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HYPERSUBSTITUTIONS IN ORTHOMODULAR LATTICES

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Abstract

It is shown that in the variety of orthomodular lattices every hypersubstitution respecting all absorption laws either leaves the lattice operations unchanged or interchanges join and meet. Further, in a variety of lattices with an involutory antiautomorphism a semigroup generated by three involutory hypersubstitutions is described.

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1. Introduction

An involutory antiautomorphism of a poset $(P; \leq)$ is a mapping $' : P \to P$ with $x \leq y \Rightarrow x' \geq y'$ and $x'' = x$. An orthomodular lattice is an algebra $(L; \vee, \wedge, ', 0, 1)$ of type $(2, 2, 1, 0, 0)$ where $(L; \vee, \wedge, 0, 1)$ is a bounded lattice, $'$ is an involutory antiautomorphism of this lattice, $x \vee x' = 1$ and the orthomodular law $x \leq y \Rightarrow y = x \vee (y \wedge x')$ holds.

Let τ be a type of algebras. By a hypersubstitution of type τ there is either meant a mapping assigning to every fundamental operation symbol of type τ a term of type τ of the same arity or there is meant the obvious extension of this mapping to the set of all terms of type τ (see e. g. [2], [3] and [4] for details). Let $\mathrm{Hyp}(\tau)$ denote the set of all hypersubstitutions of type τ. Obviously, $(\mathrm{Hyp}(\tau); \circ)$ is a submonoid of the symmetric monoid over the set of all terms of type τ (see [2] or [8]). Let V be a variety of type τ and $\sigma, \sigma_1, \sigma_2$ hypersubstitutions of type τ. σ is said to respect the equation $s = t$ of type τ with respect to V if $\sigma(s) = \sigma(t)$ holds in V. The following concept was introduced by J. Płonka (cf. [5], [6]): A hypersubstitution σ is called proper with respect to V if it respects all equations holding in V. According to [6] σ_1, σ_2 are called equivalent with respect to V if $\sigma_1(t) = \sigma_2(t)$ holds in V for all terms t of type τ.

The main result of this paper is the following:

Theorem 1.1. *Up to equivalence there is only one non-trivial hypersubstitution of type* $(2, 2, 1, 0, 0)$ *which is proper with respect to the variety of orthomodular lattices, namely the one which interchanges the binary as well as the nullary operations and leaves the unary operation fixed.*

But first we are going to characterize algebras satisfying all absorption laws.

2. Absorption algebras

Definition 2.1. By an *absorption algebra* we mean an algebra $(L; \vee, \wedge)$ of type $(2, 2)$ satisfying all eight absorption laws:

$$(x \vee y) \wedge x = x,$$

$$(x \vee y) \wedge y = y,$$

$$x \wedge (x \vee y) = x,$$

$$y \wedge (x \vee y) = y,$$

$$(x \wedge y) \vee x = x,$$

$$(x \wedge y) \vee y = y,$$

$$x \vee (x \wedge y) = x,$$

$$y \vee (x \wedge y) = y.$$

In the following let $L = (L; \vee, \wedge)$ be an absorption algebra.

Lemma 2.1. *$a \vee a = a \wedge a = a$ for all $a \in L$.*

Proof. $a \vee a = a \vee (a \wedge (a \vee a)) = a$ and $a \wedge a = a \wedge (a \vee (a \wedge a)) = a$ for all $a \in L$. ■

Lemma 2.2. *For $a, b \in L$ the following are equivalent:*

(i) $a \vee b = b$,

(ii) $b \vee a = b$,

(iii) $a \wedge b = a$,

(iv) $b \wedge a = a$.

Proof. $a \vee b = b \Rightarrow a \wedge b = a \wedge (a \vee b) = a \Rightarrow b \vee a = b \vee (a \wedge b) = b \Rightarrow b \wedge a = (b \vee a) \wedge a = a \Rightarrow a \vee b = (b \wedge a) \vee b = b$. ■

Definition 2.2. On L we define a binary relation $\leq$ by $a \leq b$ iff one of the four equivalent conditions of Lemma 2.2 is satisfied $(a, b \in L)$.

Lemma 2.3. *The relation $\leq$ is reflexive and antisymmetric, and $a \wedge b \leq a, b \leq a \vee b$ for $a, b \in L$.*

Proof. Reflexivity of $\leq$ follows from Lemma 2.1 Now let $a, b \in L$. If $a \leq b \leq a$, then $a = a \wedge b = b$. This shows antisymmetry of $\leq$. Now

$$(a \wedge b) \vee a = a \Rightarrow a \wedge b \leq a,$$

$$(a \wedge b) \vee b = b \Rightarrow a \wedge b \leq b,$$

$$(a \vee b) \wedge a = a \Rightarrow a \leq a \vee b,$$

$$(a \vee b) \wedge b = b \Rightarrow b \leq a \vee b.$$

■

Definition 2.3. Let B be a set, $\leq$ a reflexive and antisymmetric binary relation on B and $M \subseteq B$. The elements a, b of B are said to be *mutually comparable* if $a \leq b$ or $b \leq a$ (or both). Otherwise a and b are said to be *mutually incomparable.* An element c of B is called a *lower bound* of M if $c \leq d$ for all $d \in M$. Dually, c is called an *upper bound* of M if $c \geq d$ for all $d \in M$.

Definition 2.4. Let B be a set, $\leq$ be a reflexive and antisymmetric binary relation on B, and $\vee$ and $\wedge$ be binary operations on B such that for every ordered pair (a, b) of elements of B $a \vee b$ (resp. $a \wedge b$) is an upper (resp. lower) bound of a and b defined in such a way that $a \vee b = b$ and $a \wedge b = a$ provided $a \leq b$, whereas $a \vee b = a$ and $a \wedge b = b$ provided $a \geq b$. Then the quadruple $(B; \leq, \vee, \wedge)$ is called a *bound structure.*

Definition 2.5. For every absorption algebra $\mathcal{L} = (L; \vee, \wedge)$ put $B(\mathcal{L}) := (L; \leq, \vee, \wedge)$, where $\leq$ is the binary relation on L defined by $a \leq b$ iff $a \vee b = b$ for $a, b \in L$, and for every bound structure $\mathcal{B} = (B; \leq, \vee, \wedge)$ put $L(\mathcal{B}) := (B; \vee, \wedge)$.

Theorem 2.1. *The mappings $\mathcal{L} \mapsto B(\mathcal{L})$ and $\mathcal{B} \mapsto L(\mathcal{B})$ are mutually inverse bijections between the set of all absorption algebras and the set of all bound structures both over the same fixed base set.*

Proof. If $\mathcal{L}$ is an absorption algebra, then $B(\mathcal{L})$ is a bound structure according to Lemmas 2.2 and 2.3. Conversely, let $\mathcal{B} = (B; \leq, \vee, \wedge)$ be a bound structure. If $a, b \in B$, then $a \leq a \vee b$, and hence $(a \vee b) \wedge a = a$. The other seven absorption laws can be proved analogously. Hence $L(\mathcal{B})$ is an absorption algebra. If $\mathcal{L} = (L; \vee, \wedge)$ is an absorption algebra, then obviously $L(B(\mathcal{L})) = L(L; \leq \vee, \wedge) = \mathcal{L}$. Conversely, let $\mathcal{B} = (B; \leq, \vee, \wedge)$ be a bound structure, $B(L(\mathcal{B})) = (B, \sqsubseteq, \vee, \wedge)$ and $a, b \in B$. If $a \leq b$, then $a \vee b = b$ and hence $a \sqsubseteq b$. Conversely, if $a \sqsubseteq b$, then $a \vee b = b$, which together with $a \leq a \vee b$ implies $a \leq b$. Hence, $B(L(\mathcal{B})) = \mathcal{B}$ completing the proof of the theorem. ■

Remark 2.1. Theorem 2.1 says that absorption algebras may be considered as sets with a reflexive and antisymmetric binary relation such that every two elements have an upper and a lower bound.

Example 2.1. The six-element algebra $(\{0, a, b, c, d, 1\}, \vee, \wedge)$ with operation tables

$\vee$	0	a	b	c	d	1
0	0	a	b	c	d	0
a	a	a	c	c	d	1
b	b	d	b	c	d	1
c	c	c	c	c	1	1
d	d	d	d	1	d	1
1	0	1	1	1	1	1

and

$\wedge$	0	a	b	c	d	1
0	0	0	0	0	0	1
a	0	a	0	a	a	a
b	0	0	b	b	b	b
c	0	a	b	c	b	c
d	0	a	b	a	d	d
1	1	a	b	c	d	1

is an absorption algebra. The operations $\vee$ and $\wedge$ are neither commutative ($a \vee b = c \neq d = b \vee a$ and $c \wedge d = b \neq a = d \wedge c$) nor associative ($(a \vee b) \vee d = c \vee d = 1 \neq d = a \vee d = a \vee (b \vee d)$ and $(c \wedge d) \wedge a = b \wedge a = 0 \neq a = c \wedge a = c \wedge (d \wedge a)$).

Theorem 2.2. *Every absorption algebra is congruence distributive. Every finite absorption algebra has a finite basis of identities.*

Proof. It can be easily checked that $((x \vee y) \wedge (x \vee z)) \wedge (y \vee z)$ is a majority term. From this fact the first assertion follows. The rest follows by using the so-called Baker's Finite Base Theorem (see, e.g., [1], p. 135). ■

3. HYPERSUBSTITUTIONS IN ORTHOMODULAR LATTICES

In this section let τ denote the type $(2, 2, 1, 0, 0)$ with operation symbols $(\vee, \wedge, ', 0, 1)$.

It is well-known that in the variety of orthomodular lattices there are exactly 2 nullary terms, namely 0 and 1, 4 unary terms, namely 0, x, x' and 1, and 96 binary terms, namely $\bigvee_{i \in I} t_i$, $\bigvee_{i \in I} t_i \vee (c' \wedge x)$, $\bigvee_{i \in I} t_i \vee (c' \wedge x')$, $\bigvee_{i \in I} t_i \vee (c' \wedge y)$, $\bigvee_{i \in I} t_i \vee (c' \wedge y')$ and $\bigvee_{i \in I} t_i \vee c'$ where $I \subseteq \{1, \ldots, 4\}$, $t_1 := x \wedge y$, $t_2 := x \wedge y'$, $t_3 := x' \wedge y$, $t_4 := x' \wedge y'$ and $c := t_1 \vee \ldots \vee t_4$. Hence it follows that in the variety of orthomodular lattices there are up to equivalence exactly 147456 hypersubstitutions. The following assertion is a strengthening of those in [5] and [6]:

Proposition 3.1. *Within the variety of orthomodular lattices every hypersubstitution respecting all absorption laws either leaves the lattice operations unchanged or interchanges the lattice operations.*

Proof. Let $\sigma \in \mathrm{Hyp}(\tau)$ and assume that it respects all absorption laws. We consider σ only up to equivalence with respect to the variety of orthomodular lattices. Put $(\sqcup, \sqcap) := (\sigma(\vee), \sigma(\wedge))$ and let $(x \sqcup y) \wedge c = \bigvee_{i \in J} t_i$ and $(x \sqcap y) \wedge c = \bigvee_{i \in K} t_i$ with $J, K \subseteq \{1, \ldots, 4\}$. If $(L; \vee, \wedge, ', 0, 1)$ is an orthomodular lattice, then $(L; \sqcup, \sqcap)$ is an absorption algebra. Hence, it follows $x \sqcup x = x \sqcap x = x$. Because of $0 \sqcup 0 = 0 \sqcap 0 = 0$, we have $4 \notin J, K$, whereas from $1 \sqcup 1 = 1 \sqcap 1 = 1$, it follows $1 \in J, K$. Since $0 \sqcup 1 \in \{0, 1\}$, we either have both $0 \sqcup 1 = 1 \sqcup 0 = 0$ and $0 \sqcap 1 = 1 \sqcap 0 = 1$ or both $0 \sqcup 1 = 1 \sqcup 0 = 1$ and $0 \sqcap 1 = 1 \sqcap 0 = 0$. In the first case $J = \{1\}$ and $K = \{1, 2, 3\}$, whereas in the second case $J = \{1, 2, 3\}$ and $K = \{1\}$.

Let MO2:= $\{0, a, a', b, b', 1\}$ denote the six-element orthomodular lattice with atoms a, a', b, b'. Now we consider the first case. Then we have:

$x \sqcup y = t_1 \vee (c' \wedge x) \Rightarrow b \sqcup (a \sqcup b) = b \neq a = a \sqcup b$, a contradiction,

$x \sqcup y = t_1 \vee (c' \wedge x') \Rightarrow (a \sqcup b) \sqcup b = a \neq a' = a \sqcup b$, a contradiction,

$x \sqcup y = t_1 \vee (c' \wedge y) \Rightarrow (a \sqcup b) \sqcup a = a \neq b = a \sqcup b$, a contradiction,

$x \sqcup y = t_1 \vee (c' \wedge y') \Rightarrow (a \sqcup b) \sqcup a = a' \neq b' = a \sqcup b$, a contradiction,

$x \sqcap y = t_1 \vee t_2 \vee t_3 \vee (c' \wedge x) \Rightarrow b \sqcap (a \sqcap b) = b \neq a = a \sqcap b$, a contradiction,

$x \sqcap y = t_1 \vee t_2 \vee t_3 \vee (c' \wedge x') \Rightarrow (a \sqcap b) \sqcap b = a \neq a' = a \sqcap b$, a contradiction,

$x \sqcap y = t_1 \vee t_2 \vee t_3 \vee (c' \wedge y) \Rightarrow (a \sqcap b) \sqcap a = a \neq b = a \sqcap b$, a contradiction,

$x \sqcap y = t_1 \vee t_2 \vee t_3 \vee (c' \wedge y') \Rightarrow (a \sqcap b) \sqcap a = a' \neq b' = a \sqcap b$, a contradiction.

Put

$$x \bigvee y := (x \wedge y) \vee (x \wedge y') \vee (x' \wedge y),$$

$$x \bigwedge y := (x \vee y) \wedge (x \vee y') \wedge (x' \vee y).$$

Then

$$x \vee y = t_1 \vee t_2 \vee t_3 \vee c',$$

$$x \bigvee y = t_1 \vee t_2 \vee t_3,$$

$$x \wedge y = t_1,$$

$$x \bigwedge y = t_1 \vee c'.$$

From the above considerations it follows that $(\sqcup, \sqcap) \in \{\wedge, \bigwedge\} \times \{\vee, \bigvee\}$. Since in MO2, we have

$$(a \bigwedge b) \wedge a = 0 \wedge a = 0 \neq a,$$

$$(a \bigwedge b) \bigvee a = 0 \bigwedge a = 0 \neq a,$$

$$(a \bigvee b) \vee a = 1 \vee a = 1 \neq a,$$

therefore it follows $(\sqcup, \sqcap) = (\wedge, \vee)$. In the second case it follows $(\sqcup, \sqcap) = (\vee, \wedge)$ in an analogous way. ■

We are now able to prove Theorem 1.1.

***Proof of Theorem* 1.1.** Let $\sigma \in \mathrm{Hyp}(\tau)$ be proper with respect to the variety of orthomodular lattices. We consider σ only up to equivalence with respect to the variety of orthomodular lattices. Then, σ either leaves the lattice operations unchanged or it interchanges the lattice operations according to Proposition 3.1. In the first case, we have

$$x \vee 0 = x \Rightarrow x \vee \sigma(0) = x \Rightarrow \sigma(0) \neq 1 \Rightarrow \sigma(0) = 0,$$

$$x \wedge 1 = x \Rightarrow x \wedge \sigma(1) = x \Rightarrow \sigma(1) \neq 0 \Rightarrow \sigma(1) = 1,$$

$$x \vee x' = 1 \Rightarrow x \vee \sigma(x') = 1 \Rightarrow \sigma(x') \neq 0, x, \text{and}$$

$$x \wedge x' = 0 \Rightarrow x \wedge \sigma(x') = 0 \Rightarrow \sigma(x') \neq x, 1.$$

Hence $\sigma(x') = x'$. In the second case, we have

$$x \vee 0 = x \Rightarrow x \wedge \sigma(0) = x \Rightarrow \sigma(0) \neq 0 \Rightarrow \sigma(0) = 1,$$

$$x \wedge 1 = x \Rightarrow x \vee \sigma(1) = x \Rightarrow \sigma(1) \neq 1 \Rightarrow \sigma(1) = 0,$$

$$x \vee x' = 1 \Rightarrow x \wedge \sigma(x') = 0 \Rightarrow \sigma(x') \neq x, 1, \text{and}$$

$$x \wedge x' = 0 \Rightarrow x \vee \sigma(x') = 1 \Rightarrow \sigma(x') \neq 0, x.$$

From this it again follows $\sigma(x') = x'$ completing the proof of the theorem. ■

4. Hypersubstitutions in bounded lattices with an involutory antiautomorphism

In this section let τ denote the type $(2, 2, 1, 0, 0)$ with operation symbols $(\vee, \wedge, ', 0, 1)$ and W the variety of bounded lattices with an involutory dual-automorphism.

Definition 4.1. Let $\sigma_1, \ldots, \sigma_4 \in \mathrm{Hyp}(\tau)$ be defined by

$$\sigma_1(x \vee y) := (x \vee y) \wedge (x' \vee y'),$$

$$\sigma_2(x \vee y) := (x \wedge y') \vee (x' \wedge y),$$

$$\sigma_3(x \vee y) := (x \wedge y) \vee (x' \wedge y'),$$

$$\sigma_4(x \vee y) := (x \vee y') \wedge (x' \vee y),$$

$\sigma_i(x \wedge y) := x \wedge y$, $\sigma_i(x') := x'$, $\sigma_i(0) := 0$ and $\sigma_i(1) := 1$ for $i = 1, \ldots, 4$.

Theorem 4.1. *In the variety W the algebra $(\{\sigma_1, \ldots, \sigma_4\}; \circ, \sigma_1)$ is a monoid generated (as a semigroup) by $\{\sigma_3, \sigma_4\}$ and having the operation table*

$\circ$	σ_1	σ_2	σ_3	σ_4
σ_1	σ_1	σ_2	σ_3	σ_4
σ_2	σ_2	σ_2	σ_4	σ_4
σ_3	σ_3	σ_3	σ_1	σ_1
σ_4	σ_4	σ_3	σ_2	σ_1

Moreover, $\sigma_j^2 = \sigma_1$ and

$$(\sigma_i \sigma_j)\sigma_j = \sigma_i = \sigma_j(\sigma_j \sigma_i)$$

for $i = 1, \ldots, 4$ and $j = 1, 3, 4$. This semigroup has exactly six (non-empty) subsemigroups.

Proof. The first assertions can be easily checked. The identity $\sigma_j^2 = \sigma_1$ for $j = 1, 3, 4$ is evident from the table and

$$(\sigma_i \sigma_j)\sigma_j = \sigma_i(\sigma_j \sigma_j) = \sigma_i \sigma_1 = \sigma_i$$

for $i = 1, \dots, 4$ and $j = 1, 3, 4$, analogously for the second equality. It turns out that this semigroup has the following (non-empty) subsemigroups: $\{\sigma_1\}$, $\{\sigma_2\}$, $\{\sigma_1, \sigma_2\}$, $\{\sigma_1, \sigma_3\}$, $\{\sigma_1, \sigma_4\}$ and the whole semigroup $\{\sigma_1, \sigma_2, \sigma_3, \sigma_4\}$.■

For the sake of brevity, denote by $+_i$ the binary term of type τ defined by

$$x +_i y := \sigma_i(x \vee y)$$

for $i = 1, \dots, 4$. Moreover, for every $i = 1, \dots, 4$ denote by T_i the clone of terms of type τ generated by the set $\{+_i, \wedge, ', 0, 1\}$ and by id, the identity element of the monoid $\mathrm{Hyp}(\tau)$.

Theorem 4.2. *Let $\sigma_1, \dots, \sigma_4 \in \mathrm{Hyp}(\tau)$ as introduced by Definition 4.1. Then for every $i \in \{1, \dots, 4\}$ there exists a mapping ρ_i from T_i to T_i such that the identity $\rho_i \circ \sigma_i = \mathrm{id}$ holds in W. In particular, for the generators of T_i we can put*

$$\rho_1(x +_1 y) := 1 +_1 ((1 +_1 x) \wedge (1 +_1 y)),$$

$$\rho_2(x +_2 y) := 1 +_2 ((1 +_2 x) \wedge (1 +_2 y)),$$

$$\rho_3(x +_3 y) := 0 +_3 ((0 +_3 x) \wedge (0 +_3 y)),$$

$$\rho_4(x +_4 y) := 0 +_4 ((0 +_4 x) \wedge (0 +_4 y)),$$

$\rho_i(x \wedge y) := x \wedge y$, $\rho_i(x') := x'$, $\rho_i(0) := 0$ and $\rho_i(1) := 1$ for $i = 1, \dots, 4$.

Proof. It is easy to derive $1 +_1 x = (1 \vee x) \wedge (1' \vee x') = x'$ in the variety W and, analogously, also $1 +_2 x = x'$, $0 +_3 x = x'$ and $0 +_4 x = x'$. Now

$$\rho_1(\sigma_1(x \vee y)) = \rho_1(x +_1 y) = 1 +_1 ((1 +_1 x) \wedge (1 +_1 y)) = (x' \wedge y')' = x \vee y$$

in W and, of course $\rho_1(\sigma_1(x \wedge y)) = x \wedge y$, $\rho_1(\sigma_1(x')) = x'$, $\rho_1(\sigma_1(0)) = 0$ and $\rho_1(\sigma_1(1)) = 1$, i. e. $\rho_1 \circ \sigma_1 = \mathrm{id}$. The rest of the proof follows in an analogous way. ■

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