s, s^{-1}, b, b^{-1}, d, \) and \( t \) belong to \( T^K \).

Since arbitrary suitable morphisms and objects of \( K \) fulfil the identities

(D1), (D2), (D3), (D4), (D5), (D6), (D7), (T1), (T2), (T3), (T4), (T5), (T6), (T7), (T8), (T9), the sequence \((T^K, d, t)\) is an hdts-category. 

**Corollary 2.7.** Let \( K \) be any hdht\( \nabla \)s-category. Then all morphisms of the families \( 1, a, r, s, b, d, t, \nabla, \) and \( (o A, B \mid A, B \in |K|) \) possess all properties of such morphisms in a dht\( \nabla \)s-category, especially the following identities are valid:

(1) \( \nabla_I I = 1_I \otimes I \),
(2) \( t_I \otimes I = \nabla_I = l_I = r_I = d_I^{-1} \),
(3) \( d_I = r_I^{-1} = l_I^{-1} \),
(4) \( d_I \otimes d_I = d_I \otimes I \).
Lemma 2.8. Let $K$ be an hdt-$\nabla$s-category. Then one has

$$(T9) \quad \rho_{tA'}d_I = d_A(\rho_{tA'} \otimes t_A)$$

for all objects $A, A' \in |K|$ and all morphisms $\rho \in K[A,A']$.

Moreover:

(i) \quad $\forall A, A' \in |K| \ \forall \rho \in K[A,A'] \ (\rho d_{A'} = d_A(\rho \otimes \rho))$

(ii) \quad $\forall A, A' \in |K| \ \forall \rho \in K[A,A'] \ (\nabla_A \rho d_{A'} = d_A(\nabla_A \rho \otimes (\rho \otimes \rho) \nabla_{A'})$

(iii) \quad $\forall A, A' \in |K| \ \forall \rho \in K[A,A'] \ (\rho t_A = d_A(\rho t_A \otimes t_A))$

Proof. Because of $\nabla_I d_I = 1_I \otimes I$ and $\nabla_I = r_I = t_I \otimes I$ the equation

$$d_A(\rho_{tA'} \otimes t_A) = d_A(\rho_{tA'} \otimes t_A) \nabla_I d_I = d_A(\rho_{tA'} \otimes t_A) r_I d_I$$

$$= d_A(1_A \otimes t_A) d_A(\rho_{tA'} d_I) = \rho_{tA'} d_I$$

is valid for each $\rho \in K[A,A']$ and all $A, A' \in |K|$, hence $K$ fulfils condition (T9).

The condition (T9') is equivalent to (T9), since

$$d_A(\rho_{tA'} \otimes t_A) = \rho_{tA'} \nabla_I$$

by $d_I \nabla_I = 1_I$ and

$$d_A(\rho_{tA'} \otimes t_A) \nabla_I = \rho_{tA'} \nabla_I = d_A(\rho_{tA'} \otimes t_A) = \rho_{tA'} d_I$$

by $\nabla_I d_I = 1_I \otimes I$, hence property (iii) is shown.

The implications (i) and (ii) are satisfied because of the general property

$$\xi d_B = d_A(\xi \otimes \eta) \Rightarrow \xi = \xi d_B \nabla_B = d_A(\xi \otimes \eta) \nabla_B.$$  

Remark 2.9. The opposite of the implications (i) and (ii), respectively, is not true in general, since there are counterexamples in $\text{Rel}$.
Remark 2.10. As in any $dht\nabla s$-category, the morphisms

$$p_{1}^{A,B} := (1_A \otimes t_B)r_A \in K[A \otimes B, A] \cap F^K,$$

$$p_{2}^{A,B} := (t_A \otimes 1_B)l_B \in K[A \otimes B, B] \cap F^K$$

of an arbitrary $hdht\nabla s$-category $K$ are called canonical projections again and one has

$$\nabla_A = \inf \left\{ p_{1A}^{A,A}, p_{2A}^{A,A} \right\} = d_A \left( p_{1A}^{A,A} \otimes p_{2A}^{A,A} \right) \nabla_A$$

for all $A \in |K|$.

Remark that $(A \otimes B; p_{1A}^{A,B}, p_{2A}^{A,B})$ is not a categorical product in the whole category $K$, but in the subcategory $T^K$.

The family $\nabla = (\nabla_A \mid A \in |K|)$ is uniquely determined by the family $d = (d_A \mid A \in |K|)$ and the conditions $(\nabla 1)$ and $(\nabla 2)$.

Lemma 2.11. Let $K$ be an arbitrary $hdht\nabla s$-category. Then there holds:

\[ \forall A, B, C \in |K| \forall \rho, \rho' \in K[A, B] \forall \sigma, \sigma' \in K[B, C] \ (d_A(\rho \otimes \rho') \nabla_B = \rho \land d_B(\sigma \otimes \sigma') \nabla_C = \sigma) \Rightarrow d_A(\rho \otimes \rho') \nabla_B = \rho \land d_A(\sigma \otimes \sigma') \nabla_B = \sigma, \]

\[ \forall A, B \in |K| \forall \rho, \sigma \in K[A, B] \ (d_A(\rho \otimes \sigma) \nabla_B = \rho \land d_A(\sigma \otimes \sigma) = \sigma d_B) \Rightarrow d_A(\rho \otimes \rho) \nabla_B = \rho, \]

\[ \forall A, B \in |K| \forall \rho, \sigma \in K[A, B] \ (d_A(\rho \otimes \sigma) \nabla_B = \rho \land d_A(\sigma \otimes \sigma) = \sigma d_B) \Rightarrow d_A(\rho \otimes \sigma) \nabla_B = \rho, \]

\[ \forall A, B \in |K| \forall \rho \in K[A, A] \ (d_A(1_A \otimes \rho) \nabla_A = \rho \land d_A(1_A \otimes \rho) \nabla_A = \rho) \Rightarrow d_A(1_A \otimes \rho) p_1^{A,A} = d_A(1_A \otimes \rho) p_2^{A,A} = \rho. \]

Proof. Axiom $(\ast 1)$ implies condition $(\ast 2)$ because of $\rho \leq \rho' \land \sigma \leq \sigma' \Rightarrow \rho \sigma \leq \rho' \sigma'$. To show $(\ast 3)$ not that $d_A(\rho \otimes \sigma) \nabla_B = \rho \Leftrightarrow \rho \leq \sigma$ and $d_A(\sigma \otimes \sigma) = \sigma d_B \Leftrightarrow \sigma \in F^K$. So one obtains
The property (∗6) arises from (∗3) because of $1_A ∈ F^K$ for each $A ∈ |K|$. □

\[ d_A(\rho ⊗ \sigma)p^{B,B}_i = d_A(d_A(\rho ⊗ \sigma)\nabla_B ⊗ \sigma)p^{B,B}_i \]
\[ = d_A(\rho ⊗ d_A(\sigma ⊗ \sigma))a_{B,B}(\nabla_B ⊗ 1_B)p^{B,B}_i \quad (\rho ≤ \sigma) \]
\[ = d_A(\rho ⊗ \sigma)(1_B ⊗ d_B)a_{B,B}(\nabla_B ⊗ 1_B)p^{B,B}_i \quad (\sigma ∈ F_K) \]
\[ = d_A(\rho ⊗ \sigma)\nabla_B d_Bp^{B,B}_i \quad ((∗4)) \]
\[ = d_A(\rho ⊗ \sigma)\nabla_B = \rho \quad ((∗7)) \]

with respect to the axioms of an $hdht\nabla s$-category.

The property (∗4) is a consequence of (D9') and (T9'):

\[ \rho = \rho d_Bp^{B,B}_i ≤ d_A(\rho ⊗ \rho)p^{B,B}_i \quad (\rho t_B ≤ t_A) \]
\[ \Rightarrow d_A(\rho ⊗ \rho)p^{B,B}_i = d_A(\rho ⊗ \rho t_B)r_B ≤ d_A(\rho ⊗ t_A)r_B = d_A(1_A ⊗ t_A)r_A\rho = \rho \]
\[ \land \quad d_A(\rho ⊗ \rho)p^{B,B}_i = d_A(\rho t_B ⊗ \rho)l_B ≤ d_A(t_A ⊗ \rho)l_B = d_A(t_A ⊗ 1_A)l_A\rho = \rho. \]

(∗5): Using the previous results and the assumption one obtains

\[ d_A(\rho ⊗ \rho) = d_A(d_A(\rho ⊗ \sigma)p^{B,B}_2 ⊗ d_A(\rho ⊗ \sigma))p^{B,B}_2 \]
\[ = d_A(d_A ⊗ d_A)((\rho ⊗ \sigma) ⊗ (\rho ⊗ \sigma))(p^{B,B}_2 ⊗ p^{B,B}_2) \]
\[ = d_A(d_A ⊗ d_A)((\rho ⊗ \sigma) ⊗ (\rho ⊗ \sigma))(p^{B,B}_2 ⊗ p^{B,B}_2) \]
\[ = d_A(d_A(\rho ⊗ \sigma) ⊗ d_A(\sigma ⊗ \sigma))b_{B,B,B,B}(p^{B,B}_2 ⊗ p^{B,B}_2) \]
\[ = d_A(d_A(\rho ⊗ \rho) ⊗ \sigma d_B)p^{B,B,B,B}_2 \]
\[ = d_A(\rho ⊗ d_A(\rho ⊗ \sigma))a_{B,B,B,B}(1_B ⊗ d_B)p^{B,B,B,B}_2 \]
\[ = d_A(\rho ⊗ d_A(\rho ⊗ \sigma))a_{B,B,B,B}(p^{B,B}_2 ⊗ d_B) \]
\[ = d_A(\rho ⊗ d_A(\rho ⊗ \sigma))(1_B ⊗ p^{B,B}_2) \]
\[ = d_A(\rho ⊗ d_A(\rho ⊗ \sigma))p^{B,B}_2 d_B \]
\[ = d_A(\rho ⊗ \rho)p^{B,B}_2 d_B = \rho d_B. \]
Lemma 2.12. Let $K$ be a monoidal symmetric category endowed with morphisms families $d, t, (o_{A,B} | A, B \in |K|)$, and $\nabla$ such that all axioms of an $hdht\nabla s$-category without (\textsuperscript{*}1) are fulfilled. Moreover, let the condition (\textsuperscript{*}2) be valid. Then $K$ is an $hdht\nabla s$-category in the defined sense as above.

Proof. It remains to show the condition (\textsuperscript{*}1):

\[
d_A(d_A(\rho \otimes \rho')\nabla_B d_B(\sigma \otimes \sigma')\nabla_C & \otimes d_A(\rho \otimes \rho'\sigma')\nabla_C)\nabla_C
\]
\[
= d_A(\rho \otimes d_A(\rho \otimes \rho'\sigma')\nabla_C)\nabla_C
\]
\[
= d_A(1_A \otimes d_A)(\rho \otimes (\rho \otimes \rho'\sigma'))(1_C \otimes \nabla_C)\nabla_C
\]
\[
= d_A(d_A \otimes 1_A)a_{A,A,A}^{-1}(\rho \otimes (\rho \otimes \rho'\sigma'))(1_C \otimes \nabla_C)\nabla_C
\]
\[
= d_A(d_A \otimes 1_A)((\rho \otimes \rho \otimes \rho'\sigma')a_{C,C,C}^{-1}(1_C \otimes \nabla_C)\nabla_C
\]
\[
= d_A(d_A)(\rho \otimes \rho \otimes \rho'\sigma')(\nabla_C \otimes 1_C)\nabla_C
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= d_A(d_A)(\rho \otimes \rho \otimes \rho'\sigma')(\nabla_C \otimes \rho'\sigma')\nabla_C
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algebras and there are free many-sorted algebras to each generating set with respect to this variety. Especially, there are free $\text{hdht}\nabla$-theories, i.e. free algebraic theories for relational structures, by analogy with the existence of free algebraic theories for partial algebras ([3], [10]).

**Lemma 2.13.** In any $\text{hdht}\nabla$-symmetric category the following conditions are fulfilled for arbitrary morphisms $\rho$, $\sigma$:

(j) $\rho \sigma = 1_A \wedge \sigma \rho \leq 1_B \Rightarrow d_A(\rho \otimes \rho) = \rho d_B$

(jj) $\rho \sigma \leq 1_A \wedge \sigma \rho = 1_B \Rightarrow \nabla A \rho = (\rho \otimes \rho) \nabla B$

**Proof.** To show (j) we use at first the known property $\sigma d_A \leq d_B(\sigma \otimes \sigma)$. Further,

$$d_A(\rho \otimes \rho) = \rho \sigma d_A(\rho \otimes \rho) \leq \rho d_B(\sigma \otimes \sigma)(\rho \otimes \rho) \leq \rho d_B(1_B \otimes 1_B) = \rho d_B,$$

hence $d_A(\rho \otimes \rho) = \rho d_B$ by $\rho d_B \leq d_A(\rho \otimes \rho)$.

In a similar way one shows the statement (jj), namely because of $\nabla B \sigma \leq (\sigma \otimes \sigma) \nabla A$ and

$$(\rho \otimes \rho) \nabla B = (\rho \otimes \rho) \nabla B \sigma \rho \leq (\rho \sigma \otimes \rho \sigma) \nabla A \rho \leq \nabla A \rho \leq (\rho \otimes \rho) \nabla B$$

one has $\nabla A \rho = (\rho \otimes \rho) \nabla B$.

**Definition 2.14.** Morphisms $e \in K[A, A] \subseteq K$ with the property $e \leq 1_A$, i.e. $e = d_A(1_A \otimes e) \nabla A$, are called subidentities in $K$ (compare with ([7])).

**Proposition 2.15** (cf. [7]). For each morphism $\rho : A \to B$, $A, B \in |K|$, the morphism

$$\alpha(\rho) := d_A(\rho \otimes 1_A)p_{2, A}^{B, A}$$

is a subidentity of $A$ in $K$ and there holds $\alpha(\rho) \rho = \rho$. Each subidentity $e$ of $K$ fulfills $d_A(e \otimes e) = e d_A$, therefore the subidentities of $K$ are the subidentities of $E^K$ and satisfy the following conditions for all suitable morphisms and objects of $K$:
(e1) \( e \leq 1_A \quad \Rightarrow \quad ee = e \),
(e2) \( e_1, e_2 \leq 1_A \quad \Rightarrow \quad e_1 e_2 = e_2 e_1 = \inf \{e_1, e_2\} \),
(e3) \( e_1 \leq e_2 \leq 1_A \quad \Leftrightarrow \quad e_1 = e_1 e_2 \leq 1_A \),
(e4) \( e \leq 1_A \quad \Leftrightarrow \quad \alpha(e) = e \),
(e5) \( e \leq 1_A \quad \Rightarrow \quad ed_A = d_A(e \otimes e) = d_A(e \otimes 1_A) \),
(e6) \( e \leq 1_A \quad \Rightarrow \quad \nabla_A e = (e \otimes e)\nabla_A = (e \otimes 1_A)\nabla_A \),
(e7) \( \rho, \sigma \in K[A,B] \quad \Rightarrow \quad \alpha(\rho)\sigma = d_A(\rho \otimes \sigma)p_{B,2}^{A,B} \land \alpha(\sigma)\rho = d_A(\rho \otimes \sigma)p_{1,B}^{B,B} \),
(e8) \( \alpha(\rho)\sigma = \rho \quad \Rightarrow \quad \rho \leq \sigma \),
(e9) \( e\rho = \rho \land e \leq 1_A \quad \Leftrightarrow \quad \alpha(\rho) \leq e \leq 1_A \),
(e10) \( \text{cod}\rho = \text{dom}\sigma \quad \Rightarrow \quad \alpha(\rho\sigma) \leq \alpha(\rho) \),
(e11) \( e \leq 1_A \quad \Rightarrow \quad \alpha(e\rho) \leq e \),
(e12) \( e \leq 1_A \quad \Rightarrow \quad \alpha(e\rho) = e\alpha(\rho) \),
(e13) \( \rho \leq \sigma \quad \Rightarrow \quad \alpha(\rho) \leq \alpha(\sigma) \),
(e14) \( \text{cod}\rho = \text{dom}\sigma \quad \Rightarrow \quad \rho\alpha(\sigma) \leq \alpha(\rho\sigma)\rho \),
(e15) \( \text{cod}\rho = \text{dom}\sigma \quad \Rightarrow \quad \alpha(\rho\sigma) = \alpha(\rho\alpha(\sigma)) \).

**Proof.** Because of \( pt_B \leq t_A \) one obtains
\[
\alpha(\rho) = d_A(\rho \otimes 1_A)p_{2,A}^{B,A} = d_A(pt_B \otimes 1_A)l_A \leq d_A(t_A \otimes 1_A)l_A = 1_A.
\]
Using the definition of \( \alpha(\rho) \), properties \((M14), (M15)\), and \( \alpha(\rho) \leq 1_A \) one receives \( \alpha(\rho)\rho = \rho \) via
\[
\alpha(\rho)\rho = d_A(\rho \otimes 1_A)p_{2,A}^{B,A} \rho = d_A(\rho \otimes \rho)p_{2,B}^{B,B} \geq \rho d_B p_{2,B}^{B,B} = \rho = 1_A \rho \geq \alpha(\rho)\rho.
\]
Because of \( e \leq 1_A \) the property \( d_A(e \otimes e) = ed_A \) is a consequence of Lemma 2.11, \((*5)\), and the subidentities of \( K \) are exactly the subidentities of \( E^K \), therefore, all subidentities have the properties \((e1), (e2), (e3)\) and \((e4)\) (cf. [7]).
To show property (e5) use the property (e4) $e \leq 1_A \Rightarrow e = \alpha(e) = d_A(e \otimes 1_A)p_{2,A}^A$:

$$d_A(e \otimes e) = d_A(e \otimes d_A(e \otimes 1_A)p_{2,A}^A) = d_A(d_A(e \otimes e) \otimes 1_A)a_{A,A,A}^{-1}(1_A \otimes p_{2,A}^A)$$

$$= d_A(d_A(e \otimes e)p_{1,A}^A \otimes 1_A) = d_A(e \otimes 1_A).$$

The second part of the property (e6) is a consequence of (e2) and (e5) owing to $\nabla_A d_A \leq 1_A \otimes A$, $(e \otimes e) \leq 1_A \otimes A$, and $(e \otimes 1_A) \leq 1_A \otimes A$:

$$d_A(e \otimes e) = d_A(e \otimes 1_A) \Rightarrow \nabla_A d_A(e \otimes e) = \nabla_A d_A(e \otimes 1_A)$$

$$\Rightarrow (e \otimes e) \nabla_A d_A = (e \otimes 1_A) \nabla_A d_A \nabla_A \quad ((e2))$$

$$\Rightarrow (e \otimes e) \nabla_A d_A \nabla_A = (e \otimes 1_A) \nabla_A d_A \nabla_A \nabla_A$$

$$\Rightarrow (e \otimes e) \nabla_A = (e \otimes 1_A) \nabla_A. \quad ((\nabla 1))$$

Because of $(e \otimes e) \leq 1_A \otimes A$ and $\nabla_A d_A \leq 1_A \otimes A$ one has

$$(e \otimes e) \nabla_A = (e \otimes e) \nabla_A d_A \nabla_A$$

$$(d_A \nabla_A = 1_A)$$

$$= \nabla_A d_A(e \otimes e) \nabla_A$$

$$(e2)$$

$$= \nabla_A e. \quad ((\nabla D))$$

Property (e7) is an immediate consequence of $(M7)$, $(M14)$, $(M8)$, and $(M13)$.

To show (e8) take into consideration

$$\rho = \alpha(\rho) \sigma \leq 1_A \sigma = \sigma.$$

(e9): Assuming $e \rho = \rho$, $e \leq 1_A$ one gets

$$\alpha(\rho) = \alpha(e \rho) = d_A(e \rho \otimes 1_A)p_{2,B}^{A} = d_A(e \rho t_B \otimes 1_A)l_A \leq d_A(e t_A \otimes 1_A)l_A = \alpha(e) = e.$$

Conversely, $\alpha(\rho) \leq e \leq 1_A$ yields

$$\rho = \alpha(\rho) \rho \leq e \rho \leq 1_A \rho = \rho.$$

Condition (e10) is true, since

$$\alpha(\rho \sigma) = d_A(\rho \sigma \otimes 1_A)p_{2}^{C,A} = d_A(\rho \sigma t_C \otimes 1_A)l_A \leq d_A(\rho t_B \otimes 1_A)l_A = \alpha(\rho).$$
Condition (e11) arises from $\alpha(e\rho) \leq \alpha(e) = e$.

Property (e12) is a consequence of (e5) as follows:

\[
\alpha(e\rho) = d_A(e\rho \otimes 1_A)p_{2B}^{A} = d_A(e \otimes 1_A)(\rho \otimes 1_A)p_{2B}^{A}
\]
\[
= d_A(e \otimes e)(\rho \otimes 1_A)p_{2B}^{A} = e \rho d_A(\rho \otimes 1_A)p_{2B}^{A}
\]
\[
= e \alpha(\rho).
\]

To show (e13) use the definitions of $\leq$ and $\alpha(\rho)$ ($\rho : A \rightarrow B$, $\sigma : B \rightarrow C$):

\[
\alpha(\rho) = d_A(\rho \otimes 1_A)p_{2B}^{A} = d_A(d_A(\rho \otimes \sigma)\nabla B \otimes 1_A)p_{2B}^{A}
\]
\[
\leq d_A(d_A(\rho \otimes \sigma)p_{2B}^{B} \otimes 1_A)p_{2B}^{A} \quad (\nabla B \leq p_{2B}^{B})
\]
\[
= d_A(d_A(\rho \otimes 1_A)p_{2B}^{B} \otimes 1_A)p_{2B}^{A} \quad ((M14))
\]
\[
= d_A(\alpha(\rho)\sigma \otimes 1_A)p_{2B}^{A}
\]
\[
\leq d_A(\sigma \otimes 1_A)p_{2B}^{A} = \alpha(\sigma). \quad (\alpha(\rho)\sigma \leq \sigma)
\]

Assertion (e14) is true since

\[
\rho \alpha(\sigma) = \rho d_B(\sigma \otimes 1_B)p_{2C}^{B} \leq d_A(\rho \sigma \otimes \rho)p_{2C}^{B} = \alpha(\rho\sigma)\rho.
\]

Condition (e15) follows by (e10), (e13), and (e14):

Let $\rho$ and $\sigma$ be as above. Then one has

\[
\alpha(\rho\sigma) = \alpha(\rho \alpha(\sigma)\sigma) \leq \alpha(\rho\alpha(\sigma)),
\]

hence

\[
\alpha(\rho\sigma) \leq \alpha(\rho\alpha(\sigma)) \leq \alpha(\rho(\alpha(\sigma)\rho) \leq \alpha(\rho(\alpha(\sigma)\alpha(\rho))
\]

\[
\leq \alpha(\alpha(\rho\sigma)1_A) = \alpha(\rho\sigma) = \alpha(\rho\sigma).
\]

Remark that, as an easy example shows, in Red the opposite implication to (e8) is not true: Let be given $A = \{a\}, B = \{b_1, b_2\}, \rho = \{(a, b_1)\}, \sigma = \{(a, b_1), (a, b_2)\}$. Then $\rho \leq \sigma$ and $\rho < \alpha(\rho)\sigma = \sigma$.

Furthermore, the equality in (e14) is not true in general. For this let be the sets $A$ and $B$ as above and let be $C = \{x\}$. For the relations $\sigma$ as above and $\tau = \{(b_1, x)\}$ one obtains $\sigma\alpha(\tau) = \{(a, b_1)\}$ and $\sigma\tau = \{(a, x)\}$, hence $\alpha(\sigma\tau) = \{(a, a)\}$, consequently $\alpha(\sigma\tau)\sigma = \{(a, b_1), (a, b_2)\} = \sigma \neq \sigma\alpha(\tau)$.
References


Received 6 December 2000