Discussiones Mathematicae
General Algebra and Applications 21 (2001) 129–137

CARDINALITIES OF LATTICES OF TOPOLOGIES OF UNARS AND SOME RELATED TOPICS

Anna Kartashova

Department of Algebra and Geometry
Volgograd Pedagogical University
Eletskaya 7–177, 400120 Volgograd, Russia
e-mail: kvk@vspu.ru

Abstract

In this paper we find cardinalities of lattices of topologies of uncountable unars and show that the lattice of topologies of a unar cannor be countably infinite. It is proved that under some finiteness conditions the lattice of topologies of a unar is finite. Furthermore, the relations between the lattice of topologies of an arbitrary unar and its congruence lattice are established.

Keywords: unar, lattice of topologies, lattice of congruences.

2000 Mathematics Subject Classification: 08A60, 22A30, 08A30.

Let $\mathfrak{A} = \langle A, \Omega \rangle$ be an arbitrary algebra. A topology on the set A, under which every operation from Ω is continuous is called a *topology on the algebra* $\mathfrak{A}$. It is known [5] (p. 69) that the topologies on an algebra $\mathfrak{A}$ form a lattice under set inclusion. Let us call this lattice the *lattice of topologies* of the algebra $\mathfrak{A}$. Denote this lattice by $\Im(\mathfrak{A})$.

Let now $\mathfrak{A} = \langle A, f \rangle$ be a unar, i. e. an algebra with one unary operation f (see [6]). For any element $a \in A$ and any positive integer n we put $f^0(a) = a$ and $f^n(a) = f(f^{n-1}(a))$. Throughout the paper we shall denote by $\mathbb{N}$ the set of all positive integers and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$.

A unar generated by one element a is called *monogenic* and it is denoted by (a). A monogenic unar with the generator a and with defining relation $f^n(a) = f^{n+m}(a), \quad n \in \mathbb{N}_0, m \in \mathbb{N}$ is denoted by C_m^n. The unar C_m^0 is termed a *cycle of length* m. An element a of the unar $\mathfrak{A}$ is *cyclic* if the subunar generated by this element is cyclic. The set of all cyclic elements of a unar $\mathfrak{A}$ is denoted by $Z(\mathfrak{A})$. An element a of a unar $\mathfrak{A} = \langle A, f \rangle$ is *periodic* if $f^k(a) \in Z(\mathfrak{A})$ for some $k \in \mathbb{N}_0$. Otherwise it is called *torsion-free.* The union of a sequence of unars $C_m^0 \subset C_m^1 \subset C_m^2 \subset \dots$ will be denoted by C_m^∞. If a is a periodic element of a unar $\mathfrak{A} = \langle A, f \rangle$, then the least integer $n \in \mathbb{N}_0$ such that $f^n(a) \in Z(\mathfrak{A})$, is the *depth* of a. It is denoted by $d(a)$. A unar is *periodic* if each element in $\mathfrak{A}$ is periodic. A free monogenic unar is denoted by $\mathcal{F}_1$.

The disjoint union of two unars $\mathfrak{B}$ and $\mathfrak{C}$ is denoted by $\mathfrak{B} + \mathfrak{C}$. Unars $\mathfrak{B}$ and $\mathfrak{C}$ are *components* of the unar $\mathfrak{B} + \mathfrak{C}$. A unar having no proper components is called *connected.* The set of all connected components of an arbitrary unar $\mathfrak{A}$ is denoted by $c(\mathfrak{A})$.

Proposition 1. *The lattice of all topologies on the set $c(\mathfrak{A})$ of connected components of an arbitrary unar $\mathfrak{A} = \langle A, f \rangle$ is isomorphic to some principal ideal of the lattice $\Im(\mathfrak{A})$.*

Proof. Define binary relation η on the set A by setting

$$x\eta y \;\Leftrightarrow\; (\exists n, m \in \mathbb{N}_0)[f^n(x) = f^m(y)]$$

for any elements $x, y \in A$. It is clear that $\eta \in \mathrm{Con}(\mathfrak{A})$ and the factor unar $\mathfrak{A}/\eta$ is a union of one-element cycles. Moreover the lattice $\Im(\mathfrak{A}/\eta)$ of topologies of the unar $\mathfrak{A}/\eta$ coincides with the lattice of all topologies on the set A/η. By [2] (Theorem 3) the lattice of all topologies on the set $c(\mathfrak{A})$ is isomorphic to a principal ideal of $\Im(\mathfrak{A})$ because $|c(\mathfrak{A})| = |\mathfrak{A}/\eta|$. ∎

Observe that

the lattice $\mathcal{R}(Y)$ of all topologies on a nonvoid subset Y of an arbitrary set X can be embedded into the lattice $\mathcal{R}(X)$ of all topologies on the set X as a principal ideal.

In fact, fix a point $y_0 \in Y$ and define a mapping $\psi : \mathcal{R}(Y) \to \mathcal{R}(X)$ in the following way. Let $\sigma \in \mathcal{R}(Y)$. Denote be $\psi(\sigma)$ the family of subsets of the set X such that $T \in \psi(\sigma)$ if and only if either $T \in \sigma$ and $y_0 \notin T$ or $T \cap Y \in \sigma, X \smallsetminus Y \subseteq T$ and $y_0 \in T$. Then ψ is an isomorphism of $\mathcal{R}(Y)$ onto the principal ideal of $\mathcal{R}(X)$ generated by the topology $\psi(\sigma_1)$, where σ_1 is the discrete topology on X. ∎

From Proposition 1, we can deduce

Lemma 1. *Let $\mathfrak{A} = \langle A, f \rangle$ be an arbitrary unar and K be a nonvoid subset of the set $c(\mathfrak{A})$ of connected components of the unar $\mathfrak{A}$. Then the lattice $\mathcal{R}(K)$ of all topologies on the set K is isomorphic to a principal ideal of the lattice $\Im(\mathfrak{A})$.* ∎

Elements a, b of an arbitrary unar $\mathfrak{A} = \langle A, f\rangle$ are *incomparable* if $a \notin (b)$ and $b \notin (a)$.

Lemma 2. *Let $\mathfrak{A} = \langle A, f\rangle$ be an arbitrary unar and A_1 be an infinite set of pairwise non-cyclic incomparable elements of $\mathfrak{A}$. Then the lattice $\mathcal{R}(A_1)$ of all topologies on the set A_1 can be embedded into the lattice $\Im(\mathfrak{A})$.*

Proof. Denote by $\mathfrak{B} = \langle B, f\rangle$ the subunar of unar $\mathfrak{A}$ generated by set A_1. Define a binary relation $\rho = \{(a, b) \in B \times B |\ a = b \vee \{a, b\} \cap A_1 = \emptyset\}$ on the set B. Certainly the relation ρ is an equivalence.

We claim that $\rho \in \operatorname{Con}\mathfrak{B}$. In fact let $a \notin A_1$, and $f(a) \in A_1$. Since $a \in B$, there exists an element $c \in A_1$ and an integer $n \in \mathbb{N}_0$ such that $a = f^n(c)$. Hence, $f(a) = f^{n+1}(c)$. It follows that $n + 1 = 0$ and $n \notin \mathbb{N}_0$, since $f(a) \in A_1$ and $c \in A_1$. Every topology on the factor set B/ρ is a topology on the unar $\mathfrak{B}/\rho$ because either $f^{-1}(X) = \emptyset$ or $f^{-1}(X) = B/\rho$ holds for any subset X of the set B/ρ. Thus applying [2] (Theorems 2 and 3) and the equality $|A_1| = |B/\rho|$ we can conclude that the lattice $\mathcal{R}(A_1)$ of all topologies on the set A_1 can be embedded into the lattice $\Im(\mathfrak{A})$. ■

Lemma 3. *Let $\mathfrak{A} = \langle A, f\rangle$ be an arbitrary infinite unar. Then either the lattice $\Im(\mathcal{F}_1)$ or the lattice $\Im(C_1^\infty)$ can be embedded into the lattice $\Im(\mathfrak{A})$ of all topologies of the unar $\mathfrak{A}$.*

Proof. If $\mathfrak{A}$ contains a torsion-free element a, then $(a) \cong \mathcal{F}_1$, where (a) is the monogenic subunar of the unar $\mathfrak{A}$ generated by the element a. By [2] (Theorem 3) the lattice $\Im(\mathcal{F}_1)$ can be embedded into the lattice $\Im(\mathfrak{A})$.

If the set $c(\mathfrak{A})$ is infinite or the inequality $|f^{-1}(\{a\})| \geq \aleph_0$ holds for some $a \in A$, then the lattice $\mathcal{R}(X)$ of all topologies on a countable infinite set X is isomorphic to some sublattice of the lattice $\Im(\mathfrak{A})$ by Lemmas 1 and 2. On the other hand, $|\mathcal{F}_1| = \aleph_0$. Therefore, the lattice $\Im(\mathcal{F}_1)$ of all topologies of $\mathcal{F}_1$ can be embedded into the lattice $\mathcal{R}(X)$ and, hence, into the lattice $\Im(\mathfrak{A})$.

Let $\mathfrak{A}$ be periodic, $|c(\mathfrak{A})| < \aleph_0$, a set $f^{-1}(\{a\})$ finite for any element $a \in \mathfrak{A}$. Then there exists a subunar $\mathfrak{B} = \langle B, f\rangle$ of the unar $\mathfrak{A}$, which is isomorphic to C_h^∞, where $h \in \mathbb{N}$. Put $\rho = \{(a, b) \in B \times B |\ (a = b) \vee \{a, b\} \subseteq Z(\mathfrak{A})\}$. Then $\rho \in \operatorname{Con}\mathfrak{B}$ and the factor unar $\mathfrak{B}/\rho$ is isomorphic to C_1^∞. Consequently, the lattice $\Im(C_1^\infty)$ of all topologies of C_1^∞ can be embedded into the lattice $\Im(\mathfrak{A})$ by [2] (Theorems 2 and 3). ■

The least topology with respect to inclusion on the unary algebra $\mathfrak{A}$, containing a given family of subsets $\{A_\alpha \subseteq A |\, \alpha \in I\}$ will be called the *topology on the algebra* $\mathfrak{A}$ *generated by the set of elements* $\{A_\alpha |\, \alpha \in I\}$. This topology will be denoted by $t(\{A_\alpha |\, \alpha \in I\})$ and respectively by $t(U)$ if the family $\{A_\alpha |\, \alpha \in I\}$ consists of one set U.

Lemma 4. *Let $\mathfrak{A} = \langle A, f\rangle$ be isomorphic to C_1^∞. Then $t(X_1) = t(X_2) \Rightarrow X_1 = X_2$ for any nonvoid subsets X_1, X_2 of the set $A \setminus Z(\mathfrak{A})$.*

Proof. Since $t(X_1) = t(X_2)$, we conclude that $X_2 \in t(X_1)$. Hence, the set X_2 is a union of finite intersections of some sets of the form $f^{-i}(X_1)$, where $i \in \mathbb{N}_0$, because $X_2 \neq \emptyset$ and $X_2 \subseteq A \setminus Z(\mathfrak{A}) \underset{\neq}{\subset} A$.

Since $\mathfrak{A} \cong C_1^\infty$ and $X_2 \subseteq A \setminus Z(\mathfrak{A})$, there exists an element $x \in X_2$ such that

$$(\forall k \in \mathbb{N})\ [f^k(x) \notin X_2]. \tag{1}$$

Since $x \in X_2$ and $X_2 \in t(X_1)$ we have $x \in \bigcap_{i\in I} f^{-i}(X_1) \subseteq X_2$ for some finite set of indices I of the set $\mathbb{N}_0$. We claim that $I = \{0\}$. In fact, if $x \in f^{-i}(X_1)$, then $f^i(x) \in X_1$. On the other hand, since $X_1 \in t(X_2)$, the set X_1 is a union of finite intersections of some sets of the form $f^{-j}(X_2)$, where $j \in \mathbb{N}_0$. However, by (1), the condition $f^{i+j}(x) \in X_2$ implies $i+j = 0$. Hence, $i = 0$ and so $I = \{0\}$. Consequently, $X_1 \subseteq X_2$. Similarly, we can prove that $X_2 \subseteq X_1$. Thus, $X_1 = X_2$. ■

Let $\mathfrak{A}$ be an arbitrary algebra and $\theta \in \mathrm{Con}(\mathfrak{A})$. θ-congruence classes form a base of some topology $\tau(\theta)$ which we shall call the *topology generated by the congruence* θ.

Proposition 2. *There exists a set $\mathcal{H}$ of the cardinality $2^{\aleph_0}$ of different Hausdorff topologies on the unar $\mathcal{F}_1$, such that for any topology $\sigma \in \mathcal{H}$ there exist topologies $\sigma_1, \sigma_2 \in \mathcal{H}$, for which $\sigma_1 \leq \sigma$ and $\sigma \leq \sigma_2$.*

Proof. Let $x, y \in \mathcal{F}_1$ and k be an arbitrary fixed positive integer. Put

$$x \zeta_k y \iff (\exists n, m \in \mathbb{N}_0)[f^n(x) = f^m(y)\ \&\ n \equiv m (\mathrm{mod}\ k)]. \tag{2}$$

It is not hard to see that $\zeta_k \in \mathrm{Con}\,\mathcal{F}_1$. Let $P(S)$ be the set of all subsets of the set S of all primes. We claim now that the mapping $\varphi : P(S) \to \Im(\mathcal{F}_1)$ given by

$$\varphi(X) = \bigvee_{p \in X} \tau(\zeta_p), \qquad X \in P(S), \tag{3}$$

is an injection. Let $X_1, X_2 \subseteq S$ and $p \in X_1 \setminus X_2$. Denote by a the generator of the unar $\mathcal{F}_1$. Then $M = \{f^n(a) |\ n \in \mathbb{N}_0,\ p|n\} \in \varphi(X_1)$ by (3). If $M \in \varphi(X_2)$, then, by (3), we obtain that M is a union of finite intersections of sets which are open in some topology $\tau(\zeta_q)$, $q \in X_2$, the congruence ζ_q is defined according to (2).

Let $a \in L_{p_1} \cap L_{p_2} \cap \ldots \cap L_{p_k}$ and $L_{p_1} \cap L_{p_2} \cap \ldots \cap L_{p_k} \subseteq M$, where $L_{p_i} \in \tau(\zeta_{p_i})$, $p_i \in X_2$ for any $i \in \{1, 2, \ldots, k\}$. Then $f^{p_1 p_2 \ldots p_k}(a) \in M$, and $p|p_1 p_2 \ldots p_k$, i.e. there exists an index $i \in \{1, 2, \ldots, k\}$ such that $p = p_i \in X_2$. Therefore, $M \notin \varphi(X_2)$. Thus, the inequality $X_1 \neq X_2$ implies $\varphi(X_1) \neq \varphi(X_2)$.

We claim that if X is an infinite subset of the set of all primes, then $\varphi(X)$ from (3) is a Hausdorff topology one. Let $b, c \in \mathcal{F}_1$ and $b \neq c$. Then $b = f^n(a), c = f^m(a)$, where $n, m \in \mathbb{N}_0, m \neq n$ and a is the generator of the unar $\mathcal{F}_1$. Since the set X is infnite, there exists a number $p \in X$ such that $n < p$ and $m < p$. Hence $[b]_{\zeta_p} \cap [c]_{\zeta_p} = \emptyset$, because $m \neq n$. On the other hand, $[b]_{\zeta_p}, [c]_{\zeta_p} \in \varphi(X)$ by (2) and (3). It means that $\varphi(X)$ is a Hausdorff topology.

Thus, the set

$$\mathcal{H} = \{\varphi(X) |\ X \in P(S) \quad \& \quad |X| = |S \setminus X| = \aleph_0\} \tag{4}$$

consists of Hausdorff topologies and has cardinality $2^{\aleph_0}$. Let $\varphi(X) \in \mathcal{H}$. Then there exist prime numbers p_1 and p_2 such that $p_1 \in X, p_2 \in S \setminus X$. Consequently, $\varphi(X \setminus \{p_1\}), \varphi(X \cup \{p_2\}) \in \mathcal{H}$ by (4). On the other hand, (3) implies $\varphi(X \setminus \{p_1\}) \underset{\neq}{\subset} \varphi(X) \underset{\neq}{\subset} \varphi(X \cup \{p_2\})$, because the map φ is injective.

Theorem 1. *Let $\mathfrak{A} = \langle A, f \rangle$ be an arbitrary unar. Then the following conditions are equivalent:*

1. *the lattice $\Im(\mathfrak{A})$ is finite;*
2. *the lattice $\Im(\mathfrak{A})$ has a finite width;*
3. *the lattice $\Im(\mathfrak{A})$ satisfies the descending chain condition;*
4. *the lattice $\Im(\mathfrak{A})$ satisfies the ascending chain condition.*

Proof. Implications $1) \Rightarrow 2), 1) \Rightarrow 3), 1) \Rightarrow 4$ are obvious. Let the lattice $\Im(\mathfrak{A})$ of topologies of $\mathfrak{A}$ be infinite. Then $|A| \geq \aleph_0$. We shall show that $\Im(\mathfrak{A})$ is not a lattice of finite width and satisfies neither the descending chain condition nor the ascending chain condition. By Lemma 3, it suffices to consider the cases $\mathfrak{A} \cong \mathcal{F}_1$ and $\mathfrak{A} \cong C_1^{\infty}$.

If $\mathfrak{A} \cong \mathcal{F}_1$, then $\Im(\mathfrak{A})$ is not a lattice of a finite width by Theorem 1 of [2] and Theorem 4 of [1]. Furthermore, this lattice satisfies neither the descending chain condition nor the ascending chain condition by Proposition 2.

Let $\mathfrak{A} \cong C_1^{\infty}$. Put

$$X_i = \{a | a \in A \quad \& \quad d(a) \equiv 1 \pmod{i}\} \tag{5}$$

for any integer $i \in \mathbb{N}_0$. Fix arbitrary different primes i and j. We are going to show that the elements $t(X_i)$ and $t(X_j)$ of the lattice $\Im(\mathfrak{A})$ are incomparable. In fact, if $t(X_i) \leq t(X_j)$, then $X_i \in t(X_j)$. It means that the set X_i is a union of finite intersections of some sets of the form $f^{-l}(X_j)$, where $j \in \mathbb{N}_0$.

Since $\mathfrak{A} \cong C_1^{\infty}$, there exists an element $a \in \mathfrak{A}$ of the depth 1. Then (5) implies $a \in X_i$. Consequently,

$$a \in f^{-l_1}(X_j) \cap \ldots \cap f^{-l_s}(X_j) \tag{6}$$

and

$$\bigcap_{k=1}^{s} f^{-l_k}(X_j) \subseteq X_i \tag{7}$$

for some $s \in \mathbb{N}$, $\{l_1, \ldots, l_s\} \subseteq \mathbb{N}_0$. By (6), we have $f^{l_k}(a) \in X_j$ for any $k \in \{1, \ldots, s\}$. From (5), we can deduce that $l_k = 0$ for any $k \in \{1, \ldots, s\}$ and $d(a) = 1$. Applying (7), we have $X_j \subseteq X_i$ a contradiction with (5), because i and j are different primes.

Thus, the inequality $t(X_i) \leq t(X_j)$ doesn't hold. Similarly, we can prove that the inequality $t(X_j) \leq t(X_i)$ doesn't hold either. Therefore, the elements $t(X_i)$ and $t(X_j)$ of the lattice $\Im(\mathfrak{A})$ are incomparable for any prime different numbers i and j. Hence, if $\mathfrak{A} \cong C_1^{\infty}$, then $\Im(\mathfrak{A})$ is not a lattice of a finite width.

Let X be an arbitrary subset of the set $A \setminus Z(\mathfrak{A})$. Then the decreasing chain $t(X) \supset t(f^{-1}(X)) \supset t(f^{-2}(X)) \supset \ldots$ of elements of $\Im(\mathfrak{A})$ does not terminate by Lemma 4.

It remains to construct an infinite increasing chain of elements of the lattice $\Im(\mathfrak{A})$. Let $X_i = \{x | x \in A \setminus Z(\mathfrak{A}) \quad \& \quad d(x) \notin \{2, \dots, i+1\}\}$, where $i \in \mathbb{N}$. We claim that $X_i = X_{i+1} \cup f^{-(i+1)}(X_{i+1})$ for any $i \in \mathbb{N}$. If $x \in X_i$, then either $x \in X_{i+1}$ or $d(x) = i+2$. However, the equation $d(x) = i+2$ implies $d(f^{i+1}(x)) = 1$, hence $f^{i+1}(x) \in X_{i+1}$ and $x \in f^{-(i+1)}(X_{i+1})$. Therefore, $X_i \subseteq X_{i+1} \cup f^{-(i+1)}(X_{i+1})$. Let $x \in f^{-(i+1)}(X_{i+1})$. Then $f^{i+1}(x) \in X_{i+1}$, hence either $d(f^{i+1}(x)) = 1$ or $d(f^{i+1}(x)) \geq i+3$. So, $d(x) \geq i+2$, i. e. $x \in X_i$. Thus, $X_i = X_{i+1} \cup f^{-(i+1)}(X_{i+1})$. It means that $X_i \in t(X_{i+1})$.

By Lemma 4 the relation $t(X_i) \underset{\neq}{\subset} t(X_{i+1})$ is valid for any $i \in \mathbb{N}$. Finally, the lattice $\Im(\mathfrak{A})$ of topologies of the unar $\mathfrak{A}$ does not satisfy the ascending chain condition because it contains the infinite chain $t(X_1) \underset{\neq}{\subset} t(X_2) \underset{\neq}{\subset} \dots$ ■

Theorem 2. *Let $\mathfrak{A} = \langle A, f \rangle$ be an arbitrary unar. Then it holds:*

1. *the lattice $\Im(\mathfrak{A})$ isn't countably infinite;*
2. *if the set A is uncountable, then $|\Im(\mathfrak{A})| = 2^{2^{|A|}}$.*

Proof. The first statement of the theorem follows from Lemmas 3, and 4, and Proposition 2. Let us prove the second statement. Let $|A| > \aleph_0$. If $|c(\mathfrak{A})| = |A|$, then $|\Im(\mathfrak{A})| = 2^{2^{|A|}}$ by Lemma 1 and p. 380 of [7]. Now let $|c(\mathfrak{A})| < |A|$. By [3], p. 315, there exists a set A_1 of pairwise incomparable noncyclic elements of $\mathfrak{A}$ such that $|A_1| = |A|$. Hence, $|\Im(\mathfrak{A})| = 2^{2^{|A|}}$ by Lemma 2 and [7] (p. 380). ■

Corollary 1. *If a unar $\mathfrak{A} = \langle A, f \rangle$ is not a cycle, then $|\Im(\mathfrak{A})| > |\operatorname{Con}(\mathfrak{A})|$.*

Proof. If $|A| > \aleph_0$, then $|\Im(\mathfrak{A})| = 2^{2^{|A|}}$ by Theorem 2 and $|\operatorname{Con}(\mathfrak{A})| = 2^{|A|}$ by [3] (p. 312). Hence, $|\Im(\mathfrak{A})| > |\operatorname{Con}(\mathfrak{A})|$.

Let the set A be countably infinite. If the set $c(\mathfrak{A})$ of connected components of $\mathfrak{A}$ is infinite or $\mathfrak{A}$ contains some infinite set of pairwise incomparable noncyclic elements, then $|\Im(\mathfrak{A})| = 2^{2^{\aleph_0}} > 2^{\aleph_0} = |\operatorname{Con}(\mathfrak{A})|$ by Lemmas 1 and 2, and the main Theorem from [4]. Otherwise, by [4] and Theorem 2, we obtain $|\Im(\mathfrak{A})| \geq 2^{\aleph_0} > \aleph_0 = |\operatorname{Con}(\mathfrak{A})|$.

Let now the set A be finite. We claim that the mappping $\theta \mapsto \tau(\theta)$ from $\operatorname{Con}(\mathfrak{A})$ into $\Im(\mathfrak{A})$ is not surjective. Indeed, there exist such elements $a, b \in A$, that $a \notin (b)$ because the unar $\mathfrak{A}$ is not a cycle. Suppose that ρ is a congruence of $\mathfrak{A}$ such that $\tau(\rho) = t(\{a\})$. Then $[b]_\rho \in t(\{a\})$. Hence, $[b]_\rho = A$, because $b \notin \cup_{i \in \mathbb{N}_0} f^{-i}(\{a\})$. Therefore, ρ is the universal relation and the topology $t(\{a\}) = \tau(\rho)$ is anti-discrete. However, $\{a\} \in t(\{a\})$.

Consequently, the mapping τ is not surjective. On the other hand, τ is injective by Lemma 3 of [2]. Thus, $|\Im(\mathfrak{A})| > |\operatorname{Con}(\mathfrak{A})|$.

Corollary 2. *The lattice* $\Im(\mathfrak{A})$ *of topologies of an arbitrary unar* $\mathfrak{A}$ *is isomorphic to the lattice* $\operatorname{Con}(\mathfrak{A})$ *of its congruences if and only if* $\mathfrak{A}$ *is a cycle.*

Proof. The necessity of this assertion follows from the previous corollary.

Let $\mathfrak{A} \cong C_n^0$, where $n \in \mathbb{N}$. Then $\Im(\mathfrak{A}) \cong \operatorname{Con}(\mathfrak{A})$ by Corollary 2 from Theorem 1 of [2]. ∎

Corollary 3. *The following properties hold*

1. *there exist unars with isomorphic lattices of congruences, the lattices of topologies of which are not isomorphic;*
2. *there exist unars with isomorphic lattices of topologies, the lattices of congruences of which are not isomorphic.*

Proof. The lattices $\operatorname{Con}(C_1^0 + C_1^0)$ and $\operatorname{Con}(C_p^0)$, where p is a prime, are isomorphic, because they both are two-element chains. However, the lattice $\Im(C_1^0 + C_1^0)$ is four-element and $\Im(C_p^0) \cong \operatorname{Con}(C_p^0)$ according to Corollary 2. In order to prove the second assertion, we note first that $\Im(C_1^0 + C_1^0) \cong \Im(C_{p_1p_2}^0)$, where p_1, p_2 are different prime numbers. On the other hand, the lattices $\operatorname{Con}(C_1^0 + C_1^0)$ and $\operatorname{Con}(C_{p_1p_2}^0)$ are not isomorphic because $|\operatorname{Con}(C_1^0 + C_1^0)| = 2$ and $|\operatorname{Con}(C_{p_1p_2}^0)| = 4$. ∎

Acknowledgements

The author is grateful to Profesor V.A. Artamonov for fruitful discussions and regular attention to my work.

References

[1] D.P. Egorova, *Unars with congruence lattices of special kinds* (Russian), Issledovania po algebre (Saratov) **5** (1977), 3–19.

[2] A.V. Kartashova, *About some properties of lattices of topologies of algebras* (Russian), Dep. no. 404 VINITI, 21.02.00.

[3] O. Kopeček, $|EndA| = |\operatorname{Con} A| = |SubA| = 2^A$ *for any uncountable 1-unary algebra A, Algebra Universalis* **16** (1983), 312–317.

[4] O. Kopeček, *A note on some cardinal functions on unary algebras*, Contributions to General Algebra **2** (1983), 221–227.

[5] S.D. Orlov, *About the lattice of possible topologies* (Russian), Uporadočennye množestva i rešetki (Saratov) **2** (1974), 68–71.

[6] L.A. Skorniǐakov, *Unars*, p. 735–743 in: Colloq. Math. Soc. J. Bolyai, vol. **29** ("*Universal Algebra*"), North-Holland, Amsterdam 1982.

[7] A.K. Steiner, *The lattice of topologies: Structure and complementation*, Trans. Amer. Math. Soc. **122** (1966), 379–398.

Received 27 November 2000
Revised 14 December 2001