

BALANCED CONGRUENCES*

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Abstract

Let \mathcal{V} be a variety with two distinct nullary operations 0 and 1. An algebra $\mathfrak{A} \in \mathcal{V}$ is called balanced if for each $\Phi, \Psi \in \text{Con}(\mathfrak{A})$, we have $[0]\Phi = [0]\Psi$ if and only if $[1]\Phi = [1]\Psi$. The variety \mathcal{V} is called balanced if every $\mathfrak{A} \in \mathcal{V}$ is balanced. In this paper, balanced varieties are characterized by a Mal'cev condition (Theorem 3). Furthermore, some special results are given for varieties of bounded lattices.

Keywords: balanced congruence, balanced algebra, balanced variety, Mal'cev condition.

2000 Mathematics Subject Classification: Primary 08A30; Secondary 08B05.

*This paper is a result of the collaboration of the authors within the framework of the “Aktion Österreich-Tschechische Republik” (grant No. 26p2 “Local and global congruence properties”).

1. BALANCED CONGRUENCES ON BOUNDED LATTICES

Let $\mathfrak{L} = (L; \vee, \wedge, 0, 1)$ be a bounded lattice with least element 0 and greatest element 1. We study congruences on \mathfrak{L} such that the 0-class determines the 1-class and vice versa.

Let $M \subseteq L$ and $a, b \in L$. Denote by $\Theta(M)$ the least congruence on \mathfrak{L} containing the relation $M \times M$ and by $\Theta(a, b)$ the principal congruence on \mathfrak{L} generated by (a, b) , i.e. $\Theta(a, b) = \Theta(M)$ for $M = \{a, b\}$. We will use the following result of G. Grätzer and E. T. Schmidt, see [3]:

Proposition 1. *Let \mathfrak{L} be a distributive lattice and $a, b, c, d \in L$ with $c \leq d$. Then $a\Theta(c, d)b$ if and only if $a \wedge c = b \wedge c$ and $a \vee d = b \vee d$.*

Here and in the following, we write $a\Phi b$ instead of $(a, b) \in \Phi$, for any $\Phi \in \text{Con}(\mathfrak{L})$ and $a, b \in L$.

From now on, every lattice under consideration is bounded, i.e. it has 0 and 1.

Definition 1. Let \mathfrak{L} be a lattice with 0 and 1 and $\Phi \in \text{Con}(\mathfrak{L})$. Put $I = [0]\Phi$ and $F = [1]\Phi$. We say that Φ is *balanced* if

$$[0]\Phi = [0]\Theta(F) \text{ and } [1]\Phi = [1]\Theta(I).$$

Example 1. By Corollary 2.1 in [2], every complemented lattice is locally regular at 0 (in the sense of [1]), i.e. for every $\Phi, \Psi \in \text{Con}(\mathfrak{L})$, if $[a]\Phi = [a]\Psi$ for some $a \in L$, then $[0]\Phi = [0]\Psi$. Dually, it is locally regular at 1, i.e. $[a]\Phi = [a]\Psi$ implies $[1]\Phi = [1]\Psi$. Thus every congruence on every complemented lattice is balanced.

Definition 2. Let \mathfrak{L} be a lattice with 0 and 1. We say that \mathfrak{L} is a *d-lattice* if for each $a, b, c, d \in L$ the following holds:

$$\begin{aligned} a\Theta(c, 1)0 &\Rightarrow a \wedge c = 0 \\ b\Theta(d, 0)1 &\Rightarrow b \vee d = 1. \end{aligned}$$

Example 2. The lattice \mathfrak{N}_5 is a d-lattice. Of course, \mathfrak{N}_5 has just five congruences: the identical one ω , the full square $N_5 \times N_5$ and three nontrivial ones $\Theta_1, \Theta_2, \Theta_3$ defined by the following partitions (see Figure 1):

$$\begin{aligned} \Theta_1 &\dots \{0, b, c\}, \{a, 1\} \\ \Theta_2 &\dots \{0\}, \{a\}, \{b, c\}, \{1\} \\ \Theta_3 &\dots \{0, a\}, \{b, c, 1\}. \end{aligned}$$

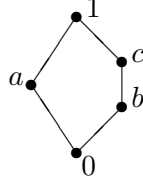


Figure 1

It is an easy exercise to verify our condition for $\Theta_1, \Theta_2, \Theta_3$; for ω and $N_5 \times N_5$ it is trivial (since $\omega = \Theta(0, 0) = \Theta(1, 1)$ and $N_5 \times N_5 = \Theta(0, 1)$).

Lemma 1. *Every bounded distributive lattice is a d-lattice.*

The proof follows immediately by Proposition 1 putting $b = 0$ and $d = 1$ in the first case and $a = 1$ and $c = 0$ in the second one. ■

Lemma 2. *Let \mathcal{L} be a bounded lattice and $\Phi \in \text{Con}(\mathcal{L})$. Take $S = \{a \in L : c \wedge a = 0 \text{ for some } c \in [1]\Phi\}$. Then $S \subseteq [0]\Phi$.*

Proof. Let $a \in S$. Then $a \wedge c = 0$ for some $c \in [1]\Phi$. Hence, $c\Phi 1$ thus also $(c \vee a)\Phi(1 \vee a) = 1$, i.e. $c \vee a \in [1]\Phi$. It yields $c\Phi(c \vee a)$ and $0 = (c \wedge a)\Phi((c \vee a) \wedge a) = a$ proving $a \in [0]\Phi$. ■

Theorem 1. *Let \mathcal{L} be a d-lattice and $\Phi \in \text{Con}(\mathcal{L})$. Then Φ is balanced if and only if*

$$\begin{aligned} [0]\Phi &= \{a \in L : c \wedge a = 0 \text{ for some } c \in [1]\Phi\} \text{ and} \\ [1]\Phi &= \{b \in L : d \vee b = 1 \text{ for some } d \in [0]\Phi\}. \end{aligned}$$

Proof. Let \mathcal{L} be a d-lattice and let $\Phi \in \text{Con}(\mathcal{L})$ be balanced. Define a set $S = \{a \in L : c \wedge a = 0 \text{ for some } c \in [1]\Phi\}$. By Lemma 2 we have $S \subseteq [0]\Phi$. Conversely, let $a \in [0]\Phi$. Since Φ is balanced, we have $0\Theta(F)a$ for $F = [1]\Phi$. Applying the fact that $\text{Con}(\mathcal{L})$ is a compactly generated lattice, there exist elements $c_1, \dots, c_k \in F$ such that $0[\Theta(1, c_1) \vee \dots \vee \Theta(1, c_k)]a$. Set $c = c_1 \wedge \dots \wedge c_k$. Then $c \leq c_i \leq 1$ whence $\Theta(1, c_i) \subseteq \Theta(1, c)$, i.e. $0\Theta(1, c)a$. Since \mathcal{L} is a d-lattice, it implies $0 = c \wedge a$. Since $c \in F$, we have $a \in S$, i.e. $[0]\Phi \subseteq S$. We have shown $[0]\Phi = S$. Dually it can be shown that $[1]\Phi = \{b \in L : d \vee b = 1 \text{ for some } d \in [0]\Phi\}$.

Conversely, let $\Phi \in \text{Con}(\mathfrak{L})$, and suppose that $[0]\Phi$, $[1]\Phi$ are described as in Theorem 1. Set $I = [0]\Phi$, $F = [1]\Phi$ and $\Psi_1 = \Theta(F)$, $\Psi_2 = \Theta(I)$. Evidently, $[1]\Psi_1 = F$ and $[0]\Psi_2 = I$.

It gives immediately

$$\begin{aligned} [0]\Phi &= \{a \in L : c \wedge a = 0 \text{ for some } c \in F\} = \\ &= \{a \in L : c \wedge a = 0 \text{ for some } c \in [1]\Psi_1\} \end{aligned}$$

thus $I \subseteq [0]\Psi_1$ by Lemma 2. However, both Φ and Ψ_1 have the 1-class F and Ψ_1 is the least congruence with this property, i.e. $\Psi_1 \subseteq \Phi$ whence $[0]\Psi_1 \subseteq I$. We have shown that $[0]\Phi = I = [0]\Theta(F)$.

Analogously one can prove $[1]\Phi = [1]\Theta(I)$, i.e. Φ is balanced. \blacksquare

Example 3. Consider the distributive lattice \mathfrak{L} visualized in Figure 2. Consider the congruences Φ_1 and Φ_2 determined by the following partitions:

$$\begin{aligned} \Phi_1 &\dots \{0, a\}, \{p, q\}, \{c, 1\} \\ \Phi_2 &\dots \{0, p\}, \{a, q\}, \{c\}, \{1\}. \end{aligned}$$

Then Φ_1 is balanced since

$$\begin{aligned} [0]\Phi_1 &= \{0, a\} = \{x \in L : c \wedge x = 0\}, \\ [1]\Phi_1 &= \{c, 1\} = \{y \in L : a \vee y = 1\}. \end{aligned}$$

On the contrary, Φ_2 is not balanced since $[1]\Phi_2 = \{1\} = [1]\omega$, whereas $[0]\Phi_2 \neq [0]\omega$.

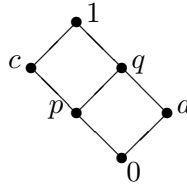


Figure 2

2. BALANCED ALGEBRAS WITH TWO NULLARY OPERATIONS

Definition 3. Let τ be a type containing two distinct nullary operations denoted by 0 and 1 and let \mathfrak{A} be an algebra of type τ . A congruence $\Phi \in \text{Con}(\mathfrak{A})$ is called *balanced* if

$$[0]\Phi = [0]\Theta(F) \text{ and } [1]\Phi = [1]\Theta(I)$$

where, as in Definition 1, $I = [0]\Phi, F = [1]\Phi$ and for $M \subseteq A$, $\Theta(M)$ denotes the least congruence on \mathfrak{A} containing $M \times M$. The algebra \mathfrak{A} is called *balanced* if each $\Phi \in \text{Con}(\mathfrak{A})$ is balanced. A variety \mathcal{V} of type τ is called *balanced* if every $\mathfrak{A} \in \mathcal{V}$ has this property.

Lemma 3. *An algebra \mathfrak{A} with nullary operations 0 and 1 is balanced if and only if for each $\Phi, \Psi \in \text{Con}(\mathfrak{A})$ the following condition holds:*

$$[0]\Phi = [0]\Psi \text{ if and only if } [1]\Phi = [1]\Psi.$$

Proof. Of course, if Φ and Ψ are balanced and $[0]\Phi = [0]\Psi = I$, then $[1]\Phi = [1]\Theta(I) = [1]\Psi$ and vice versa. Conversely, suppose that $[0]\Phi = [0]\Psi$ if and only if $[1]\Phi = [1]\Psi$ for each $\Phi, \Psi \in \text{Con}(\mathfrak{A})$. Then $[0]\Phi = I = [0]\Theta(I)$, thus also $[1]\Phi = [1]\Theta(I)$ and similarly, $[0]\Phi = [0]\Theta(F)$ with $F = [1]\Phi$, i.e. every congruence on \mathfrak{A} is balanced. So \mathfrak{A} is balanced. ■

Example 4. The lattice \mathfrak{N}_5 (see Figure 1) is balanced, since for Θ_2 (of Example 2), we have $[0]\Theta_2 = [0]\omega$ and $[1]\Theta_2 = [1]\omega$. We have $\Theta_2 \neq \omega$, so it is a nontrivial example.

The consequence of Example 1 is that every complemented lattice is balanced (this yields another proof that \mathfrak{N}_5 is balanced). However, this condition is not necessary. For example, every simple lattice is balanced but it need not be complemented, see e.g. Figure 3.

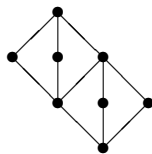


Figure 3

Problem. If \mathfrak{L} is a bounded distributive lattice (or a d-lattice), is it true that \mathfrak{L} is balanced if and only if \mathfrak{L} is complemented?

Lemma 4. *Let \mathfrak{A} be an algebra with nullary operations 0 and 1. If \mathfrak{A} is balanced then for each $\Phi \in \text{Con}(\mathfrak{A})$ the following property holds:*

(S) $[0]\Phi$ is a singleton if and only if $[1]\Phi$ is a singleton .

Proof. Suppose, e.g., that $[1]\Phi = \{1\}$. Then $[1]\Phi = [1]\omega$ for the identical congruence $\omega \in \text{Con}(\mathfrak{A})$, thus, by Lemma 3, also $[0]\Phi = [0]\omega = \{0\}$. The converse can be shown analogously. ■

For varieties, the converse statement is also valid:

Theorem 2. *Let \mathcal{V} be a variety of type τ containing two distinct nullary operations 0 and 1. \mathcal{V} is balanced if and only if for each $\mathfrak{A} \in \mathcal{V}$ and every $\Phi \in \text{Con}(\mathfrak{A})$ property (S) holds.*

Proof. Let \mathcal{V} satisfy condition (S), let $\mathfrak{A} \in \mathcal{V}$ and $\Phi, \Psi \in \text{Con}(\mathfrak{A})$. Suppose $[1]\Phi = [1]\Psi$. Then clearly also $[1]\Phi = [1]\Phi \wedge \Psi$, so we can assume $\Psi \subseteq \Phi$ without loss of generality. Consider the factor algebra \mathfrak{A}/Ψ and the factor congruence $\Phi/\Psi \in \text{Con}(\mathfrak{A}/\Psi)$. Since $[1]\Psi = [1]\Phi$, the class of Φ/Ψ containing the element $[1]\Psi \in \mathfrak{A}/\Psi$ is a singleton. Thus, by (S), also the class $[[0]\Psi]\Phi/\Psi$ is a singleton whence $[0]\Psi = [0]\Phi$.

Analogously we can show that $[0]\Phi = [0]\Psi \Rightarrow [1]\Phi = [1]\Psi$, thus \mathfrak{A} and hence also \mathcal{V} is balanced, by Lemma 3. The converse assertion follows directly by Lemma 4. ■

3. A CHARACTERIZATION OF BALANCED VARIETIES

The following Theorem characterizes balanced varieties by a Mal'cev condition.

Theorem 3. *Let \mathcal{V} be a variety of type τ which contains two distinct nullary operations 0 and 1. The following conditions are equivalent:*

1. \mathcal{V} is balanced.
2. There exist – for some m, n, k and h – unary terms $p_1, \dots, p_m, q_1, \dots, q_n$, $(2m+1)$ -ary terms r_1, \dots, r_k and $(2n+1)$ -ary terms s_1, \dots, s_h such that the following identities hold in \mathcal{V} :

$$\begin{aligned}
& p_1(0) = \dots = p_m(0) = 1, q_1(1) = \dots = q_n(1) = 0, \text{ and} \\
& x = r_1(p_1(x), \dots, p_m(x), 1, \dots, 1, x), \\
& r_i(1, \dots, 1, p_1(x), \dots, p_m(x), x) = r_{i+1}(p_1(x), \dots, p_m(x), 1, \dots, 1, x), \\
& i = 1, \dots, k-1, \\
& 0 = r_k(1, \dots, 1, p_1(x), \dots, p_m(x), x), \text{ and} \\
& x = s_1(q_1(x), \dots, q_n(x), 0, \dots, 0, x), \\
& s_j(0, \dots, 0, q_1(x), \dots, q_n(x), x) = s_{j+1}(q_1(x), \dots, q_n(x), 0, \dots, 0, x), \\
& j = 1, \dots, h-1, \\
& 1 = s_h(0, \dots, 0, q_1(x), \dots, q_n(x), x).
\end{aligned}$$

3. *There exist – for some m, n – unary terms $p_1, \dots, p_m, q_1, \dots, q_n$ and 3-ary terms $r_1, \dots, r_m, s_1, \dots, s_n$ such that the following identities hold in \mathcal{V} :*

$$\begin{aligned}
& p_1(0) = \dots = p_m(0) = 1, q_1(1) = \dots = q_n(1) = 0, \text{ and} \\
& x = r_1(p_1(x), 1, x), \\
& r_i(1, p_i(x), x) = r_{i+1}(p_{i+1}(x), 1, x), i = 1, \dots, m-1, \\
& 0 = r_m(1, p_m(x), x), \text{ and} \\
& x = s_1(q_1(x), 0, x), \\
& s_j(0, q_j(x), x) = s_{j+1}(q_{j+1}(x), 0, x), j = 1, \dots, n-1, \\
& 1 = s_n(0, q_n(x), x).
\end{aligned}$$

4. *There exist – for some m, n – unary terms $p_1, \dots, p_m, q_1, \dots, q_n$ such that*

$$\begin{aligned}
& [p_1(x) = 1 \& \dots \& p_m(x) = 1] \Leftrightarrow x = 0, \\
& [q_1(x) = 0 \& \dots \& q_n(x) = 0] \Leftrightarrow x = 1.
\end{aligned}$$

Proof. (1) \Rightarrow (2): Let $\mathfrak{A} = \mathfrak{F}_{\mathcal{V}}(x)$ be a free algebra of \mathcal{V} with one free generator x , let $\Psi = \Theta(x, 0) \in \text{Con}(\mathfrak{A})$ and $B = [1]\Psi$. Set $\Phi = \Theta(B)$, i.e., Φ is the least congruence on \mathfrak{A} having the 1-class equal to B . Then $[1]\Psi = [1]\Phi$ thus, by (1), also $[0]\Psi = [0]\Phi$, i.e., $0\Phi x$. As the lattice $\text{Con}(\mathfrak{A})$ is compactly generated, there exist $b_1, \dots, b_m \in B$ such that

$$(*) \quad x[\Theta(1, b_1) \vee \dots \vee \Theta(1, b_m)]0.$$

Since $b_i \in \mathfrak{F}_{\mathcal{V}}(x)$, there exist unary terms $p_i(x)$ such that $b_i = p_i(x)$ for $i = 1, \dots, m$. Since $b_i \in [1]\Theta(x, 0)$, it implies immediately

$$p_i(0) = 1 \text{ for } i = 1, \dots, m.$$

Applying the well-known Mal'cev scheme (see [4]) on (*), we obtain

$$\begin{aligned}
x &= r_1(p_1(x), \dots, p_m(x), 1, \dots, 1, x), \\
r_j(1, \dots, 1, p_1(x), \dots, p_m(x), x) &= r_{j+1}(p_1(x), \dots, p_m(x), 1, \dots, 1, x), \\
0 &= r_k(1, \dots, 1, p_1(x), \dots, p_m(x), x)
\end{aligned}$$

for some $(2m + 1)$ -ary terms r_1, \dots, r_k and $j = 1, \dots, k - 1$.

If we set $\Psi = \Theta(x, 1)$ and $B = [0]\Psi$, then the constants 0 and 1 only interchange their roles and we obtain the remaining identities of (2) analogously.

(2) \Rightarrow (3): For $i = 2, \dots, m - 1$ and $j = 1, \dots, k$ put

$$\begin{aligned}
r_{j1}(u, v, x) &= r_j(u, p_2(x), \dots, p_m(x), v, 1, \dots, 1, x) \\
r_{ji}(u, v, x) &= r_j(1, \dots, 1, u, p_{i+1}(x), \dots, p_m(x), p_1(x), \dots, p_{i-1}(x), v, 1, \dots, 1, x),
\end{aligned}$$

where there are $i - 1$ nullary operations 1 at the beginning and $m - i$ nullary operations 1 after v , and

$$r_{jm}(u, v, x) = r_j(1, \dots, 1, u, p_1(x), \dots, p_{m-1}(x), v, x).$$

Furthermore, put $p_{ji}(x) = p_i(x)$ for $i = 1, \dots, m$ and $j = 1, \dots, k$.

Then we have for $i = 1, \dots, m - 1$ and $j = 1, \dots, k$:

$$\begin{aligned}
&r_{ji}(1, p_{ji}(x), x) = \\
&= r_j(1, \dots, 1, 1, p_{i+1}(x), \dots, p_m(x), p_1(x), \dots, p_{i-1}(x), p_i(x), 1, \dots, 1, x) = \\
&= r_{j,i+1}(p_{j,i+1}(x), 1, x)
\end{aligned}$$

and for $j = 1, \dots, k - 1$:

$$\begin{aligned}
r_{jm}(1, p_{jm}(x), x) &= r_j(1, \dots, 1, 1, p_1(x), \dots, p_{m-1}(x), p_m(x), x) = \\
&= r_{j+1}(p_1(x), \dots, p_m(x), 1, \dots, 1, x) = \\
&= r_{j+1,1}(p_{j+1,1}(x), 1, x),
\end{aligned}$$

where the second identity follows from (2).

Furthermore we have, again by (2):

$$\begin{aligned}
r_{11}(p_{11}(x), 1, x) &= r_1(p_1(x), p_2(x), \dots, p_m(x), 1, 1, \dots, 1, x) = x, \\
r_{km}(1, p_{km}(x), x) &= r_k(1, \dots, 1, 1, p_1(x), \dots, p_{m-1}(x), p_m(x), x) = 0.
\end{aligned}$$

By writing m instead of km , r_1, \dots, r_m instead of r_{11}, \dots, r_{km} and p_1, \dots, p_m instead of p_{11}, \dots, p_{km} (both in lexicographic order), we obtain the first part of (3). The second part of (3) can be shown analogously.

(3) \Rightarrow (4): Take the same terms p_i, q_j as in (3). Then $p_i(0) = 1, q_j(1) = 0$ for $i = 1, \dots, m$ and $j = 1, \dots, n$.

If we suppose $p_i(x) = 1$ for $i = 1, \dots, m$, then, by (3), we obtain

$$\begin{aligned} x &= r_1(p_1(x), 1, x) = r_1(1, 1, x) = \\ &= r_2(1, 1, x) = \dots = r_k(1, 1, x) = \\ &= r_m(1, p_m(x), x) = 0, \end{aligned}$$

i.e., $x = 0$, thus we have the first implication of (4). The second one can be shown analogously.

(4) \Rightarrow (1): Let $\mathfrak{A} \in \mathcal{V}$ and $\Phi \in \text{Con}(\mathfrak{A})$. Suppose, e.g., $[1]\Phi = \{1\}$ and take $b \in [0]\Phi$. Then $[b]\Phi = [0]\Phi$ and, by (4), we have in the factor algebra \mathfrak{A}/Φ

$$[p_i(b)]\Phi = p_i([b]\Phi) = p_i([0]\Phi) = [1]\Phi = \{1\}$$

whence $p_i(b) = 1$ for $i = 1, \dots, m$. Applying (4) again, we conclude $b = 0$, thus $[0]\Phi = \{0\}$. Analogously we can show

$$[0]\Phi = \{0\} \Rightarrow [1]\Phi = \{1\}.$$

By Theorem 2, \mathcal{V} is balanced. ■

Remark. The equivalence of (2) and (3) can also be proved directly since (3) \Rightarrow (2) can be seen as follows:

Take the r_i in (3) and define the $(2m + 1)$ -ary terms \bar{r}_i by

$$\bar{r}_i(x_1, \dots, x_m, y_1, \dots, y_m, x) = r_i(x_i, y_i, x), i = 1, \dots, m,$$

then we have

$$\begin{aligned} \bar{r}_i(p_1(x), \dots, p_m(x), 1, \dots, 1, x) &= r_i(p_i(x), 1, x) \text{ and} \\ \bar{r}_i(1, \dots, 1, p_1(x), \dots, p_m(x), x) &= r_i(1, p_i(x), x), \end{aligned}$$

and we obtain the identities of (2) with $m = k$ and \bar{r}_i instead of r_i . Similarly, we obtain the identities of (2) with $n = h$ and some \bar{s}_i instead of s_i .

Example 5. The variety of all ortholattices $(L; \vee, \wedge, \perp, 0, 1)$ is balanced. We can take $m = 1 = n$ and $p_1(x) = q_1(x) = x^\perp$. Then $p_1(x) = 1$ if and only if $x = 0$ and $q_1(x) = 0$ if and only if $x = 1$.

Example 6. Every variety of double p -algebras is balanced (a double p -algebra is an algebra $(L; \vee, \wedge, *, ^+, 0, 1)$ of type $(2, 2, 1, 1, 0, 0)$, where $(L; \vee, \wedge, 0, 1)$ is a bounded lattice, a^* is a pseudocomplement of $a \in L$ and a^+ is a dual pseudocomplement). We can take $m = 1 = n$ and $p_1(x) = x^*$, $q_1(x) = x^+$. Of course, $0^* = 1$, $1^+ = 0$, and $x^* = 1 \Rightarrow x = 0$, $x^+ = 0 \Rightarrow x = 1$.

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Received 27 March 2000

Revised 2 August 2000