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CARDINALITIES OF LATTICES OF TOPOLOGIES OF UNARS AND SOME RELATED TOPICS

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Abstract

In this paper we find cardinalities of lattices of topologies of uncountable unars and show that the lattice of topologies of a unar cannor be countably infinite. It is proved that under some finiteness conditions the lattice of topologies of a unar is finite. Furthermore, the relations between the lattice of topologies of an arbitrary unar and its congruence lattice are established.

Keywords: unar, lattice of topologies, lattice of congruences.

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Let $\mathfrak{A} = \langle A, \Omega \rangle$ be an arbitrary algebra. A topology on the set A, under which every operation from Ω is continuous is called a *topology on the algebra* $\mathfrak{A}$. It is known [5] (p. 69) that the topologies on an algebra $\mathfrak{A}$ form a lattice under set inclusion. Let us call this lattice the *lattice of topologies* of the algebra $\mathfrak{A}$. Denote this lattice by $\Im(\mathfrak{A})$.

Let now $\mathfrak{A} = \langle A, f \rangle$ be a unar, i. e. an algebra with one unary operation f (see [6]). For any element $a \in A$ and any positive integer n we put $f^0(a) = a$ and $f^n(a) = f(f^{n-1}(a))$. Throughout the paper we shall denote by $\mathbb{N}$ the set of all positive integers and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$.

A unar generated by one element a is called *monogenic* and it is denoted by (a). A monogenic unar with the generator a and with defining relation $f^n(a) = f^{n+m}(a), \quad n \in \mathbb{N}_0, m \in \mathbb{N}$ is denoted by C_m^n. The unar C_m^0 is termed a *cycle of length m*. An element a of the unar $\mathfrak{A}$ is *cyclic* if the subunar generated by this element is cyclic. The set of all cyclic elements of a unar $\mathfrak{A}$ is denoted by $Z(\mathfrak{A})$. An element a of a unar $\mathfrak{A} = \langle A, f\rangle$ is *periodic* if $f^k(a) \in Z(\mathfrak{A})$ for some $k \in \mathbb{N}_0$. Otherwise it is called *torsion-free.* The union of a sequence of unars $C_m^0 \subset C_m^1 \subset C_m^2 \subset \dots$ will be denoted by C_m^∞. If a is a periodic element of a unar $\mathfrak{A} = \langle A, f\rangle$, then the least integer $n \in \mathbb{N}_0$ such that $f^n(a) \in Z(\mathfrak{A})$, is the *depth* of a. It is denoted by $d(a)$. A unar is *periodic* if each element in $\mathfrak{A}$ is periodic. A free monogenic unar is denoted by $\mathcal{F}_1$.

The disjoint union of two unars $\mathfrak{B}$ and $\mathfrak{C}$ is denoted by $\mathfrak{B} + \mathfrak{C}$. Unars $\mathfrak{B}$ and $\mathfrak{C}$ are *components* of the unar $\mathfrak{B} + \mathfrak{C}$. A unar having no proper components is called *connected.* The set of all connected components of an arbitrary unar $\mathfrak{A}$ is denoted by $c(\mathfrak{A})$.

Proposition 1. *The lattice of all topologies on the set $c(\mathfrak{A})$ of connected components of an arbitrary unar $\mathfrak{A} = \langle A, f\rangle$ is isomorphic to some principal ideal of the lattice $\Im(\mathfrak{A})$.*

Proof. Define binary relation η on the set A by setting

$$x\eta y \;\Leftrightarrow\; (\exists n, m \in \mathbb{N}_0)[f^n(x) = f^m(y)]$$

for any elements $x, y \in A$. It is clear that $\eta \in \mathrm{Con}(\mathfrak{A})$ and the factor unar $\mathfrak{A}/\eta$ is a union of one-element cycles. Moreover the lattice $\Im(\mathfrak{A}/\eta)$ of topologies of the unar $\mathfrak{A}/\eta$ coincides with the lattice of all topologies on the set A/η. By [2] (Theorem 3) the lattice of all topologies on the set $c(\mathfrak{A})$ is isomorphic to a principal ideal of $\Im(\mathfrak{A})$ because $|c(\mathfrak{A})| = |\mathfrak{A}/\eta|$. ■

Observe that

the lattice $\mathcal{R}(Y)$ of all topologies on a nonvoid subset Y of an arbitrary set X can be embedded into the lattice $\mathcal{R}(X)$ of all topologies on the set X as a principal ideal.

In fact, fix a point $y_0 \in Y$ and define a mapping $\psi : \mathcal{R}(Y) \to \mathcal{R}(X)$ in the following way. Let $\sigma \in \mathcal{R}(Y)$. Denote be $\psi(\sigma)$ the family of subsets of the set X such that $T \in \psi(\sigma)$ if and only if either $T \in \sigma$ and $y_0 \notin T$ or $T \cap Y \in \sigma, X \setminus Y \subseteq T$ and $y_0 \in T$. Then ψ is an isomorphism of $\mathcal{R}(Y)$ onto the principal ideal of $\mathcal{R}(X)$ generated by the topology $\psi(\sigma_1)$, where σ_1 is the discrete topology on X. ■

From Proposition 1, we can deduce

Lemma 1. *Let $\mathfrak{A} = \langle A, f\rangle$ be an arbitrary unar and K be a nonvoid subset of the set $c(\mathfrak{A})$ of connected components of the unar $\mathfrak{A}$. Then the lattice $\mathcal{R}(K)$ of all topologies on the set K is isomorphic to a principal ideal of the lattice $\Im(\mathfrak{A})$.* ■

Elements a, b of an arbitrary unar $\mathfrak{A} = \langle A, f\rangle$ are *incomparable* if $a \notin (b)$ and $b \notin (a)$.

Lemma 2. *Let $\mathfrak{A} = \langle A, f\rangle$ be an arbitrary unar and A_1 be an infinite set of pairwise non-cyclic incomparable elements of $\mathfrak{A}$. Then the lattice $\mathcal{R}(A_1)$ of all topologies on the set A_1 can be embedded into the lattice $\Im(\mathfrak{A})$.*

Proof. Denote by $\mathfrak{B} = \langle B, f\rangle$ the subunar of unar $\mathfrak{A}$ generated by set A_1. Define a binary relation $\rho = \{(a, b) \in B \times B |\ a = b \vee \{a, b\} \cap A_1 = \emptyset\}$ on the set B. Certainly the relation ρ is an equivalence.

We claim that $\rho \in \operatorname{Con}\mathfrak{B}$. In fact let $a \notin A_1$, and $f(a) \in A_1$. Since $a \in B$, there exists an element $c \in A_1$ and an integer $n \in \mathbb{N}_0$ such that $a = f^n(c)$. Hence, $f(a) = f^{n+1}(c)$. It follows that $n + 1 = 0$ and $n \notin \mathbb{N}_0$, since $f(a) \in A_1$ and $c \in A_1$. Every topology on the factor set B/ρ is a topology on the unar $\mathfrak{B}/\rho$ because either $f^{-1}(X) = \emptyset$ or $f^{-1}(X) = B/\rho$ holds for any subset X of the set B/ρ. Thus applying [2] (Theorems 2 and 3) and the equality $|A_1| = |B/\rho|$ we can conclude that the lattice $\mathcal{R}(A_1)$ of all topologies on the set A_1 can be embedded into the lattice $\Im(\mathfrak{A})$. ∎

Lemma 3. *Let $\mathfrak{A} = \langle A, f\rangle$ be an arbitrary infinite unar. Then either the lattice $\Im(\mathcal{F}_1)$ or the lattice $\Im(C_1^\infty)$ can be embedded into the lattice $\Im(\mathfrak{A})$ of all topologies of the unar $\mathfrak{A}$.*

Proof. If $\mathfrak{A}$ contains a torsion-free element a, then $(a) \cong \mathcal{F}_1$, where (a) is the monogenic subunar of the unar $\mathfrak{A}$ generated by the element a. By [2] (Theorem 3) the lattice $\Im(\mathcal{F}_1)$ can be embedded into the lattice $\Im(\mathfrak{A})$.

If the set $c(\mathfrak{A})$ is infinite or the inequality $|f^{-1}(\{a\})| \geq \aleph_0$ holds for some $a \in A$, then the lattice $\mathcal{R}(X)$ of all topologies on a countable infinite set X is isomorphic to some sublattice of the lattice $\Im(\mathfrak{A})$ by Lemmas 1 and 2. On the other hand, $|\mathcal{F}_1| = \aleph_0$. Therefore, the lattice $\Im(\mathcal{F}_1)$ of all topologies of $\mathcal{F}_1$ can be embedded into the lattice $\mathcal{R}(X)$ and, hence, into the lattice $\Im(\mathfrak{A})$.

Let $\mathfrak{A}$ be periodic, $|c(\mathfrak{A})| < \aleph_0$, a set $f^{-1}(\{a\})$ finite for any element $a \in \mathfrak{A}$. Then there exists a subunar $\mathfrak{B} = \langle B, f\rangle$ of the unar $\mathfrak{A}$, which is isomorphic to C_h^∞, where $h \in \mathbb{N}$. Put $\rho = \{(a, b) \in B \times B |\ (a = b) \vee \{a, b\} \subseteq Z(\mathfrak{A})\}$. Then $\rho \in \operatorname{Con}\mathfrak{B}$ and the factor unar $\mathfrak{B}/\rho$ is isomorphic to C_1^∞. Consequently, the lattice $\Im(C_1^\infty)$ of all topologies of C_1^∞ can be embedded into the lattice $\Im(\mathfrak{A})$ by [2] (Theorems 2 and 3). ∎

The least topology with respect to inclusion on the unary algebra $\mathfrak{A}$, containing a given family of subsets $\{A_\alpha \subseteq A|\, \alpha \in I\}$ will be called the *topology on the algebra* $\mathfrak{A}$ *generated by the set of elements* $\{A_\alpha|\, \alpha \in I\}$. This topology will be denoted by $t(\{A_\alpha|\, \alpha \in I\})$ and respectively by $t(U)$ if the family $\{A_\alpha|\, \alpha \in I\}$ consists of one set U.

Lemma 4. *Let* $\mathfrak{A} = \langle A, f\rangle$ *be isomorphic to* C_1^∞. *Then* $t(X_1) = t(X_2) \Rightarrow X_1 = X_2$ *for any nonvoid subsets* X_1, X_2 *of the set* $A \setminus Z(\mathfrak{A})$.

Proof. Since $t(X_1) = t(X_2)$, we conclude that $X_2 \in t(X_1)$. Hence, the set X_2 is a union of finite intersections of some sets of the form $f^{-i}(X_1)$, where $i \in \mathbb{N}_0$, because $X_2 \neq \emptyset$ and $X_2 \subseteq A \setminus Z(\mathfrak{A}) \underset{\neq}{\subset} A$.

Since $\mathfrak{A} \cong C_1^\infty$ and $X_2 \subseteq A \setminus Z(\mathfrak{A})$, there exists an element $x \in X_2$ such that

$$(1) \qquad (\forall k \in \mathbb{N})\ [f^k(x) \notin X_2].$$

Since $x \in X_2$ and $X_2 \in t(X_1)$ we have $x \in \bigcap\limits_{i \in I} f^{-i}(X_1) \subseteq X_2$ for some finite set of indices I of the set $\mathbb{N}_0$. We claim that $I = \{0\}$. In fact, if $x \in f^{-i}(X_1)$, then $f^i(x) \in X_1$. On the other hand, since $X_1 \in t(X_2)$, the set X_1 is a union of finite intersections of some sets of the form $f^{-j}(X_2)$, where $j \in \mathbb{N}_0$. However, by (1), the condition $f^{i+j}(x) \in X_2$ implies $i+j = 0$. Hence, $i = 0$ and so $I = \{0\}$. Consequently, $X_1 \subseteq X_2$. Similarly, we can prove that $X_2 \subseteq X_1$. Thus, $X_1 = X_2$. ∎

Let $\mathfrak{A}$ be an arbitrary algebra and $\theta \in \mathrm{Con}(\mathfrak{A})$. θ-congruence classes form a base of some topology $\tau(\theta)$ which we shall call the *topology generated by the congruence* θ.

Proposition 2. *There exists a set* $\mathcal{H}$ *of the cardinality* $2^{\aleph_0}$ *of different Hausdorff topologies on the unar* $\mathcal{F}_1$, *such that for any topology* $\sigma \in \mathcal{H}$ *there exist topologies* $\sigma_1, \sigma_2 \in \mathcal{H}$, *for which* $\sigma_1 \leq \sigma$ *and* $\sigma \leq \sigma_2$.

Proof. Let $x, y \in \mathcal{F}_1$ and k be an arbitrary fixed positive integer. Put

$$(2) \qquad x\zeta_k y \iff (\exists n, m \in \mathbb{N}_0)[f^n(x) = f^m(y)\ \&\ n \equiv m (\mathrm{mod}\ k)].$$

It is not hard to see that $\zeta_k \in \mathrm{Con}\,\mathcal{F}_1$. Let $P(S)$ be the set of all subsets of the set S of all primes. We claim now that the mapping $\varphi : P(S) \to \Im(\mathcal{F}_1)$ given by

$$\varphi(X) = \bigvee_{p \in X} \tau(\zeta_p), \qquad X \in P(S), \tag{3}$$

is an injection. Let $X_1, X_2 \subseteq S$ and $p \in X_1 \setminus X_2$. Denote by a the generator of the unar $\mathcal{F}_1$. Then $M = \{f^n(a) |\ n \in \mathbb{N}_0,\ p|n\} \in \varphi(X_1)$ by (3). If $M \in \varphi(X_2)$, then, by (3), we obtain that M is a union of finite intersections of sets which are open in some topology $\tau(\zeta_q)$, $q \in X_2$, the congruence ζ_q is defined according to (2).

Let $a \in L_{p_1} \cap L_{p_2} \cap \ldots \cap L_{p_k}$ and $L_{p_1} \cap L_{p_2} \cap \ldots \cap L_{p_k} \subseteq M$, where $L_{p_i} \in \tau(\zeta_{p_i})$, $\quad p_i \in X_2$ for any $i \in \{1, 2, \ldots, k\}$. Then $f^{p_1 p_2 \cdots p_k}(a) \in M$, and $p | p_1 p_2 \ldots p_k$, i.e. there exists an index $i \in \{1, 2, \ldots, k\}$ such that $p = p_i \in X_2$. Therefore, $M \notin \varphi(X_2)$. Thus, the inequality $X_1 \neq X_2$ implies $\varphi(X_1) \neq \varphi(X_2)$.

We claim that if X is an infinite subset of the set of all primes, then $\varphi(X)$ from (3) is a Hausdorff topology one. Let $b, c \in \mathcal{F}_1$ and $b \neq c$. Then $b = f^n(a), c = f^m(a)$, where $n, m \in \mathbb{N}_0, m \neq n$ and a is the generator of the unar $\mathcal{F}_1$. Since the set X is infnite, there exists a number $p \in X$ such that $n < p$ and $m < p$. Hence $[b]_{\zeta_p} \cap [c]_{\zeta_p} = \emptyset$, because $m \neq n$. On the other hand, $[b]_{\zeta_p}, [c]_{\zeta_p} \in \varphi(X)$ by (2) and (3). It means that $\varphi(X)$ is a Hausdorff topology.

Thus, the set

$$\mathcal{H} = \{\varphi(X) |\ X \in P(S) \quad \& \quad |X| = |S \setminus X| = \aleph_0\} \tag{4}$$

consists of Hausdorff topologies and has cardinality $2^{\aleph_0}$. Let $\varphi(X) \in \mathcal{H}$. Then there exist prime numbers p_1 and p_2 such that $p_1 \in X, p_2 \in S \setminus X$. Consequently, $\varphi(X \setminus \{p_1\}), \varphi(X \cup \{p_2\}) \in \mathcal{H}$ by (4). On the other hand, (3) implies $\varphi(X \setminus \{p_1\}) \underset{\neq}{\subset} \varphi(X) \underset{\neq}{\subset} \varphi(X \cup \{p_2\})$, because the map φ is injective.

Theorem 1. *Let $\mathfrak{A} = \langle A, f\rangle$ be an arbitrary unar. Then the following conditions are equivalent:*

1. *the lattice $\Im(\mathfrak{A})$ is finite;*
2. *the lattice $\Im(\mathfrak{A})$ has a finite width;*
3. *the lattice $\Im(\mathfrak{A})$ satisfies the descending chain condition;*
4. *the lattice $\Im(\mathfrak{A})$ satisfies the ascending chain condition.*

Proof. Implications $1) \Rightarrow 2), 1) \Rightarrow 3), 1) \Rightarrow 4$ are obvious. Let the lattice $\Im(\mathfrak{A})$ of topologies of $\mathfrak{A}$ be infinite. Then $|A| \geq \aleph_0$. We shall show that $\Im(\mathfrak{A})$ is not a lattice of finite width and satisfies neither the descending chain condition nor the ascending chain condition. By Lemma 3, it suffices to consider the cases $\mathfrak{A} \cong \mathcal{F}_1$ and $\mathfrak{A} \cong C_1^\infty$.

If $\mathfrak{A} \cong \mathcal{F}_1$, then $\Im(\mathfrak{A})$ is not a lattice of a finite width by Theorem 1 of [2] and Theorem 4 of [1]. Furthermore, this lattice satisfies neither the descending chain condition nor the ascending chain condition by Proposition 2.

Let $\mathfrak{A} \cong C_1^\infty$. Put

$$X_i = \{a | a \in A \quad \& \quad d(a) \equiv 1 \pmod{i}\} \tag{5}$$

for any integer $i \in \mathbb{N}_0$. Fix arbitrary different primes i and j. We are going to show that the elements $t(X_i)$ and $t(X_j)$ of the lattice $\Im(\mathfrak{A})$ are incomparable. In fact, if $t(X_i) \leq t(X_j)$, then $X_i \in t(X_j)$. It means that the set X_i is a union of finite intersections of some sets of the form $f^{-l}(X_j)$, where $j \in \mathbb{N}_0$.

Since $\mathfrak{A} \cong C_1^\infty$, there exists an element $a \in \mathfrak{A}$ of the depth 1. Then (5) implies $a \in X_i$. Consequently,

$$a \in f^{-l_1}(X_j) \cap \ldots \cap f^{-l_s}(X_j) \tag{6}$$

and

$$\bigcap_{k=1}^{s} f^{-l_k}(X_j) \subseteq X_i \tag{7}$$

for some $s \in \mathbb{N}$, $\{l_1, \ldots, l_s\} \subseteq \mathbb{N}_0$. By (6), we have $f^{l_k}(a) \in X_j$ for any $k \in \{1, \ldots, s\}$. From (5), we can deduce that $l_k = 0$ for any $k \in \{1, \ldots, s\}$ and $d(a) = 1$. Applying (7), we have $X_j \subseteq X_i$ a contradiction with (5), because i and j are different primes.

Thus, the inequality $t(X_i) \leq t(X_j)$ doesn't hold. Similarly, we can prove that the inequality $t(X_j) \leq t(X_i)$ doesn't hold either. Therefore, the elements $t(X_i)$ and $t(X_j)$ of the lattice $\Im(\mathfrak{A})$ are incomparable for any prime different numbers i and j. Hence, if $\mathfrak{A} \cong C_1^\infty$, then $\Im(\mathfrak{A})$ is not a lattice of a finite width.

Let X be an arbitrary subset of the set $A \setminus Z(\mathfrak{A})$. Then the decreasing chain $t(X) \supset t(f^{-1}(X)) \supset t(f^{-2}(X)) \supset \ldots$ of elements of $\Im(\mathfrak{A})$ does not terminate by Lemma 4.

It remains to construct an infinite increasing chain of elements of the lattice $\Im(\mathfrak{A})$. Let $X_i = \{x | x \in A \setminus Z(\mathfrak{A}) \quad \& \quad d(x) \notin \{2, \ldots, i+1\}\}$, where $i \in \mathbb{N}$. We claim that $X_i = X_{i+1} \cup f^{-(i+1)}(X_{i+1})$ for any $i \in \mathbb{N}$. If $x \in X_i$, then either $x \in X_{i+1}$ or $d(x) = i+2$. However, the equation $d(x) = i+2$ implies $d(f^{i+1}(x)) = 1$, hence $f^{i+1}(x) \in X_{i+1}$ and $x \in f^{-(i+1)}(X_{i+1})$. Therefore, $X_i \subseteq X_{i+1} \cup f^{-(i+1)}(X_{i+1})$. Let $x \in f^{-(i+1)}(X_{i+1})$. Then $f^{i+1}(x) \in X_{i+1}$, hence either $d(f^{i+1}(x)) = 1$ or $d(f^{i+1}(x)) \geq i+3$. So, $d(x) \geq i+2$, i. e. $x \in X_i$. Thus, $X_i = X_{i+1} \cup f^{-(i+1)}(X_{i+1})$. It means that $X_i \in t(X_{i+1})$.

By Lemma 4 the relation $t(X_i) \underset{\neq}{\subset} t(X_{i+1})$ is valid for any $i \in \mathbb{N}$. Finally, the lattice $\Im(\mathfrak{A})$ of topologies of the unar $\mathfrak{A}$ does not satisfy the ascending chain condition because it contains the infinite chain $t(X_1) \underset{\neq}{\subset} t(X_2) \underset{\neq}{\subset} \ldots$ ■

Theorem 2. *Let $\mathfrak{A} = \langle A, f \rangle$ be an arbitrary unar. Then it holds:*

1. *the lattice $\Im(\mathfrak{A})$ isn't countably infinite;*

2. *if the set A is uncountable, then $|\Im(\mathfrak{A})| = 2^{2^{|A|}}$.*

Proof. The first statement of the theorem follows from Lemmas 3, and 4, and Proposition 2. Let us prove the second statement. Let $|A| > \aleph_0$. If $|c(\mathfrak{A})| = |A|$, then $|\Im(\mathfrak{A})| = 2^{2^{|A|}}$ by Lemma 1 and p. 380 of [7]. Now let $|c(\mathfrak{A})| < |A|$. By [3], p. 315, there exists a set A_1 of pairwise incomparable noncyclic elements of $\mathfrak{A}$ such that $|A_1| = |A|$. Hence, $|\Im(\mathfrak{A})| = 2^{2^{|A|}}$ by Lemma 2 and [7] (p. 380). ■

Corollary 1. *If a unar $\mathfrak{A} = \langle A, f \rangle$ is not a cycle, then $|\Im(\mathfrak{A})| > |\operatorname{Con}(\mathfrak{A})|$.*

Proof. If $|A| > \aleph_0$, then $|\Im(\mathfrak{A})| = 2^{2^{|A|}}$ by Theorem 2 and $|\operatorname{Con}(\mathfrak{A})| = 2^{|A|}$ by [3] (p. 312). Hence, $|\Im(\mathfrak{A})| > |\operatorname{Con}(\mathfrak{A})|$.

Let the set A be countably infinite. If the set $c(\mathfrak{A})$ of connected components of $\mathfrak{A}$ is infinite or $\mathfrak{A}$ contains some infinite set of pairwise incomparable noncyclic elements, then $|\Im(\mathfrak{A})| = 2^{2^{\aleph_0}} > 2^{\aleph_0} = |\operatorname{Con}(\mathfrak{A})|$ by Lemmas 1 and 2, and the main Theorem from [4]. Otherwise, by [4] and Theorem 2, we obtain $|\Im(\mathfrak{A})| \geq 2^{\aleph_0} > \aleph_0 = |\operatorname{Con}(\mathfrak{A})|$.

Let now the set A be finite. We claim that the mappping $\theta \mapsto \tau(\theta)$ from $\operatorname{Con}(\mathfrak{A})$ into $\Im(\mathfrak{A})$ is not surjective. Indeed, there exist such elements $a, b \in A$, that $a \notin (b)$ because the unar $\mathfrak{A}$ is not a cycle. Suppose that ρ is a congruence of $\mathfrak{A}$ such that $\tau(\rho) = t(\{a\})$. Then $[b]_\rho \in t(\{a\})$. Hence, $[b]_\rho = A$, because $b \notin \cup_{i \in \mathbb{N}_0} f^{-i}(\{a\})$. Therefore, ρ is the universal relation and the topology $t(\{a\}) = \tau(\rho)$ is anti-discrete. However, $\{a\} \in t(\{a\})$.

Consequently, the mapping τ is not surjective. On the other hand, τ is injective by Lemma 3 of [2]. Thus, $|\Im(\mathfrak{A})| > |\operatorname{Con}(\mathfrak{A})|$.

Corollary 2. *The lattice $\Im(\mathfrak{A})$ of topologies of an arbitrary unar $\mathfrak{A}$ is isomorphic to the lattice $\operatorname{Con}(\mathfrak{A})$ of its congruences if and only if $\mathfrak{A}$ is a cycle.*

Proof. The necessity of this assertion follows from the previous corollary.

Let $\mathfrak{A} \cong C_n^0$, where $n \in \mathbb{N}$. Then $\Im(\mathfrak{A}) \cong \operatorname{Con}(\mathfrak{A})$ by Corollary 2 from Theorem 1 of [2]. ∎

Corollary 3. *The following properties hold*

1. *there exist unars with isomorphic lattices of congruences, the lattices of topologies of which are not isomorphic;*

2. *there exist unars with isomorphic lattices of topologies, the lattices of congruences of which are not isomorphic.*

Proof. The lattices $\operatorname{Con}(C_1^0 + C_1^0)$ and $\operatorname{Con}(C_p^0)$, where p is a prime, are isomorphic, because they both are two-element chains. However, the lattice $\Im(C_1^0 + C_1^0)$ is four-element and $\Im(C_p^0) \cong \operatorname{Con}(C_p^0)$ according to Corollary 2. In order to prove the second assertion, we note first that $\Im(C_1^0 + C_1^0) \cong \Im(C_{p_1p_2}^0)$, where p_1, p_2 are different prime numbers. On the other hand, the lattices $\operatorname{Con}(C_1^0 + C_1^0)$ and $\operatorname{Con}(C_{p_1p_2}^0)$ are not isomorphic because $|\operatorname{Con}(C_1^0 + C_1^0)| = 2$ and $|\operatorname{Con}(C_{p_1p_2}^0)| = 4$. ∎

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