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ON THE STRUCTURE OF HALFDIAGONAL-HALFTERMINAL-SYMMETRIC CATEGORIES WITH DIAGONAL INVERSIONS

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Dedicated to Hans-Jürgen Hoehnke on the occasion of his 75th birthday.

Abstract

The category of all binary relations between arbitrary sets turns out to be a certain symmetric monoidal category Rel with an additional structure characterized by a family $d = (d_A : A \to A \otimes A \mid A \in |\mathrm{Rel}|)$ of diagonal morphisms, a family $t = (t_A : A \to I \mid A \in |\mathrm{Rel}|)$ of terminal morphisms, and a family $\nabla = (\nabla_A : A \otimes A \to A \mid A \in |\mathrm{Rel}|)$ of diagonal inversions having certain properties. Using this properties in [11] was given a system of axioms which characterizes the abstract concept of a halfdiagonal-halfterminal-symmetric monoidal category with diagonal inversions ($hdht\nabla s$-category). Besides of certain identities this system of axioms contains two identical implications. In this paper is shown that there is an equivalent characterizing system of axioms for $hdht\nabla s$-categories consisting of identities only. Therefore, the class of all small $hdht\nabla$-symmetric categories (interpreted as hetrogeneous algebras of a certain type) forms a variety and hence there are free theories for relational structures.

Keywords: halfdiagonal-halfterminal-symmetric category, diagonal inversion, partial order relation, subidentity, equation.

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1. Defining conditions

Let $K^\bullet$ be any symmetric monoidal category in the sense of Eilenberg-Kelly ([2]) with the object class $|K|$, the morphism class K, the distinguished object I, the bifunctor $\otimes : K \times K \to K$, and the families a, r, l, s of isomorphisms of K such that the following axioms are valid for all objects and all morphisms of K. By $K[A,B]$ we denote the set of all morphisms $\rho \in K$ with the domain (source) $\operatorname{dom}\rho = A$ and the codomain (target) $\operatorname{codom}\rho = B$.

Bifunctor properties:

(F1) $\operatorname{dom}(\rho \otimes \rho') = \operatorname{dom}\rho \otimes \operatorname{dom}\rho'$,

(F2) $\operatorname{codom}(\rho \otimes \rho') = \operatorname{codom}\rho \otimes \operatorname{codom}\rho'$,

(F3) $1_{A\otimes B} = 1_A \otimes 1_B$,

(F4) $(\rho \otimes \rho')(\sigma \otimes \sigma') = \rho\sigma \otimes \rho'\sigma'$.

Conditions of monoidality:

(M1) $a_{A,B,C\otimes D}a_{A\otimes B,C,D} = (1_A \otimes a_{A,B,C})a_{A,B\otimes C,D}(a_{A,B,C} \otimes 1_D)$,

(M2) $a_{A,I,B}(r_A \otimes 1_B) = 1_A \otimes l_B$,

(M3) $a_{A,B,C}s_{A\otimes B,C}a_{C,A,B} = (1_A \otimes s_{B,C})a_{A,C,B}(s_{A,C} \otimes 1_B)$,

(M4) $s_{A,B}s_{B,A} = 1_{A\otimes B}$,

(M5) $s_{A,I}l_A = r_A$,

(M6) $a_{A,B,C}((\rho \otimes \sigma) \otimes \tau) = (\rho \otimes (\sigma \otimes \tau))a_{A',B',C'}$,

(M7) $r_A\rho = (\rho \otimes 1_I)r_{A'}$,

(M8) $s_{A,B}(\sigma \otimes \rho) = (\rho \otimes \sigma)s_{A',B'}$.

Remark that the validity of an equation containing morphism compositions includes that they are defined on both sides.

An immediate consequence of the conditions above is the validity of

(M9) $\forall A,\ B \in |K|\ (a_{I,A,B}(l_A \otimes 1_B) = l_{A\otimes B})$,

(M10) $\forall A,\ B \in |K|\ (a_{A,B,I}r_{A\otimes B} = 1_A \otimes r_B)$,

(M11) $r_I = l_I$,

(M12) $s_{I,I} = 1_{I\otimes I}$,

(M13) $\forall A \in |K|\ (s_{I,A} r_A = l_A)$,

(M14) $\forall A \in |K|\ (l_A \rho = (1_I \otimes \rho) l_{A'})$.

Using the denotation

$$b_{A,B,C,D} := a_{A\otimes B,C,D}(a^{-1}_{A,B,C}(1_A \otimes s_{B,C})a_{A,C,B} \otimes 1_D)a^{-1}_{A\otimes C,B,D}$$

one obtains the following properties for all objects $A, A', B, B', C, C', D, D'$ of K and all morphisms $\rho \in K[A, A']$, $\sigma \in K[B, B']$, $\lambda \in K[C, C']$, $\mu \in K[D, D']$:

(M15) $b_{A,B,C,D}((\rho \otimes \sigma) \otimes (\lambda \otimes \mu) = ((\rho \otimes \lambda) \otimes (\sigma \otimes \mu) b_{A',B',C'D'}$,

(M16) $b_{A,I,I,B} = 1_{A\otimes I} \otimes 1_{I\otimes B}$,

(M17) $b_{A,B,C,D} b_{A,C,B,D} = 1_{A\times B} \otimes 1_{C\otimes D}$,

(M18) $b_{A,B,C,D}(s_{A,C} \otimes s_{B,D}) = s_{A\otimes B,C\otimes D} b_{C,D,A,B}$.

Obviously, all morphisms $b_{A,B,C,D}$ are isomorphims in the category $K^\bullet$.

Definition 1.1 ([1]). A *diagonal-terminal-symmetric category* (shortly *dts*-category) $\underline{K} = (K^\bullet, d, t)$ is defined as a symmetric monoidal category endowed with morphism families

$$d = (d_A : A \to A \otimes A \mid A \in |K|) \ \text{ and } \ t = (t_A : A \to I \mid A \in |K|)$$

satisfying the following conditions for all objects $A, B, A' \in |K|$ and all morphisms $\rho \in K[A, A']$.

Diagonality:

(D1) $d_A(d_A \otimes 1_A) = d_A(1_A \otimes d_A)a_{A,A,A}$,

(D2) $d_A s_{A,A} = d_A$,

(D3) $d_{A\otimes B} = (d_A \otimes d_B) b_{A,A,B,B}$,

(D4) $d_A(\rho \otimes \rho) = \rho d_{A'}$.

Terminality:

(T1) $d_A(1_A \otimes t_A) r_A = 1_A$,

(T2) $t_I = 1_I$,

(T3) $\rho t_{A'} = t_A$.

Let $A,\ A',\ B$ be arbitrary objects in K and let $\rho \in K[A,A']$ be any morphism in K. Then the properties

(D5) $d_A(d_A \otimes d_A) = d_A d_{A\otimes A}$,

(D6) $d_A(d_A \otimes d_A) = d_A(d_A \otimes d_A)b_{A,A,A,A}$,

(D7) $t_A d_I = d_A(t_A \otimes t_A)$,

(D9) $\rho d_{A'} d_{A'\otimes A'} = d_A(\rho d_{A'} \otimes d_A(\rho \otimes \rho)))$,

(T4) $d_A(t_A \otimes 1_A)l_A = 1_A$,

(T5) $d_{A\otimes B}((1_A \otimes t_B)r_A \otimes (t_A \otimes 1_B)l_B) = 1_{A\otimes B}$,

(T6) $t_{A\otimes B} = (t_A \otimes t_B)t_{I\otimes I}$,

(T7) $r_I = t_{I\otimes I}$,

(T8) $d_A t_{A\otimes A} = t_A$,

(T9) $\rho t_{A'} d_I = d_A(\rho t_{A'} \otimes t_A)$

are consequences of the conditions above ([1]).

The category *Set* of all total functions between arbitrary sets is a model of a *dts*-category by

$I := \{\emptyset\},\quad A \otimes B := \{\langle a,b\rangle \mid a \in A \ \wedge\ b \in B\}$,

$\rho \in Set[A,B] :\Leftrightarrow \rho = \{(a,b) \mid a \in A \ \wedge\ b = \rho(a) \in B\}$,

$$\forall\, a \in A\ \exists!!\ b \in B\ (b = \rho(a)),$$

$\rho \in Set[A,B],\ \sigma \in Set[B,C] \ \Rightarrow\ \rho \circ \sigma := \{(a,c) \mid a \in A \ \wedge\ c = \sigma(\rho(a))\}$,

$$(a,c) \in \rho \circ \sigma \Leftrightarrow \exists\, b \in B\ ((a,b) \in \rho \ \wedge\ (b,c) \in \sigma),$$

$\rho \in Set[A,B], \rho' \in Set[A',B'] \Rightarrow \rho \otimes \rho' := \{(\langle a,a'\rangle, \langle \rho(a), \rho'(a')\rangle) \mid a \in A, a' \in A'\}$,

$a_{A,B,C} := \{(\langle a,\langle b,c\rangle\rangle, \langle\langle a,b\rangle,c\rangle) \mid a \in A,\ b \in B,\ c \in C\}$,

$s_{A,B} := \{(\langle a,b\rangle, \langle b,a\rangle) \mid a \in A,\ b \in B\}$,

$r_A := \{(\langle a,\emptyset\rangle, a) \mid a \in A\}$,

$l_A := \{(\langle \emptyset,a\rangle, a) \mid a \in A\}$,

$d_A := \{(a, \langle a,a\rangle) \mid a \in A\}$,

$t_A := \{(a,\emptyset) \mid a \in A\}$.

Remark that I is a *terminal object* in any *dts*-category $\underline{K}$ and $(A \otimes B;\, p_1^{A,B}, p_2^{A,B})$ forms a *categorical product* of the objects A, B in the category K, where $p_1^{A,B} := (1_A \otimes t_B)r_A$ and $p_2^{A,B} := (t_A \otimes 1_B)l_B$.

Moreover, $d_A(\rho \otimes \sigma) = \rho d_B$ is equivalent to $\rho = \sigma$ for all $A,\ B \in |K|$ and all $\rho, \sigma \in K[A, B]$ because of

$$\sigma = \sigma d_B p_2^{B,B} = d_A(\sigma t_B \otimes \sigma)l_B = d_A(t_A \otimes \sigma)l_B$$

$$= d_A(\rho t_B \otimes \sigma)l_B = d_A(\rho \otimes \sigma)p_2^{B,B} = \rho d_B p_2^{B,B} = \rho.$$

The morphisms $p_1^{A,B}$ and $p_2^{A,B}$ are called *canonical projections* in the category K.

Conditions (D9) and (T9) are equivalent to

$\rho d_{A'} = d_A(\rho d_{A'} \otimes d_A(\rho \otimes \rho))p_2^{A',A'}$ and $\rho t_{A'} = d_A(\rho t_{A'} \otimes t_A)p_2^{I,I}$, respectively.

Definition 1.2. Let $K^\bullet$ be again a symmetric monoidal category endowed with morhism families d and t as above. Then $\underline{K} = (K^\bullet, d, t)$ is called *halfdiagonal-terminal-symmetric category* (shortly *hdts-category*), if the conditions

(D1), (D2), (D3), (D5), (D7), (T1), (T2), (T3)

hold identically.

As above, the identities (T4), (T5), (T6), (T7), (T8), (T9) follow from the defining conditions in an *hdts*-category.

Definition 1.3. A *diagonal-halfterminal-symmetric category* (shortly *dhts*-category) ([3], [7], [10]) is defined as a sequence $\underline{K} := (K^\bullet; d, t, O, o)$ such that $K^\bullet$ is again a symmetric monoidal category, d and t are families as above, O is a distinguished zero-object of $K^\bullet$, $o : I \to O$ is a distinguished morphism of $K^\bullet$, and the following equations are fulfilled for all objects $A, B, A', B' \in |K|$ and all morphisms $\rho \in K[A, A']$, $\sigma \in K[B, B']$, $\lambda \in K[A, O]$, $\kappa \in K[O, A]$:

(D4), (T1), (T4), (T5), (T6), and

(o1) $\quad t_A o = \lambda,$

(o2) $\quad (1_A \otimes t_O)r_A = \kappa,$

(O1) $\quad A \otimes O = O \otimes A = O.$

Remark that the conditions

(D1), (D2), (D3), (D5), (D6), (D7), (D9), (T2), (T7), (T8), (T9), and

(B1) $\qquad b_{A,B,C,D}(1_{A\otimes C}\otimes t_{B\otimes D})r_{A\otimes C}=(1_A\otimes t_B)r_A\otimes(1_C\otimes t_D)r_C,$

(B2) $\qquad b_{A,B,C,D}(t_{A\otimes C}\otimes 1_{B\otimes D})l_{B\otimes D}=(t_A\otimes 1_B)l_B\otimes(t_C\otimes 1_D)l_D$

are consequences of the other conditions ([3], [7], [10]).

Formulas (o1), (o2), and (O1) explain that the morphism sets $K[A,O]$ and $K[O,A]$ both consist of exactly one element $o_{A,O}$ and $o_{O,A}$, respectively, and O is a zero object in K. In any *dhts*-category there is a so-called *zero-morphism* $o_{A,B}$ to each pair of objects $A,B\in|K|$ with the properties

(o3) $\qquad \forall\rho\in K[A,A'],\ \sigma\in K[B,B']\ (\rho o_{A,B}=o_{A',B}\ \wedge\ o_{A,B}\sigma=o_{A,B'}),$

(o4) $\qquad \forall\xi,\eta\in K\ (o_{A,B}\otimes\xi=o_{A,B}=\eta\otimes o_{A,B}),$

(o5) $\qquad o_{O,A}=(1_A\otimes t_O)r_A=(t_O\otimes 1_A)l_A.$

The category *Par* of all partial functions between arbitrary sets is a model of a *dhts*-category by the same fixations as above and $O=\emptyset$ (the empty set) and $o:I\to O,\ o_{A,O}:A\to O,\ o_{O,A}:O\to A,\ o_{A,B}:A\to B$ as the empty functions. The morphisms are given by

$$\begin{aligned}\rho\in K[A,B]\quad :\Leftrightarrow\quad &\rho=\{(a,\rho(a))\mid a\in D(\rho)\ \wedge\ \rho(a)\in B\},\\ &\forall\ a\in D(\rho)\subseteq A\ \exists!!\ b\in B\ (b=\rho(a)).\end{aligned}$$

The following fact is of importance for the consideration of *dhts*-categories.

Lemma 1.4. *Let $\underline{K}$ be a symmetric monoidal category endowed with morphism families d and t as above which fulfil conditions* (D4), (T1) *and* (T6). *Then conditions* (T4) *and* (T5) *are consequences of the validity of* (D2) *and* (D3) *in $\underline{K}$.*

Proof. Using (T1) and (D2) one obtains (T4) as follows:

$$1_A=d_A(1_A\otimes t_A)r_A=d_As_{A,A}(1_A\otimes t_A)r_A=d_A(t_A\otimes 1_A)s_{I,A}r_A=d_A(t_A\otimes 1_A)l_A.$$

The calculation

$$
\begin{aligned}
& d_{A\otimes B}((1_A\otimes t_B)r_A\otimes(t_A\otimes 1_B)l_B) \\
&\quad = (d_A\otimes d_B)b_{A,A,B,B}((1_A\otimes t_B)r_A\otimes(t_A\otimes 1_B)l_B) && ((\mathrm{D3})) \\
&\quad = (d_A(1_A\otimes t_A)\otimes d_B(t_B\otimes 1_B))b_{A,I,I,B}(r_A\otimes l_B) && ((\mathrm{M15})) \\
&\quad = (d_A(1_A\otimes t_A)\otimes d_B(t_B\otimes 1_B))(1_{A\otimes I}\otimes 1_{I\otimes B})(r_A\otimes l_B) && ((\mathrm{M16})) \\
&\quad = (d_A(1_A\otimes t_A)\otimes d_B(t_B\otimes 1_B))(r_A\otimes l_B) && ((\mathrm{F3})) \\
&\quad = (d_A(1_A\otimes t_A)r_A\otimes d_B(t_B\otimes 1_B)l_B) && ((\mathrm{F4})) \\
&\quad = 1_A\otimes 1_B && ((\mathrm{T1}),(\mathrm{T4}))
\end{aligned}
$$

shows the validity of (T5). ■

Let $\underline{K}$ be an arbitrary *dhts*-category. Then all morphisms $\rho\in K[A,A']$, $A,A'\in|K|$, fulfilling $\rho t_{A'}=t_A$, form a subcategory $\underline{M}^K$ of $\underline{K}$ which is even a *dts*-caregory. Denoting by $\underline{M}_K$ the smallest *dts*-subcategory of $\underline{M}^K$ containing all morphisms of the families $a,\ r,\ l,\ s,\ d,\ t$ one has

$$\underline{M}_K\subseteq\underline{\mathrm{Iso}}(K)\subseteq\underline{\mathrm{Cor}}(K)\subseteq\underline{M}^K,$$

where $\underline{\mathrm{Iso}}(K)$ ($\underline{\mathrm{Cor}}(K)$) is a *dts*-subcategory of $\underline{M}^K$ generated by all isomorphisms (coretractions) of K together with all terminal morphisms of K, since all coretractions and all terminal morphisms fulfil the condition (T3) (see [7], [10]).

The object $I\in|K|$ is a terminal object in the subcategories $\underline{M}_K$, $\underline{\mathrm{Iso}}(K)$, $\underline{\mathrm{Cor}}(K)$, and $\underline{M}^K$ but not in the whole category $\underline{K}$. Morphisms of the kind $p_1^{A,B}=(1_A\otimes t_B)r_A$ and $p_2^{A,B}=(t_A\otimes 1_B)l_B$ are called *canonincal projections* again and $(A\otimes B;\ p_1^{A,B},p_2^{A,B})$ is a *categorical product* of A and B in $\underline{M}^K$, but in general not in the whole category.

Schreckenberger had proved ([7]) that

$$\rho\leq\sigma:\Leftrightarrow d_A(\rho\otimes\sigma)=\rho d_{A'}\qquad(\rho,\sigma\in K[A,A'])$$

defines a partial order relation which is stable under composition and $\otimes$-operation. Moreover, the following are equivaent:

(i) $$d_A(\rho \otimes \sigma) = \rho d_{A'},$$

(ii) $$d_A(\rho \otimes \sigma)p_2^{A',A'} = \rho,$$

(iii) $$d_A(\sigma \otimes \rho)p_1^{A',A'} = \rho.$$

Hoehnke had shown ([3]) the validity of the identical implication

$$\rho = d_A(\rho \otimes \sigma)p_2^{A',A'} \Rightarrow \rho = d_A(\rho \otimes \sigma)p_1^{A',A'}.$$

The relation $\leq$ in the *dhts*-category $\underline{Par}$ describes exactly the usual inclusion $\subseteq$.

Morphisms $e_A \in K[A, A]$ of any *dhts*-category $\underline{K}$ fulfilling $e_A \leq 1_A$ for any $A \in |K|$ are called *subidentities* ([7]). Especially, for each $\rho \in K[A, B]$, the morphism

$$\alpha(\rho) := d_A(\rho \otimes 1_A)p_2^{B,A}(= d_A(1_A \otimes \rho)p_1^{A,B})$$

is a subidentity of $A \in |K|$, since

$$\begin{aligned} d_A(d_A(\rho\otimes 1_A)p_2^{B,A}\otimes 1_A)p_2^{A,A} &= d_A(\rho \otimes d_A(1_A \otimes 1_A))a_{B,A,A}(p_2^{B,A} \otimes 1_A)p_2^{A,A} \\ &= d_A(\rho \otimes d_A)(1_B \otimes p_2^{A,A})p_2^{B,A} \\ &= d_A(\rho \otimes d_A p_2^{A,A})p_2^{B,A} = d_A(\rho \otimes 1_A)p_2^{B,A}. \end{aligned}$$

Important properties of subidentities are described in [7], [13], [15].

Definition 1.5. *A diagonal-halfterminal-symmetric category with diagonal inversion* ∇ (shortly *dht*∇*s-category*, [10]) is, by definition, a sequence $\underline{K} := (K^\bullet; d, t, \nabla, O, o)$ such that $(K^\bullet; d, t, O, o)$ is a *dhts*-category endowed with a morphism family $\nabla = (\nabla_A |\ A \in |K|)$ satisfying the following for all $A \in |K|$:

(∇1) $d_A \nabla_A = 1_A,$

(∇2) $\nabla_A d_A d_{A\otimes A} = d_{A\otimes A}(\nabla_A d_A \otimes 1_{A\otimes A}).$

The category Par is also a model of a *dht*∇*s*-category, where

$$\nabla_A := \{(\langle a, a\rangle, a) |\ a \in A\},\ A \in |Par|.$$

The properties

(D8) $\nabla_A d_A = d_{A\otimes A}(\nabla_A \otimes \nabla_A)$,

(D9') $\rho d_{A'} = d_A(\rho d_{A'} \otimes d_A(\rho \otimes \rho))\nabla_{A'\otimes A'}$,

(T9') $\rho t_{A'} = d_A(\rho t_{A'} \otimes t_A)\nabla_I$,

(∇3) $a_{A,A,A}(\nabla_A \otimes 1_A)\nabla_A = (1_A \otimes \nabla_A)\nabla_A$,

(∇4) $s_{A,A}\nabla_A = \nabla_A$,

(∇5) $\nabla_{A\otimes B} = b_{A,B,A,B}(\nabla_A \otimes \nabla_B)$,

(∇6) $\nabla_A d_A = (d_A \otimes 1_A)a^{-1}_{A,A,A}(1_A \otimes \nabla_A)$,

(∇7) $\nabla_A d_A = (1_A \otimes d_A)a_{A,A,A}(\nabla_A \otimes 1_A)$,

(∇8) $\nabla_A d_A = (d_A \otimes d_A)\nabla_{A\otimes A}$,

(∇9) $\nabla_A \rho d_{A'} = d_{A\otimes A}(\nabla_A \rho \otimes (\rho \otimes \rho)\nabla_{A'})$,

(∇9') $\nabla_A \rho = d_{A\otimes A}(\nabla_A \rho \otimes (\rho \otimes \rho)\nabla_{A'})\nabla_{A'}$,

(∇10) $\nabla_{A\otimes A}\nabla_A = (\nabla_A \otimes \nabla_A)\nabla_A$,

(D∇) $\rho = d_A(\rho \otimes \rho)\nabla_{A'}$

follow from the axioms and the other properties of a $dht\nabla s$-category for all $A,\ A',\ B \in |K|$ and all $\rho \in K[A, A']$ (see [13]).

By the definition of the partial order relation, (T9) is equivalent to $\rho t_{A'} \leq t_A$, (∇2) is equivalent to $\nabla_A d_A \leq 1_{A^2}$, and ($\nabla$9) is equivalent to $\nabla_A \rho \leq (\rho \otimes \rho)\nabla_{A'}$ for $\rho \in K[A, A']$.

Moreover, one has the following important property in any $dht\nabla s$-category $\underline{K}$ ([11]):

(P∇) $\forall\, A, A' \in |K|\ \forall\, \rho, \sigma \in K[A, A']\ (d_A(\rho\otimes\sigma)p_2^{A',A'} = \rho \Leftrightarrow d_A(\rho\otimes\sigma)\nabla_{A'} = \rho)$.

In any $dht\nabla s$-category, conditions (D9), (T9), and (∇9) result in (D9'), (T9'), and (∇9'), respectively.

2. $hdht\nabla s$-categories

Definition 2.1 ([10]). A sequence $\underline{K} = (K^\bullet; d, t, \nabla, o)$ is called *halfdiagonal-halfterminal-symmetric monoidal category with diagonal inversion* ∇ (shortly *$hdht\nabla s$-category*), iff $K^\bullet$ is a symmetric monoidal category as above,

$(d_A : A \to A \otimes A \mid A \in |K|)$, $(t_A : A \to I \mid A \in |K|)$, $(\nabla_A : A \otimes A \to A \mid A \in |K|)$ are families of morphisms of K, and $o : I \to O \quad (I \neq O \in |K|)$ is a distinguished morphism of K such that for all objects and all morphisms of the underlying category K the conditions

(D1), (D2), (D3), (D5), (D7), (D8),

(T1), (T2), (T6), (T9'),

(∇1), (∇2), (∇3), (∇4), (∇5), (D∇),

(o1), (o2), (O1),

and

(*1) $\quad d_A(\rho \otimes \rho')\nabla_B d_B(\sigma \otimes \sigma')\nabla_C$

$$= d_A(d_A(\rho \otimes \rho')\nabla_B d_B(\sigma \otimes \sigma')\nabla_C \otimes d_A(\rho\sigma \otimes \rho'\sigma')\nabla_C)\nabla_C$$

are fulfilled.

The system of axioms given in this definition is free of contradictions, because the category $\underline{Rel}$ of all binary relations between sets is a model of it, i.e. $\underline{Rel}$ fulfils all the axioms of an $hdht\nabla s$-category, where $|Rel|$ is the class of all sets, the morphisms are characterized by

$$\rho \in Rel[A, A'] :\Leftrightarrow \rho = \{(a, a') \mid a \in D(\rho) \subseteq A \wedge a' \in W(\rho) \subseteq A' \wedge H(a, a')\},$$

where $H(x, y)$ is a sentence form in two variables, the distinguished objects are $I = \{\emptyset\}$ and $O = \emptyset$, the operation $\otimes$ for objects is given as in Set, the composition and the $\otimes$-operation of morphisms are described by

$$\rho \in Rel[A, B], \sigma \in Rel[B, C] \Rightarrow \rho \circ \sigma = \{(a, c) \mid \exists\, b \in B \ ((a, b) \in \rho \wedge (b, c) \in \sigma)\},$$

$$\rho \in Rel[A, B], \rho' \in Rel[A', B'] \Rightarrow \rho \otimes \rho' = \{(\langle a, a'\rangle, \langle b, b'\rangle) \mid (a, b) \in \rho \wedge (a', b') \in \rho'\},$$

and the morphisms of the families a, r, l, s, b, d, t, ∇, $(0_{A,B} \mid A, B \in |Rel|)$ are as in Par.

Lemma 2.2. *The relation $\leq$ defined by*

$$\rho \leq \sigma :\Leftrightarrow d_A(\rho \otimes \sigma)\nabla_B = \rho$$

is a partial order relation in any hdht∇-symmetric category which is compatible with compostion and $\otimes$-operation for morphisms. Moreover, the greatest *lower bound of two morphisms $\lambda,\ \mu \in K[A,B]$ with respect to the canonical order relation $\leq$ is given by*

$$d_A(\lambda \otimes \mu)\nabla_B = inf\{\lambda, \mu\}.$$

Proof. Condition $(D\nabla)$ shows the reflexivity of $\leq$. The relation is antisymmetric because of

$$\begin{aligned}
\rho \leq \sigma \ \wedge \ \sigma \leq \rho \quad \Rightarrow \quad \sigma &= d_A(\sigma \otimes \rho)\nabla_B \\
&= d_A s_{A,A}(\sigma \otimes \rho)\nabla_B && ((D2)) \\
&= d_A(\rho \otimes \sigma)s_{B,B}\nabla_B && ((M8)) \\
&= d_A(\rho \otimes \sigma)\nabla_B && ((\nabla 4)) \\
&= \rho.
\end{aligned}$$

The implication

$$\begin{aligned}
\rho \leq \sigma \wedge \sigma \leq \tau \Rightarrow \rho &= d_A(\rho \otimes \sigma)\nabla_B \\
&= d_A(\rho \otimes d_A(\sigma \otimes \tau)\nabla_B)\nabla_B \\
&= d_A(1_A \otimes d_A)(\rho \otimes (\sigma \otimes \tau))(1_B \otimes \nabla_B)\nabla_B \\
&= d_A(d_A \otimes 1_A)((\rho \otimes \sigma) \otimes \tau)a^{-1}_{B,B,B}(1_B \otimes \nabla_B)\nabla_B && ((M6), (D1)) \\
&= d_A(d_A(\rho \otimes \sigma) \otimes \tau)(\nabla_B \otimes 1_B)\nabla_B && ((\nabla 3)) \\
&= d_A(d_A(\rho \otimes \sigma)\nabla_B \otimes \tau)\nabla_B \\
&= d_A(\rho \otimes \tau)\nabla_B \\
&\Rightarrow \rho \leq \tau
\end{aligned}$$

yields the transitivity of the relation $\leq$.

Now suppose $\rho \leq \sigma,\ \ \lambda \leq \mu$, and cod $\rho =$ dom λ. Then $\rho\lambda \leq \sigma\mu$ follows via the definition of $\leq$ by condition $(*1)$:

$$\begin{aligned}
\rho\leq\sigma \ \wedge \ \lambda\leq\mu\Rightarrow \quad & d_A(\rho\otimes\sigma)\nabla_B=\rho \ \wedge \ d_B(\lambda\otimes\mu)\nabla_C=\lambda\\
\Rightarrow \rho\lambda= \quad & d_A(\rho\otimes\sigma)\nabla_B d_B(\lambda\otimes\mu)\nabla_C\\
= \quad & d_A(d_A(\rho\otimes\sigma)\nabla_B d_B(\lambda\otimes\mu)\nabla_C\otimes d_A(\rho\lambda\otimes\sigma\mu)\nabla_C)\nabla_C\\
= \quad & d_A(\rho\lambda\otimes d_A(\rho\lambda\otimes\sigma\mu)\nabla_C)\nabla_C\\
= \quad & d_A(d_A(\rho\lambda\otimes\rho\lambda)\otimes\sigma\mu)a^{-1}_{C,C,C}(1_C\otimes\nabla_C)\nabla_C\\
= \quad & d_A(\rho\lambda\otimes\rho\lambda)\nabla_C\otimes\sigma\mu)\nabla_C\\
= \quad & d_A(\rho\lambda\otimes\sigma\mu)\nabla_C\\
\Rightarrow \quad & \rho\lambda\leq\sigma\mu.
\end{aligned}$$

For morphisms $\rho\leq\sigma\in K[A,B]$ and $\rho'\leq\sigma'\in K[A',B']$ one obtains

$$\rho=d_A(\rho\otimes\sigma)\nabla_B \ \text{ and } \ \rho'=d_{A'}(\rho'\otimes\sigma')\nabla_{B'},$$

hence

$$\begin{aligned}
\rho\otimes\rho' &= d_A(\rho\otimes\sigma)\nabla_B\otimes d_{A'}(\rho'\otimes\sigma')\nabla_{B'}\\
&= (d_A\otimes d_{A'})((\rho\otimes\sigma)\otimes(\rho'\otimes\sigma'))(\nabla_B\otimes\nabla_{B'})\\
&= d_{A\otimes A'}((\rho\otimes\rho')\otimes(\sigma\otimes\sigma'))b_{B,B',B,B'}(\nabla_B\otimes\nabla_{B'}) && ((D3),(M18))\\
&= d_{A\otimes A'}((\rho\otimes\rho')\otimes(\sigma\otimes\sigma'))\nabla_{B\otimes B'} && ((\nabla 5)),
\end{aligned}$$

therefore $\rho\otimes\rho'\leq\sigma\otimes\sigma'$.

Now let λ and μ be morphisms from A into B. Then

$$\begin{aligned}
d_A(\lambda\otimes\mu)\nabla_B &= d_A(d_A(\lambda\otimes\lambda)\nabla_B\otimes\mu)\nabla_B && ((D\nabla))\\
&= d_A(\lambda\otimes d_A(\lambda\otimes\mu)\nabla_B)\nabla_B && ((D1),(M6),(\nabla 3))\\
&= d_A s_{A,A}(\lambda\otimes d_A(\lambda\otimes\mu)\nabla_B)\nabla_B && ((D2))\\
&= d_A(d_A(\lambda\otimes\mu)\nabla_B\otimes\lambda)s_{B,B}\nabla_B && ((M8))\\
&= d_A(d_A(\lambda\otimes\mu)\nabla_B\otimes\lambda)\nabla_B && ((\nabla 4)),
\end{aligned}$$

hence $d_A(\lambda \otimes \mu)\nabla_B \leq \lambda$. In the same manner one shows $d_A(\lambda \otimes \mu)\nabla_B \leq \mu$.

Further let be $\tau \leq \lambda$ and $\tau \leq \mu$. Then it follows

$$\tau = d_A(\tau \otimes \mu)\nabla_B = d_A(d_A(\tau \otimes \lambda)\nabla_B \otimes \mu)\nabla_B = d_A(\tau \otimes d_A(\lambda \otimes \mu)\nabla_B)\nabla_B,$$

therefore $\tau \leq d_A(\lambda \otimes \mu)\nabla_B$. Consequently, $d_A(\lambda \otimes \mu)\nabla_B$ is the greatest lower bound of λ and μ with respect to the partial order relation. ∎

Lemma 2.3. *Any hdht∇s-category $\underline{K}$ has the following properties:*

$$\begin{array}{ll} \forall\ A \in |K| & (\nabla_A d_a \leq 1_{A\otimes A}), \\ \forall\ A, A' \in |K|\ \forall\ \rho \in K[A, A'] & (\rho d_{A'} \leq d_A(\rho \otimes \rho)), \\ \forall\ A, A' \in |K|\ \forall\ \rho \in K[A, A'] & (\nabla_A \rho \leq (\rho \otimes \rho)\nabla_{A'}). \end{array}$$

Proof. Composing the equation in condition (∇2) with $\nabla_{A',A'}$ and using (∇1) one obtains

$$\nabla_A d_A = \nabla_A d_A d_{A\otimes A} \nabla_{A'\otimes A'} = d_{A\otimes A}(\nabla_A d_A \otimes 1_{A\otimes A})\nabla_{A\otimes A},$$

hence $\nabla_A d_A \leq 1_{A\otimes A}$ by the definition of $\leq$.

Condition (D∇) gives rise to

$$\rho d_{A'} = (d_A(\rho \otimes \rho)\nabla_{A'})d_{A'} = (d_A(\rho \otimes \rho))(\nabla_{A'} d_{A'}) \leq d_A(\rho \otimes \rho) \qquad \text{and}$$

$$\nabla_A \rho = \nabla_A(d_A(\rho \otimes \rho)\nabla_{A'}) = (\nabla_A d_A)((\rho \otimes \rho)\nabla_{A'}) \leq (\rho \otimes \rho)\nabla_{A'},$$

respectively. ∎

Corollary 2.4. *By the definition of the partial order relation,*

(D9') $\quad \rho d_{A'} = d_A(\rho d_{A'} \otimes d_A(\rho \otimes \rho))\nabla_{A'\otimes A'}$ *and*

(∇9') $\quad \nabla_A \rho = d_{A\otimes A}(\nabla_A \rho \otimes (\rho \otimes \rho)\nabla_{A'})\nabla_{A'}$

are identities in each hdht∇s-category $\underline{K}$. ∎

Theorem 2.5. *Let $\underline{K}$ be an hdht∇s-category as defined above. Then the class*

$$F^K := \{\rho \in K \mid d_{\operatorname{dom}\rho}(\rho \otimes \rho) = \rho d_{\operatorname{cod}\rho}\}$$

of so-called functional morphisms forms an hdht∇s-subcategory $\underline{F}^K$ of $\underline{K}$ which is even a dht∇s-category.

The partial order relation in the dht∇-symmetric category $\underline{F}^K$ is the restriction of $\leq$ in the hdht∇-symmetric category $\underline{K}$.

Proof. The conditions (D5), (D7), and (D8) show that the class F^K contains all morphisms of the families d, t, and ∇, respectively.

Let $\rho \in K[A,B]$ be an isomorphism in $\underline{K}$. Then there is a $\rho^{-1} \in K[B,A]$ such that $\rho^{-1} d_A \le d_B(\rho^{-1} \otimes \rho^{-1})$ and $\rho d_B \le d_A(\rho \otimes \rho)$, hence $d_A(\rho \otimes \rho) \le \rho d_B \le d_A(\rho \otimes \rho)$, i.e. $\rho d_B = d_A(\rho \otimes \rho)$. Therefore, each isomorphism of $\underline{K}$ belongs to F^K, especially, all identities and all morphisms of the families $a,\ a^{-1},\ r,\ r^{-1},\ l,\ l^{-1},\ s,\ s^{-1},\ b,\ b^{-1}$ are in F^K. All zero morphisms $o_{A,B},\ A, B \in |K|,\ o = o_{I,O}$, are elements of F^K since $o_{A,B} d_B = o_{A,B\otimes B} = d_A(o_{A,B} \otimes o_{A,B})$.

Let $\rho \in K[A,B] \cap F^K$ and $\sigma \in K[B,C] \cap F^K$. Then

$$(\rho\sigma)d_C = \rho(\sigma d_C) = \rho(d_B(\sigma\otimes\sigma)) = (\rho d_B)(\sigma\otimes\sigma) = d_A(\rho\otimes\rho)(\sigma\otimes\sigma) = d_A(\rho\sigma\otimes\rho\sigma),$$

hence F^K is closed under composition.

If $\rho \in K[A,B]$ and $\rho' \in K[A',B']$ are morphisms of F^K, then $(\rho \otimes \rho') \in K[A \otimes A', B \otimes B']$ is in F^K too, since

$$\begin{aligned}
(\rho \otimes \rho')d_{B\otimes B'} &= (\rho \otimes \rho')(d_B \otimes d_{B'})b_{B,B,B',B'} \\
&= (d_A(\rho \otimes \rho) \otimes d_{A'}(\rho' \otimes \rho')b_{B,B,B',B'} \\
&= (d_A \otimes d_{A'})b_{A,A,A',A'}((\rho \otimes \rho') \otimes (\rho \otimes \rho')) \\
&= d_{A\otimes A'}((\rho \otimes \rho') \otimes (\rho \otimes \rho')).
\end{aligned}$$

With respect to the axioms of an $hdht\nabla s$-category, which are identities only, and because of the defining condition of $F^K \subseteq K$, one has a $dht\nabla s$-category $\underline{F}^K$.

The partial order relation $\le$ in $\underline{K}$ is defined by $\rho \le \sigma \Leftrightarrow \rho = d_A(\rho \otimes \sigma)\nabla_{A'}$ for morphisms $\rho, \sigma \in K[A, A']$. By property (P$\nabla$), this condition is equivalent to $\rho = d_A(\rho \otimes \sigma)p_2^{A',A'}$ for morphisms ρ, σ of F^K, hence $\rho \le \sigma$ with respect to the partial order relation in the $dht\nabla s$-category $\underline{F}^K$. ■

Proposition 2.6. *All morphisms $\rho \in K[A,B]$, $A, B \in |K|$, of an $hdht\nabla s$-category $\underline{K}$ fulfilling the condition $\rho t_B = t_A$ (so-called total morphisms) form a symmetric monoidal subcategory $T^{K\bullet}$ which contains all coretractions of $\underline{K}$ and all morphims t_A, $A \in |K|$.*

Moreover, $\underline{T}^K := (T^{K\bullet}, d, t)$ is an hdts-category.

Proof. Obviously, all identity morphisms 1_A, $A \in |K|$, are in T^K.
Because of

$$\rho t_B = t_A \ \wedge\ \sigma t_C = t_B \Rightarrow (\rho\sigma)t_c = \rho(\sigma t_C) = \rho t_B = t_A$$

and

$$\rho t_B{=}t_A\wedge\rho' t_{B'}{=}t_{A'}\Rightarrow(\rho\otimes\rho')t_{B\otimes B'}{=}(\rho\otimes\rho')(t_B\otimes t_{B'})t_{I\otimes I}{=}(t_A\otimes t_{A'})t_{I\otimes I}{=}t_{A\otimes A'}$$

the class T^K is closed under composition and $\otimes$-operation.

Let $\rho \in K[A,B]$ be a coretraction in $\underline{K}$. Then there is $\rho^* \in K[B,A]$ such that $\rho\rho^* = 1_A$. So, one has (see [6], p. 12)

$$\begin{aligned}
\rho t_B = 1_A\rho t_B &= d_A(1_A\otimes t_A)r_A\rho t_B && ((T1))\\
&= d_A(\rho t_B\otimes t_A)r_I && ((M7))\\
&= d_A(\rho\otimes\rho)(t_B\otimes\rho^* t_A)r_I && ((\rho\rho^*=1_A))\\
&\geq \rho d_B(t_B\otimes 1_B)(1_I\otimes\rho^* t_A)l_I && ((2.3))\\
&= \rho d_b(t_B\otimes 1_B)l_B\rho^* t_A && ((M14))\\
&= \rho 1_B\rho^* t_A && ((T4))\\
&= t_A \geq \rho t_B,
\end{aligned}$$

therefore $\rho t_B = t_A$, hence $\rho \in T^K$.

Because of $t_A t_I = t_A 1_I = t_A$, $A \in |K|$, $d_A\nabla_A = 1_A$, $A \in |K|$, and each isomorphism is just a coretraction, all morphisms of the families a, a^{-1}, r, r^{-1}, l, l^{-1}, s, s^{-1}, b, b^{-1}, d, and t belong to T^K.

Since arbitrary suitable morphisms and objects of $\underline{K}$ fulfil the identities (D1), (D2), (D3), (D5), (D6), (D7), (T1), (T2), (T3), (T4), (T5), (T6), (T7), (T8), (T9), the sequence $(T^{K\bullet}, d, t)$ is an *hdts*-category. ■

Corollary 2.7. *Let $\underline{K}$ be any hdht∇s-category. Then all morphisms of the families* 1, a, r, l, s, b, d, t, ∇, *and* $(o_{A,B} \mid A,B \in |K|)$ *possess all properties of such morhisms in a dht∇s-category, especially the following identities are valid:*

(D8), (T4), (T5), (T7), (T8), (B1), (B2), (o3), (o4), (o5),

(∇6), ∇7), (∇8), (∇10),

(I1) $\quad \nabla_I d_I = 1_{I\otimes I}$,

(I2) $\quad t_{I\otimes I} = \nabla_I = l_I = r_I = d_I^{-1}$,

(I3) $\quad d_I = r_I^{-1} = l_I^{-1}$,

(I4) $\quad d_I\otimes d_I = d_{I\otimes I}$. ■

Lemma 2.8. *Let $\underline{K}$ be an hdht∇s-category. Then one has*

(T9) $\rho t_{A'} d_I = d_A(\rho t_{A'} \otimes t_A)$

for all objects $A, A' \in |K|$ and all morphisms $\rho \in K[A, A']$.

Moreover:

(i) $\forall\ A, A' \in |K|\ \forall\ \rho \in K[A, A']\ (\rho d_{A'} d_{A' \otimes A'} = d_A(\rho d_{A'} \otimes d_A(\rho \otimes \rho))$
$$\Rightarrow \rho d_{A'} = d_A(\rho d_{A'} \otimes d_A(\rho \otimes \rho))\nabla_{A' \otimes A'}),$$

(ii) $\forall\ A, A' \in |K|\ \forall\ \rho \in K[A, A']\ (\nabla_A \rho d_{A'} = d_A(\nabla_A \rho \otimes (\rho \otimes \rho)\nabla_{A'})$
$$\Rightarrow \nabla_A \rho = d_A(\nabla_A \rho \otimes (\rho \otimes \rho)\nabla_{A'})\nabla_{A'}),$$

(iii) $\forall\ A, A' \in |K|\ \forall\ \rho \in K[A, A']\ (\rho t_{A'} d_I = d_A(\rho t_{A'} \otimes t_A)$
$$\Leftrightarrow \rho t_{A'} = d_A(\rho t_{A'} \otimes t_A)\nabla_I).$$

Proof. Because of $\nabla_I d_I = 1_{I \otimes I}$ and $\nabla_I = r_I = l_I = t_{I \otimes I}$ the equation

$$d_A(\rho t_{A'} \otimes t_A) = d_A(\rho t_{A'} \otimes t_A)\nabla_I d_I = d_A(\rho t_{A'} \otimes t_A) r_I d_I$$
$$= d_A(1_A \otimes t_A) r_A \rho t_{A'} d_I = \rho t_{A'} d_I$$

is valid for each $\rho \in K[A, A']$ and all $A, A' \in |K|$, hence $\underline{K}$ fulfils condition (T9).

The condition (T9') is equivalent to (T9), since

$$d_A(\rho t_{A'} \otimes t_A) = \rho t_{A'} d_I \Rightarrow d_A(\rho t_{A'} \otimes t_A)\nabla_I = \rho t_{A'}$$

by $d_I \nabla_I = 1_I$ and

$$d_A(\rho t_{A'} \otimes t_A)\nabla_I = \rho t_{A'} \Rightarrow d_A(\rho t_{A'} \otimes t_A) = \rho t_{A'} d_I$$

by $\nabla_I d_I = 1_{I \otimes I}$, hence property (iii) is shown.

The implications (i) and (ii) are satisfied because of the general property

$$\xi d_B = d_A(\xi \otimes \eta) \Rightarrow \xi = \xi d_B \nabla_B = d_A(\xi \otimes \eta)\nabla_B.$$

■

Remark 2.9. The opposite of the implications (i) and (ii), respectively, is not true in general, since there are conterexamples in $\underline{Rel}$.

Remark 2.10. As in any $dht\nabla s$-category, the morphisms

$$p_1^{A,B} := (1_A \otimes t_B)r_A \in K[A \otimes B, A] \cap F^K,$$

$$p_2^{A,B} := (t_A \otimes 1_B)l_B \in K[A \otimes B, B] \cap F^K$$

of an arbitrary $hdht\nabla s$-category $\underline{K}$ are called *canonical projections* again and one has

$$\nabla_A = \inf\left\{p_1^{A,A}, p_2^{A,A}\right\} = d_A\left(p_1^{A,A} \otimes p_2^{A,A}\right)\nabla_A$$

for all $A \in |K|$.

Remark that $(A \otimes B; p_1^{A,B}, p_2^{A,B})$ is not a categorical product in the whole category $\underline{K}$, but in the subcategory T^K

The family $\nabla = (\nabla_A \mid A \in |K|)$ is uniquely determined by the family $d = (d_A \mid A \in |K|)$ and the conditions (∇1) and (∇2).

Lemma 2.11. *Let $\underline{K}$ be an arbitrary $hdht\nabla s$-category. Then there holds:*

(∗2) $\forall A, B, C \in |K| \ \forall \rho, \rho' \in K[A, B] \ \forall \sigma, \sigma' \in K[B, C] \ (d_A(\rho \otimes \rho')\nabla_B = \rho$
$\wedge \ d_B(\sigma \otimes \sigma')\nabla_C = \sigma \Rightarrow d_A(\rho\sigma \otimes \rho'\sigma')\nabla_C = \rho\sigma)$,

(∗3) $\forall A, B \in |K| \ \forall \rho, \sigma \in K[A, B] \ (d_A(\rho \otimes \sigma)\nabla_B = \rho \ \wedge \ d_A(\sigma \otimes \sigma) = \sigma d_B$
$\Rightarrow d_A(\rho \otimes \sigma)p_i^{B,B} = \rho \quad (i \in \{1, 2\}))$,

(∗4) $\forall A, B \in |K| \ \forall \rho \in K[A, B] \ (d_A(\rho \otimes \rho)p_i^{B,B} = \rho \quad (i \in \{1, 2\}))$,

(∗5) $\forall A, B \in |K| \ \forall \rho, \sigma \in K[A, B] \ (d_A(\rho \otimes \sigma)\nabla_B = \rho \ \wedge \ d_A(\sigma \otimes \sigma) = \sigma d_B$
$\Rightarrow d_A(\rho \otimes \rho) = \rho d_B)$,

(∗6) $\forall A \in |K| \ \forall \rho \in K[A, A] \ (d_A(1_A \otimes \rho)\nabla_A = \rho$
$\Rightarrow d_A(1_A \otimes \rho)p_1^{A,A} = d_A(1_A \otimes \rho)p_2^{A,A} = \rho)$.

Proof. Axiom (∗1) implies condition (∗2) because of $\rho \leq \rho' \quad \wedge \quad \sigma \leq \sigma' \Rightarrow \rho\sigma \leq \rho'\sigma'$. To show (∗3) not that $d_A(\rho \otimes \sigma)\nabla_B = \rho \Leftrightarrow \rho \leq \sigma$ and $d_A(\sigma \otimes \sigma) = \sigma d_B \Leftrightarrow \sigma \in F^K$. So one obtains

$$\begin{aligned} d_A(\rho\otimes\sigma)p_i^{B,B} &= d_A(d_A(\rho\otimes\sigma)\nabla_B\otimes\sigma)p_i^{B,B} && (\rho\leq\sigma)\\ &= d_A(\rho\otimes d_A(\sigma\otimes\sigma))a_{B,B,B}(\nabla_B\otimes 1_B)p_i^{B,B} && (\sigma\in F_K)\\ &= d_A(\rho\otimes\sigma)(1_B\otimes d_B)a_{B,B,B}(\nabla_B\otimes 1_B)p_i^{B,B} && ((F4))\\ &= d_A(\rho\otimes\sigma)\nabla_B d_B p_i^{B,B} && ((\nabla 7))\\ &= d_A(\rho\otimes\sigma)\nabla_B=\rho \end{aligned}$$

with respect to the axioms of an $hdht\nabla s$-category.

The property $(*4)$ is a consequence of (D9') and (T9'):

$$\begin{aligned} &\rho=\rho d_B p_i^{B,B}\leq d_A(\rho\otimes\rho)p_i^{B,B}\ \wedge\ \rho t_B\leq t_A\\ &\quad\Rightarrow d_A(\rho\otimes\rho)p_1^{B,B}=d_A(\rho\otimes\rho t_B)r_B\leq d_A(\rho\otimes t_A)r_B=d_A(1_A\otimes t_A)r_A\rho=\rho\\ &\quad\wedge\ \ d_A(\rho\otimes\rho)p_2^{B,B}=d_A(\rho t_B\otimes\rho)l_B\leq d_A(t_A\otimes\rho)l_B=d_A(t_A\otimes 1_A)l_A\rho=\rho. \end{aligned}$$

(*5): Using the previous results and the assumption one obtains

$$\begin{aligned} d_A(\rho\otimes\rho)&=d_A(d_A(\rho\otimes\sigma)p_2^{B,B}\otimes d_A(\rho\otimes\sigma))p_2^{B,B})\\ &=d_A(d_A\otimes d_A)((\rho\otimes\sigma)\otimes(\rho\otimes\sigma)(p_2^{B,B}\otimes p_2^{B,B})\\ &=d_A d_{A\otimes A}((\rho\otimes\sigma)\otimes(\rho\otimes\sigma))(p_2^{B,B}\otimes p_2^{B,B})\\ &=d_A(d_A(\rho\otimes\rho)\otimes d_A(\sigma\otimes\sigma))b_{B,B,B,B}(p_2^{B,B}\otimes p_2^{B,B})\\ &=d_A(d_A(\rho\otimes\rho)\otimes\sigma d_B)p_2^{B\otimes B,B\otimes B}\\ &=d_A(\rho\otimes d_A(\rho\otimes\sigma))a_{B,B,B}(1_{B\otimes B}\otimes d_B)p_2^{B\otimes B,B\otimes B}\\ &=d_A(\rho\otimes d_A(\rho\otimes\sigma))a_{B,B,B}p_2^{B\otimes B,B}d_B\\ &=d_A(\rho\otimes d_A(\rho\otimes\sigma))(1_B\otimes p_2^{B,B})p_2^{B,B}d_B\\ &=d_A(\rho\otimes d_A(\rho\otimes\sigma)p_2^{B,B})p_2^{B,B}d_B\\ &=d_A(\rho\otimes\rho)p_2^{B,B}d_B=\rho d_B. \end{aligned}$$

The property (*6) arises from (*3) because of $1_A\in F^K$ for each $A\in|K|$. ■

Lemma 2.12. *Let $\underline{K}$ be a monoidal symmetric category endowed with morphisms families d, t, $(o_{A,B} \mid A, B \in |K|)$, and ∇ such that all axioms of an hdht∇s-category without* (*1) *are fulfilled. Moreover, let the condition* (*2) *be valid. Then $\underline{K}$ is an hdht∇s-category in the defined sense as above.*

Proof. It remains to show the condition (*1):

$$\begin{aligned}
&d_A(d_A(\rho\otimes\rho')\nabla_B d_B(\sigma\otimes\sigma')\nabla_C \otimes d_A(\rho\sigma\otimes\rho'\sigma')\nabla_C)\nabla_C \\
&\quad = d_A(\rho\sigma\otimes d_A(\rho\sigma\otimes\rho'\sigma')\nabla_C)\nabla_C && ((*2))\\
&\quad = d_A(1_A\otimes d_A)(\rho\sigma\otimes(\rho\sigma\otimes\rho'\sigma'))(1_C\otimes\nabla_C)\nabla_C && ((F4))\\
&\quad = d_A(d_A\otimes 1_A)a^{-1}_{A,A,A}(\rho\sigma\otimes(\rho\sigma\otimes\rho'\sigma'))(1_C\otimes\nabla_C)\nabla_C && ((D3))\\
&\quad = d_A(d_A\otimes 1_A)((\rho\sigma\otimes\rho\sigma)\otimes\rho'\sigma')a^{-1}_{C,C,C}(1_C\otimes\nabla_C)\nabla_C && ((M6))\\
&\quad = d_A(d_A)(\rho\sigma\otimes\rho\sigma)\otimes\rho'\sigma'))(\nabla_C\otimes 1_C)\nabla_C && ((\nabla 3))\\
&\quad = d_A(d_A(\rho\sigma\otimes\rho\sigma)\nabla_C\otimes\rho'\sigma'))\nabla_C && ((F4))\\
&\quad = d_A(\rho\sigma\otimes\rho'\sigma')\nabla_C && ((D\nabla))\\
&\quad = \rho\sigma && ((*2))\\
&\quad = d_A(\rho\otimes\rho')\nabla_B d_B(\sigma\otimes\sigma')\nabla_C && ((*2))
\end{aligned}$$

■

The results of the last both lemmata are important for the axiomization of hdht∇s- categories. The system of axioms for an hdht∇s-category given in [11] contains two identical implications, namely (21) (⇔ (*2)) and (20) (⇔ (*6)). The property (*6) is a consequence of the other properties and the conditions (*1) and (*2) are equivalent in a monoidal symmetric category $\underline{K}$ endowed with morphisms families d, t, $(o_{A,B} \mid A, B \in |K|)$, and ∇ such that

(D1), (D2), (D3), (D5), (D7), (D8), (T1), (T2), (T6), (T9'),

(∇1), (∇2), (∇3), (∇4), (∇5), (∇6), (∇7), (D∇),

(o1), (o2), (O1)

are fulfilled. Therefore, hdht∇s-categories are axiomatizable by identities only, hence all small hdht∇s-categories form a variety of many-sorted total algebras and there are free many-sorted algebras to each generating set with respect to this variety. Especially, there are free $hdht\nabla s$-theories, i.e. free algebraic theories for relational structures, by analogy with the existence of free algebraic theories for partial algebras ([3], [10]).

Lemma 2.13. *In any $hdht\nabla$-symmetric category the following conditions are fulfilled for arbitrary morphisms ρ, σ:*

$$\text{(j)} \qquad \rho\sigma = 1_A \quad \wedge \quad \sigma\rho \leq 1_B \quad \Rightarrow \quad d_A(\rho \otimes \rho) = \rho d_B$$

$$\text{(jj)} \qquad \rho\sigma \leq 1_A \quad \wedge \quad \sigma\rho = 1_B \quad \Rightarrow \quad \nabla_A \rho = (\rho \otimes \rho)\nabla_B$$

Proof. To show (j) we use at first the known property $\sigma d_A \leq d_B(\sigma \otimes \sigma)$. Further,

$$d_A(\rho \otimes \rho) = \rho\sigma d_A(\rho \otimes \rho) \leq \rho d_B(\sigma \otimes \sigma)(\rho \otimes \rho) \leq \rho d_B(1_B \otimes 1_B) = \rho d_B,$$

hence $d_A(\rho \otimes \rho) = \rho d_B$ by $\rho d_B \leq d_A(\rho \otimes \rho)$.

In a similar way one shows the statement (jj), namely because of $\nabla_B \sigma \leq (\sigma \otimes \sigma)\nabla_A$ and

$$(\rho \otimes \rho)\nabla_B = (\rho \otimes \rho)\nabla_B \sigma\rho \leq (\rho\sigma \otimes \rho\sigma)\nabla_A \rho \leq \nabla_A \rho \leq (\rho \otimes \rho)\nabla_B$$

one has $\nabla_A \rho = (\rho \otimes \rho)\nabla_B$. ∎

Definition 2.14. Morphisms $e \in K[A, A] \subseteq K$ with the property $e \leq 1_A$, i.e. $e = d_A(1_A \otimes e)\nabla_A$, are called *subidentities* in $\underline{K}$ (compare with ([7])).

Proposition 2.15 (cf. [7]). *For each morphism $\rho : A \to B$, $A, B \in |K|$, the morphism*

$$\alpha(\rho) := d_A(\rho \otimes 1_A)p_2^{B,A}$$

is a subidentity of A in $\underline{K}$ and there holds $\alpha(\rho)\rho = \rho$. Each subidentity e of $\underline{K}$ fulfils $d_A(e \otimes e) = ed_A$, therefore the subidentities of $\underline{K}$ are the subidentities of $\underline{F}^K$ and satisfy the following conditions for all suitable morphims and objects of K:

$$
\begin{array}{lll}
\text{(e1)} & e \leq 1_A & \Rightarrow \quad ee = e, \\
\text{(e2)} & e_1, e_2 \leq 1_A & \Rightarrow \quad e_1e_2 = e_2e_1 = \mathit{inf}\{e_1, e_2\}, \\
\text{(e3)} & e_1 \leq e_2 \leq 1_A & \Leftrightarrow \quad e_1 = e_1e_2 \leq 1_A, \\
\text{(e4)} & e \leq 1_A & \Leftrightarrow \quad \alpha(e) = e, \\
\text{(e5)} & e \leq 1_A & \Rightarrow \quad ed_A = d_A(e \otimes e) = d_A(e \otimes 1_A), \\
\text{(e6)} & e \leq 1_A & \Rightarrow \quad \nabla_A e = (e \otimes e)\nabla_A = (e \otimes 1_A)\nabla_A, \\
\text{(e7)} & \rho, \sigma \in K[A, B] & \Rightarrow \alpha(\rho)\sigma = d_A(\rho\otimes\sigma)p_2^{B,B} \wedge \alpha(\sigma)\rho = d_A(\rho\otimes\sigma)p_1^{B,B}, \\
\text{(e8)} & \alpha(\rho)\sigma = \rho & \Rightarrow \quad \rho \leq \sigma, \\
\text{(e9)} & e\rho = \rho \;\wedge\; e \leq 1_A & \Leftrightarrow \quad \alpha(\rho) \leq e \leq 1_A, \\
\text{(e10)} & cod\rho = dom\sigma & \Rightarrow \alpha(\rho\sigma) \leq \alpha(\rho), \\
\text{(e11)} & e \leq 1_A & \Rightarrow \alpha(e\rho) \leq e, \\
\text{(e12)} & e \leq 1_A & \Rightarrow \alpha(e\rho) = e\alpha(\rho), \\
\text{(e13)} & \rho \leq \sigma & \Rightarrow \quad \alpha(\rho) \leq \alpha(\sigma), \\
\text{(e14)} & cod\rho = dom\sigma & \Rightarrow \rho\alpha(\sigma) \leq \alpha(\rho\sigma)\rho, \\
\text{(e15)} & cod\rho = dom\sigma & \Rightarrow \alpha(\rho\sigma) = \alpha(\rho\alpha(\sigma)),
\end{array}
$$

Proof. Because of $\rho t_B \leq t_A$ one obtains

$$\alpha(\rho) = d_A(\rho \otimes 1_A)p_2^{B,A} = d_A(\rho t_B \otimes 1_A)l_A \leq d_A(t_A \otimes 1_A)l_A = 1_A.$$

Using the definition of $\alpha(\rho)$, properties $(M14)$, $(M15)$, and $\alpha(\rho) \leq 1_A$ one receives $\alpha(\rho)\rho = \rho$ via

$$\alpha(\rho)\rho = d_A(\rho \otimes 1_A)p_2^{B,A}\rho = d_A(\rho \otimes \rho)p_2^{B,B} \geq \rho d_B p_2^{B,B} = \rho = 1_A\rho \geq \alpha(\rho)\rho.$$

Because of $e \leq 1_A$ the property $d_A(e \otimes e) = ed_A$ is a consequence of Lemma 2.11, (*5), and the subidentities of $\underline{K}$ are exactly the subidentities of $\underline{F}^K$, therefore, all subidentities have the properties (e1), (e2), (e3) and (e4) (cf. [7]).

To show property (e5) use the property (e4) $e \leq 1_A \Rightarrow e = \alpha(e) = d_A(e \otimes 1_A)p_2^{A,A}$:

$$d_A(e \otimes e) = d_A(e \otimes d_A(e \otimes 1_A)p_2^{A,A}) = d_A(d_A(e \otimes e) \otimes 1_A)a_{A,A,A}^{-1}(1_A \otimes p_2^{A,A})$$
$$= d_A(d_A(e \otimes e)p_1^{A,A} \otimes 1_A) = d_A(e \otimes 1_A).$$

The second part of the property (e6) is a consequence of (e2) and (e5) owing to $\nabla_A d_A \leq 1_{A\otimes A}$, $(e \otimes e) \leq 1_{A\otimes A}$, and $(e \otimes 1_A) \leq 1_{A\otimes A}$:

$$\begin{aligned} d_A(e \otimes e) = d_A(e \otimes 1_A) &\Rightarrow \nabla_A d_A(e \otimes e) = \nabla_A d_A(e \otimes 1_A) \\ &\Rightarrow (e \otimes e)\nabla_A d_A = (e \otimes 1_A)\nabla_A d_A && ((\mathrm{e}2)) \\ &\Rightarrow (e \otimes e)\nabla_A d_A \nabla_A = (e \otimes 1_A)\nabla_A d_A \nabla_A \\ &\Rightarrow (e \otimes e)\nabla_A = (e \otimes 1_A)\nabla_A. && ((\nabla 1)) \end{aligned}$$

Because of $(e \otimes e) \leq 1_{A\otimes A}$ and $\nabla_A d_A \leq 1_{A\otimes A}$ one has

$$\begin{aligned} (e \otimes e)\nabla_A &= (e \otimes e)\nabla_A d_A \nabla_A && (d_A\nabla_A = 1_A) \\ &= \nabla_A d_A (e \otimes e)\nabla_A && ((\mathrm{e}2)) \\ &= \nabla_A e. && ((D\nabla)) \end{aligned}$$

Property (e7) is an immediate consequence of $(M7)$, $(M14)$, $(M8)$, and $(M13)$.

To show (e8) take into consideration

$$\rho = \alpha(\rho)\sigma \leq 1_A\sigma = \sigma.$$

(e9): Assuming $e\rho = \rho$, $e \leq 1_A$ one gets

$$\alpha(\rho) = \alpha(e\rho) = d_A(e\rho \otimes 1_A)p_2^{B,A} = d_A(e\rho t_B \otimes 1_A)l_A \leq d_A(et_A \otimes 1_A)l_A = \alpha(e) = e.$$

Conversely, $\alpha(\rho) \leq e \leq 1_A$ yields

$$\rho = \alpha(\rho)\rho \leq e\rho \leq 1_A\rho = \rho.$$

Condition (e10) is true, since

$$\alpha(\rho\sigma) = d_A(\rho\sigma \otimes 1_A)p_2^{C,A} = d_A(\rho\sigma t_C \otimes 1_A)l_A \leq d_A(\rho t_B \otimes 1_A)l_A = \alpha(\rho).$$

Condition (e11) arises from $\alpha(e\rho) \leq \alpha(e) = e$.

Property (e12) is a consequence of (e5) as follows:

$$\begin{aligned}\alpha(e\rho) &= d_A(e\rho \otimes 1_A)p_2^{B,A} = d_A(e \otimes 1_A)(\rho \otimes 1_A)p_2^{B,A} \\ &= d_A(e \otimes e)(\rho \otimes 1_A)p_2^{B,A} = ed_A(\rho \otimes 1_A)p_2^{B,A} \\ &= e\alpha(\rho).\end{aligned}$$

To show (e13) use the definitions of $\leq$ and $\alpha(\rho)$ ($\rho : A \to B$, $\sigma : B \to C$):

$$\begin{aligned}\alpha(\rho) = d_A(\rho \otimes 1_A)p_2^{B,A} &= d_A(d_A(\rho \otimes \sigma)\nabla_B \otimes 1_A)p_2^{B,A} && (\rho \leq \sigma) \\ &\leq d_A(d_A(\rho \otimes \sigma)p_2^{B,B} \otimes 1_A)p_2^{B,A} && (\nabla_B \leq p_2^{B,B}) \\ &= d_A(d_A(\rho \otimes 1_A)p_2^{B,A}\sigma \otimes 1_A)p_2^{B,A} && ((M14)) \\ &= d_A(\alpha(\rho)\sigma \otimes 1_A)p_2^{B,A} \\ &\leq d_A(\sigma \otimes 1_A)p_2^{B,A} = \alpha(\sigma). && (\alpha(\rho)\sigma \leq \sigma)\end{aligned}$$

Assertion (e14) is true since

$$\rho\alpha(\sigma) = \rho d_B(\sigma \otimes 1_B)p_2^{C,B} \leq d_A(\rho\sigma \otimes \rho)p_2^{C,B} = \alpha(\rho\sigma)\rho.$$

Condition (e15) follows by (e10), (e13), and (e14):

Let ρ and σ be as above. Then one has

$$\alpha(\rho\sigma) = \alpha(\rho\alpha(\sigma)\sigma) \leq \alpha(\rho\alpha(\sigma)),$$

hence

$$\begin{aligned}\alpha(\rho\sigma) \leq \alpha(\rho\alpha(\sigma)) \quad &\leq \alpha(\alpha(\rho\sigma)\rho) \leq \alpha(\alpha(\rho\sigma)\alpha(\rho)) \\ &\leq \alpha(\alpha(\rho\sigma)1_A) = \alpha(\alpha(\rho\sigma)) \quad = \alpha(\rho\sigma).\end{aligned}$$ ■

Remark that, as an easy example shows, in *Rel* the opposite implication to (e8) is not true: Let be given $A = \{a\}, B = \{b_1, b_2\}, \rho = \{(a, b_1)\}, \sigma = \{(a, b_1), (a, b_2)\}$. Then $\rho \leq \sigma$ and $\rho < \alpha(\rho)\sigma = \sigma$.

Furthermore, the equality in (e14) is not true in general. For this let be the sets A and B as above and let be $C = \{x\}$. For the relations σ as above and $\tau = \{(b_1, x)\}$ one obtains $\sigma\alpha(\tau) = \{(a, b_1)\}$ and $\sigma\tau = \{(a, x)\}$, hence $\alpha(\sigma\tau) = \{(a, a)\}$, consequently $\alpha(\sigma\tau)\sigma = \{(a, b_1), (a, b_2)\} = \sigma \neq \sigma\alpha(\tau)$.

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