# HYPERSUBSTITUTIONS IN ORTHOMODULAR LATTICES 

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#### Abstract

It is shown that in the variety of orthomodular lattices every hypersubstitution respecting all absorption laws either leaves the lattice operations unchanged or interchanges join and meet. Further, in a variety of lattices with an involutory antiautomorphism a semigroup generated by three involutory hypersubstitutions is described.


Keywords: hypersubstitution, proper hypersubstitution, orthomodular lattice, absorption algebra.

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## 1. Introduction

An involutory antiautomorphism of a poset $(P ; \leq)$ is a mapping ${ }^{\prime}: P \rightarrow P$ with $x \leq y \Rightarrow x^{\prime} \geq y^{\prime}$ and $x^{\prime \prime}=x$. An orthomodular lattice is an algebra $\left(L ; \vee, \wedge,^{\prime}, 0,1\right)$ of type $(2,2,1,0,0)$ where $(L ; \vee, \wedge, 0,1)$ is a bounded lattice, $'$ is an involutory antiautomorphism of this lattice, $x \vee x^{\prime}=1$ and the orthomodular law $x \leq y \Rightarrow y=x \vee\left(y \wedge x^{\prime}\right)$ holds.

Let $\tau$ be a type of algebras. By a hypersubstitution of type $\tau$ there is either meant a mapping assigning to every fundamental operation symbol of type $\tau$ a term of type $\tau$ of the same arity or there is meant the obvious extension of this mapping to the set of all terms of type $\tau$ (see e. g. [2], [3] and [4] for details). Let $\operatorname{Hyp}(\tau)$ denote the set of all hypersubstitutions of type $\tau$. Obviously, $(\operatorname{Hyp}(\tau) ; \circ)$ is a submonoid of the symmetric monoid over the set of all terms of type $\tau$ (see [2] or [8]). Let $V$ be a variety of type $\tau$ and $\sigma, \sigma_{1}, \sigma_{2}$ hypersubstitutions of type $\tau . \sigma$ is said to respect the equation $s=t$ of type $\tau$ with respect to $V$ if $\sigma(s)=\sigma(t)$ holds in $V$. The following concept was introduced by J. Płonka (cf. [5], [6]): A hypersubstitution $\sigma$ is called proper with respect to $V$ if it respects all equations holding in $V$. According to [6] $\sigma_{1}, \sigma_{2}$ are called equivalent with respect to $V$ if $\sigma_{1}(t)=\sigma_{2}(t)$ holds in $V$ for all terms $t$ of type $\tau$.

The main result of this paper is the following:
Theorem 1.1. Up to equivalence there is only one non-trivial hypersubstitution of type $(2,2,1,0,0)$ which is proper with respect to the variety of orthomodular lattices, namely the one which interchanges the binary as well as the nullary operations and leaves the unary operation fixed.

But first we are going to characterize algebras satisfying all absorption laws.

## 2. Absorption algebras

Definition 2.1. By an absorption algebra we mean an algebra ( $L ; \vee, \wedge$ ) of type $(2,2)$ satisfying all eight absorption laws:

$$
\begin{aligned}
& (x \vee y) \wedge x=x \\
& (x \vee y) \wedge y=y \\
& x \wedge(x \vee y)=x \\
& y \wedge(x \vee y)=y
\end{aligned}
$$

$$
\begin{aligned}
& (x \wedge y) \vee x=x, \\
& (x \wedge y) \vee y=y, \\
& x \vee(x \wedge y)=x, \\
& y \vee(x \wedge y)=y .
\end{aligned}
$$

In the following let $L=(L ; \vee, \wedge)$ be an absorption algebra.
Lemma 2.1. $a \vee a=a \wedge a=a$ for all $a \in L$.
Proof. $a \vee a=a \vee(a \wedge(a \vee a))=a$ and $a \wedge a=a \wedge(a \vee(a \wedge a))=a$ for all $a \in L$.

Lemma 2.2. For $a, b \in L$ the following are equivalent:
(i) $a \vee b=b$,
(ii) $b \vee a=b$,
(iii) $a \wedge b=a$,
(iv) $b \wedge a=a$.

Proof. $a \vee b=b \Rightarrow a \wedge b=a \wedge(a \vee b)=a \Rightarrow b \vee a=b \vee(a \wedge b)=b \Rightarrow$ $b \wedge a=(b \vee a) \wedge a=a \Rightarrow a \vee b=(b \wedge a) \vee b=b$.

Definition 2.2. On $L$ we define a binary relation $\leq$ by $a \leq b$ iff one of the four equivalent conditions of Lemma 2.2 is satisfied $(a, b \in L)$.

Lemma 2.3. The relation $\leq$ is reflexive and antisymmetric, and $a \wedge b \leq$ $a, b \leq a \vee b$ for $a, b \in L$.

Proof. Reflexivity of $\leq$ follows from Lemma 2.1 Now let $a, b \in L$. If $a \leq b \leq a$, then $a=a \wedge b=b$. This shows antisymmetry of $\leq$. Now

$$
\begin{aligned}
& (a \wedge b) \vee a=a \Rightarrow a \wedge b \leq a, \\
& (a \wedge b) \vee b=b \Rightarrow a \wedge b \leq b, \\
& (a \vee b) \wedge a=a \Rightarrow a \leq a \vee b, \\
& (a \vee b) \wedge b=b \Rightarrow b \leq a \vee b .
\end{aligned}
$$

Definition 2.3. Let $B$ be a set, $\leq$ a reflexive and antisymmetric binary relation on $B$ and $M \subseteq B$. The elements $a, b$ of $B$ are said to be mutually comparable if $a \leq b$ or $b \leq a$ (or both). Otherwise $a$ and $b$ are said to be mutually incomparable. An element $c$ of $B$ is called a lower bound of $M$ if $c \leq d$ for all $d \in M$. Dually, $c$ is called an upper bound of $M$ if $c \geq d$ for all $d \in M$.

Definition 2.4. Let $B$ be a set, $\leq$ be a reflexive and antisymmetric binary relation on $B$, and $\vee$ and $\wedge$ be binary operations on $B$ such that for every ordered pair $(a, b)$ of elements of $B a \vee b$ (resp. $a \wedge b$ ) is an upper (resp. lower) bound of $a$ and $b$ defined in such a way that $a \vee b=b$ and $a \wedge b=a$ provided $a \leq b$, whereas $a \vee b=a$ and $a \wedge b=b$ provided $a \geq b$. Then the quadruple $(B ; \leq, \vee, \wedge)$ is called a bound structure.

Definition 2.5. For every absorption algebra $\mathcal{L}=(L ; \vee, \wedge)$ put $B(\mathcal{L}):=$ $(L ; \leq, \vee, \wedge)$, where $\leq$ is the binary relation on $L$ defined by $a \leq b$ iff $a \vee b=b$ for $a, b \in L$, and for every bound structure $\mathcal{B}=(B ; \leq, \vee, \wedge)$ put $L(\mathcal{B}):=$ $(B ; \vee, \wedge)$.

Theorem 2.1. The mappings $\mathcal{L} \mapsto B(\mathcal{L})$ and $\mathcal{B} \mapsto L(\mathcal{B})$ are mutually inverse bijections between the set of all absorption algebras and the set of all bound structures both over the same fixed base set.

Proof. If $\mathcal{L}$ is an absorption algebra, then $B(\mathcal{L})$ is a bound structure according to Lemmas 2.2 and 2.3. Conversely, let $\mathcal{B}=(B ; \leq, \vee, \wedge)$ be a bound structure. If $a, b \in B$, then $a \leq a \vee b$, and hence $(a \vee b) \wedge a=a$. The other seven absorption laws can be proved analogously. Hence $L(\mathcal{B})$ is an absorption algebra. If $\mathcal{L}=(L ; \vee, \wedge)$ is an absorption algebra, then obviously $L(B(\mathcal{L}))=L(L ; \leq \vee, \wedge)=\mathcal{L}$. Conversely, let $\mathcal{B}=(B ; \leq, \vee, \wedge)$ be a bound structure, $B(L(\mathcal{B}))=(B, \sqsubseteq, \vee, \wedge)$ and $a, b \in B$. If $a \leq b$, then $a \vee b=b$ and hence $a \sqsubseteq b$. Conversely, if $a \sqsubseteq b$, then $a \vee b=b$, which together with $a \leq a \vee b$ implies $a \leq b$. Hence, $B(L(\mathcal{B}))=\mathcal{B}$ completing the proof of the theorem.

Remark 2.1. Theorem 2.1 says that absorption algebras may be considered as sets with a reflexive and antisymmetric binary relation such that every two elements have an upper and a lower bound.

Example 2.1. The six-element algebra $(\{0, a, b, c, d, 1\}, \vee, \wedge)$ with operation tables

| $\vee$ | 0 | $a$ | $b$ | $c$ | $d$ | 1 |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 0 | 0 | $a$ | $b$ | $c$ | $d$ | 0 |
| $a$ | $a$ | $a$ | $c$ | $c$ | $d$ | 1 |
| $b$ | $b$ | $d$ | $b$ | $c$ | $d$ | 1 |
| $c$ | $c$ | $c$ | $c$ | $c$ | 1 | 1 |
| $d$ | $d$ | $d$ | $d$ | 1 | $d$ | 1 |
| 1 | 0 | 1 | 1 | 1 | 1 | 1 |

and

| $\wedge$ | 0 | $a$ | $b$ | $c$ | $d$ | 1 |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 0 | 0 | 0 | 0 | 0 | 0 | 1 |
| $a$ | 0 | $a$ | 0 | $a$ | $a$ | $a$ |
| $b$ | 0 | 0 | $b$ | $b$ | $b$ | $b$ |
| $c$ | 0 | $a$ | $b$ | $c$ | $b$ | $c$ |
| $d$ | 0 | $a$ | $b$ | $a$ | $d$ | $d$ |
| 1 | 1 | $a$ | $b$ | $c$ | $d$ | 1 |

is an absorption algebra. The operations $\vee$ and $\wedge$ are neither commutative $(a \vee b=c \neq d=b \vee a$ and $c \wedge d=b \neq a=d \wedge c)$ nor associative $((a \vee b) \vee d=c \vee d=1 \neq d=a \vee d=a \vee(b \vee d)$ and $(c \wedge d) \wedge a=b \wedge a=$ $0 \neq a=c \wedge a=c \wedge(d \wedge a))$.

Theorem 2.2. Every absorption algebra is congruence distributive. Every finite absorption algebra has a finite basis of identities.

Proof. It can be easily checked that $((x \vee y) \wedge(x \vee z)) \wedge(y \vee z)$ is a majority term. From this fact the first assertion follows. The rest follows by using the so-called Baker's Finite Base Theorem (see, e.g., [1], p. 135).

## 3. Hypersubstitutions in orthomodular lattices

In this section let $\tau$ denote the type $(2,2,1,0,0)$ with operation symbols $\left(\vee, \wedge,{ }^{\prime}, 0,1\right)$.

It is well-known that in the variety of orthomodular lattices there are exactly 2 nullary terms, namely 0 and 1,4 unary terms, namely $0, x, x^{\prime}$ and 1 , and 96 binary terms, namely $\bigvee_{i \in I} t_{i}, \bigvee_{i \in I} t_{i} \vee\left(c^{\prime} \wedge x\right), \bigvee_{i \in I} t_{i} \vee\left(c^{\prime} \wedge x^{\prime}\right)$, $\bigvee_{i \in I} t_{i} \vee\left(c^{\prime} \wedge y\right), \bigvee_{i \in I} t_{i} \vee\left(c^{\prime} \wedge y^{\prime}\right)$ and $\bigvee_{i \in I} t_{i} \vee c^{\prime}$ where $I \subseteq\{1, \ldots, 4\}$, $t_{1}:=x \wedge y, t_{2}:=x \wedge y^{\prime}, t_{3}:=x^{\prime} \wedge y, t_{4}:=x^{\prime} \wedge y^{\prime}$ and $c:=t_{1} \vee \ldots \vee t_{4}$. Hence it follows that in the variety of orthomodular lattices there are up to equivalence exactly 147456 hypersubstitutions. The following assertion is a strengthening of those in [5] and [6]:

Proposition 3.1. Within the variety of orthomodular lattices every hypersubstitution respecting all absorption laws either leaves the lattice operations unchanged or interchanges the lattice operations.

Proof. Let $\sigma \in \operatorname{Hyp}(\tau)$ and assume that it respects all absorption laws. We consider $\sigma$ only up to equivalence with respect to the variety of orthomodular lattices. Put $(\sqcup, \sqcap):=(\sigma(\vee), \sigma(\wedge))$ and let $(x \sqcup y) \wedge c=\bigvee_{i \in J} t_{i}$ and $(x \sqcap y) \wedge c=\bigvee_{i \in K} t_{i}$ with $J, K \subseteq\{1, \ldots, 4\}$. If $\left(L ; \vee, \wedge,^{\prime}, 0,1\right)$ is an orthomodular lattice, then $(L ; \sqcup, \sqcap)$ is an absorption algebra. Hence, it follows $x \sqcup x=x \sqcap x=x$. Because of $0 \sqcup 0=0 \sqcap 0=0$, we have $4 \notin J, K$, whereas from $1 \sqcup 1=1 \sqcap 1=1$, it follows $1 \in J, K$. Since $0 \sqcup 1 \in\{0,1\}$, we either have both $0 \sqcup 1=1 \sqcup 0=0$ and $0 \sqcap 1=1 \sqcap 0=1$ or both $0 \sqcup 1=1 \sqcup 0=1$ and $0 \sqcap 1=1 \sqcap 0=0$. In the first case $J=\{1\}$ and $K=\{1,2,3\}$, whereas in the second case $J=\{1,2,3\}$ and $K=\{1\}$.

Let MO2: $=\left\{0, a, a^{\prime}, b, b^{\prime}, 1\right\}$ denote the six-element orthomodular lattice with atoms $a, a^{\prime}, b, b^{\prime}$. Now we consider the first case. Then we have:
$x \sqcup y=t_{1} \vee\left(c^{\prime} \wedge x\right) \Rightarrow b \sqcup(a \sqcup b)=b \neq a=a \sqcup b$, a contradiction, $x \sqcup y=t_{1} \vee\left(c^{\prime} \wedge x^{\prime}\right) \Rightarrow(a \sqcup b) \sqcup b=a \neq a^{\prime}=a \sqcup b$, a contradiction, $x \sqcup y=t_{1} \vee\left(c^{\prime} \wedge y\right) \Rightarrow(a \sqcup b) \sqcup a=a \neq b=a \sqcup b$, a contradiction, $x \sqcup y=t_{1} \vee\left(c^{\prime} \wedge y^{\prime}\right) \Rightarrow(a \sqcup b) \sqcup a=a^{\prime} \neq b^{\prime}=a \sqcup b$, a contradiction, $x \sqcap y=t_{1} \vee t_{2} \vee t_{3} \vee\left(c^{\prime} \wedge x\right) \Rightarrow b \sqcap(a \sqcap b)=b \neq a=a \sqcap b$, a contradiction, $x \sqcap y=t_{1} \vee t_{2} \vee t_{3} \vee\left(c^{\prime} \wedge x^{\prime}\right) \Rightarrow(a \sqcap b) \sqcap b=a \neq a^{\prime}=a \sqcap b$, a contradiction, $x \sqcap y=t_{1} \vee t_{2} \vee t_{3} \vee\left(c^{\prime} \wedge y\right) \Rightarrow(a \sqcap b) \sqcap a=a \neq b=a \sqcap b$, a contradiction, $x \sqcap y=t_{1} \vee t_{2} \vee t_{3} \vee\left(c^{\prime} \wedge y^{\prime}\right) \Rightarrow(a \sqcap b) \sqcap a=a^{\prime} \neq b^{\prime}=a \sqcap b$, a contradiction. Put

$$
\begin{aligned}
& x \bigvee y:=(x \wedge y) \vee\left(x \wedge y^{\prime}\right) \vee\left(x^{\prime} \wedge y\right), \\
& x \bigwedge y:=(x \vee y) \wedge\left(x \vee y^{\prime}\right) \wedge\left(x^{\prime} \vee y\right) .
\end{aligned}
$$

Then

$$
\begin{aligned}
& x \vee y=t_{1} \vee t_{2} \vee t_{3} \vee c^{\prime}, \\
& x \bigvee y=t_{1} \vee t_{2} \vee t_{3}, \\
& x \wedge y=t_{1}, \\
& x \bigwedge y=t_{1} \vee c^{\prime} .
\end{aligned}
$$

From the above considerations it follows that $(\sqcup, \sqcap) \in\{\wedge, \wedge\} \times\{\vee, \bigvee\}$. Since in MO2, we have

$$
\begin{aligned}
& (a \bigwedge b) \wedge a=0 \wedge a=0 \neq a \\
& (a \bigwedge b) \bigvee a=0 \bigwedge a=0 \neq a \\
& (a \bigvee b) \vee a=1 \vee a=1 \neq a
\end{aligned}
$$

therefore it follows $(\sqcup, \sqcap)=(\wedge, \vee)$. In the second case it follows $(\sqcup, \sqcap)=$ $(\vee, \wedge)$ in an analogous way.

We are now able to prove Theorem 1.1.

Proof of Theorem 1.1. Let $\sigma \in \operatorname{Hyp}(\tau)$ be proper with respect to the variety of orthomodular lattices. We consider $\sigma$ only up to equivalence with respect to the variety of orthomodular lattices. Then, $\sigma$ either leaves the lattice operations unchanged or it interchanges the lattice operations according to Proposition 3.1. In the first case, we have

$$
\begin{aligned}
& x \vee 0=x \Rightarrow x \vee \sigma(0)=x \Rightarrow \sigma(0) \neq 1 \Rightarrow \sigma(0)=0, \\
& x \wedge 1=x \Rightarrow x \wedge \sigma(1)=x \Rightarrow \sigma(1) \neq 0 \Rightarrow \sigma(1)=1, \\
& x \vee x^{\prime}=1 \Rightarrow x \vee \sigma\left(x^{\prime}\right)=1 \Rightarrow \sigma\left(x^{\prime}\right) \neq 0, x, \text { and } \\
& x \wedge x^{\prime}=0 \Rightarrow x \wedge \sigma\left(x^{\prime}\right)=0 \Rightarrow \sigma\left(x^{\prime}\right) \neq x, 1 .
\end{aligned}
$$

Hence $\sigma\left(x^{\prime}\right)=x^{\prime}$. In the second case, we have

$$
\begin{aligned}
& x \vee 0=x \Rightarrow x \wedge \sigma(0)=x \Rightarrow \sigma(0) \neq 0 \Rightarrow \sigma(0)=1, \\
& x \wedge 1=x \Rightarrow x \vee \sigma(1)=x \Rightarrow \sigma(1) \neq 1 \Rightarrow \sigma(1)=0, \\
& x \vee x^{\prime}=1 \Rightarrow x \wedge \sigma\left(x^{\prime}\right)=0 \Rightarrow \sigma\left(x^{\prime}\right) \neq x, 1, \text { and } \\
& x \wedge x^{\prime}=0 \Rightarrow x \vee \sigma\left(x^{\prime}\right)=1 \Rightarrow \sigma\left(x^{\prime}\right) \neq 0, x .
\end{aligned}
$$

From this it again follows $\sigma\left(x^{\prime}\right)=x^{\prime}$ completing the proof of the theorem.

## 4. Hypersubstitutions in bounded lattices with an involutory ANTIAUTOMORPHISM

In this section let $\tau$ denote the type $(2,2,1,0,0)$ with operation symbols $\left(\vee, \wedge,^{\prime}, 0,1\right)$ and $W$ the variety of bounded lattices with an involutory dual-automorphism.

Definition 4.1. Let $\sigma_{1}, \ldots, \sigma_{4} \in \operatorname{Hyp}(\tau)$ be defined by

$$
\begin{gathered}
\sigma_{1}(x \vee y):=(x \vee y) \wedge\left(x^{\prime} \vee y^{\prime}\right), \\
\sigma_{2}(x \vee y):=\left(x \wedge y^{\prime}\right) \vee\left(x^{\prime} \wedge y\right), \\
\sigma_{3}(x \vee y):=(x \wedge y) \vee\left(x^{\prime} \wedge y^{\prime}\right), \\
\sigma_{4}(x \vee y):=\left(x \vee y^{\prime}\right) \wedge\left(x^{\prime} \vee y\right), \\
\sigma_{i}(x \wedge y):=x \wedge y, \sigma_{i}\left(x^{\prime}\right):=x^{\prime}, \sigma_{i}(0):=0 \text { and } \sigma_{i}(1):=1 \text { for } i=1, \ldots, 4
\end{gathered}
$$

Theorem 4.1. In the variety $W$ the algebra $\left(\left\{\sigma_{1}, \ldots, \sigma_{4}\right\} ; \circ, \sigma_{1}\right)$ is a monoid generated (as a semigroup) by $\left\{\sigma_{3}, \sigma_{4}\right\}$ and having the operation table

| $\circ$ | $\sigma_{1}$ | $\sigma_{2}$ | $\sigma_{3}$ | $\sigma_{4}$ |
| :---: | :---: | :---: | :---: | :---: |
| $\sigma_{1}$ | $\sigma_{1}$ | $\sigma_{2}$ | $\sigma_{3}$ | $\sigma_{4}$ |
| $\sigma_{2}$ | $\sigma_{2}$ | $\sigma_{2}$ | $\sigma_{4}$ | $\sigma_{4}$ |
| $\sigma_{3}$ | $\sigma_{3}$ | $\sigma_{3}$ | $\sigma_{1}$ | $\sigma_{1}$ |
| $\sigma_{4}$ | $\sigma_{4}$ | $\sigma_{3}$ | $\sigma_{2}$ | $\sigma_{1}$ |

Moreover, $\sigma_{j}^{2}=\sigma_{1}$ and

$$
\left(\sigma_{i} \sigma_{j}\right) \sigma_{j}=\sigma_{i}=\sigma_{j}\left(\sigma_{j} \sigma_{i}\right)
$$

for $i=1, \ldots, 4$ and $j=1,3,4$. This semigroup has exactly six (non-empty) subsemigroups.

Proof. The first assertions can be easily checked. The identity $\sigma_{j}^{2}=\sigma_{1}$ for $j=1,3,4$ is evident from the table and

$$
\left(\sigma_{i} \sigma_{j}\right) \sigma_{j}=\sigma_{i}\left(\sigma_{j} \sigma_{j}\right)=\sigma_{i} \sigma_{1}=\sigma_{i}
$$

for $i=1, \ldots, 4$ and $j=1,3,4$, analogously for the second equality. It turns out that this semigroup has the following (non-empty) subsemigroups: $\left\{\sigma_{1}\right\}$, $\left\{\sigma_{2}\right\},\left\{\sigma_{1}, \sigma_{2}\right\},\left\{\sigma_{1}, \sigma_{3}\right\},\left\{\sigma_{1}, \sigma_{4}\right\}$ and the whole semigroup $\left\{\sigma_{1}, \sigma_{2}, \sigma_{3}, \sigma_{4}\right\}$.

For the sake of brevity, denote by $+_{i}$ the binary term of type $\tau$ defined by

$$
x+{ }_{i} y:=\sigma_{i}(x \vee y)
$$

for $i=1, \ldots, 4$. Moreover, for every $i=1, \ldots, 4$ denote by $T_{i}$ the clone of terms of type $\tau$ generated by the set $\left\{+_{i}, \wedge_{,}^{\prime}, 0,1\right\}$ and by id, the identity element of the monoid $\operatorname{Hyp}(\tau)$.

Theorem 4.2. Let $\sigma_{1}, \ldots, \sigma_{4} \in \operatorname{Hyp}(\tau)$ as introduced by Definition 4.1. Then for every $i \in\{1, \ldots, 4\}$ there exists a mapping $\rho_{i}$ from $T_{i}$ to $T_{i}$ such that the identity $\rho_{i} \circ \sigma_{i}=\mathrm{id}$ holds in $W$. In particular, for the generators of $T_{i}$ we can put

$$
\begin{gathered}
\rho_{1}\left(x+{ }_{1} y\right):=1+{ }_{1}\left(\left(1+{ }_{1} x\right) \wedge\left(1+{ }_{1} y\right)\right) \\
\rho_{2}\left(x+{ }_{2} y\right):=1+{ }_{2}\left(\left(1+{ }_{2} x\right) \wedge\left(1+{ }_{2} y\right)\right) \\
\rho_{3}\left(x+{ }_{3} y\right):=0+{ }_{3}\left(\left(0+{ }_{3} x\right) \wedge\left(0+{ }_{3} y\right)\right) \\
\rho_{4}\left(x+{ }_{4} y\right):=0+_{4}\left(\left(0+_{4} x\right) \wedge\left(0+_{4} y\right)\right) \\
\rho_{i}(x \wedge y):=x \wedge y, \rho_{i}\left(x^{\prime}\right):=x^{\prime}, \rho_{i}(0):=0 \text { and } \rho_{i}(1):=1 \text { for } i=1, \ldots, 4
\end{gathered}
$$

Proof. It is easy to derive $1+{ }_{1} x=(1 \vee x) \wedge\left(1^{\prime} \vee x^{\prime}\right)=x^{\prime}$ in the variety $W$ and, analogously, also $1+2 x=x^{\prime}, 0+3 x=x^{\prime}$ and $0+{ }_{4} x=x^{\prime}$. Now
$\rho_{1}\left(\sigma_{1}(x \vee y)\right)=\rho_{1}\left(x+{ }_{1} y\right)=1+{ }_{1}\left(\left(1+{ }_{1} x\right) \wedge\left(1+{ }_{1} y\right)\right)=\left(x^{\prime} \wedge y^{\prime}\right)^{\prime}=x \vee y$
in $W$ and, of course $\rho_{1}\left(\sigma_{1}(x \wedge y)\right)=x \wedge y, \rho_{1}\left(\sigma_{1}\left(x^{\prime}\right)\right)=x^{\prime}, \rho_{1}\left(\sigma_{1}(0)\right)=0$ and $\rho_{1}\left(\sigma_{1}(1)\right)=1$, i. e. $\rho_{1} \circ \sigma_{1}=$ id. The rest of the proof follows in an analogous way.

## References

[1] I. Chajda and K. Głazek, A Basic Course on General Algebra, Technical University Press, Zielona Góra 2000.
[2] K. Denecke, D. Lau, R. Pöschel, and D. Schweigert, Hyperidentities, hyperequational classes and clone congruences, Contribution to General Algebra 7 (1991), 97-118.
[3] E. Graczyńska and D. Schweigert, Hypervarieties of a given type, Algebra Universalis 27 (1990), 305-318.
[4] R. Padmanabhan and P. Penner, Bases of hyperidentities of lattices, C.R. Math. Rep. Acad. Sci. Canada 4 (1982), 9-14.
[5] J. Płonka, Proper and inner hypersubstitutions of varieties, p. 106-115 in: "General Algebra and Ordered Sets (Horni Lipová 1994)", Palacký University, Olomouc 1994.
[6] J. Płonka, On hyperidentities of some varieties, p. 199-213 in:"General Algebra and Discrete Mathematics (Potsdam 1993)", Heldermann-Verlag, Lemgo 1995.
[7] Z. Szylicka, Proper hypersubstitutions of normalizations and externalizations of varieties, p. 144-155 in:"General Algebra and Ordered Sets (Horni Lipová 1994)", Palacký University, Olomouc 1994.
[8] W. Taylor, Hyperidentities and hypervarieties, Aequationes Math. 23 (1981), 30-49.


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