Discussiones Mathematicae
General Algebra and Applications 21 (2001) 67–82

THE VECTOR CROSS PRODUCT FROM AN ALGEBRAIC POINT OF VIEW

Götz Trenkler*

Department of Statistics, University of Dortmund
Vogelpothsweg 87, D–44221 Dortmund, Germany
e-mail: trenkler@amadeus.statistik.uni-dortmund.de

Abstract

The usual vector cross product of the three-dimensional Euclidian space is considered from an algebraic point of view. It is shown that many proofs, known from analytical geometry, can be distinctly simplified by using the matrix oriented approach. Moreover, by using the concept of generalized matrix inverse, we are able to facilitate the analysis of equations involving vector cross products.

Keywords: vector cross product, generalized inverse, Moore-Penrose inverse, linear equations.

2000 Mathematics Subject Classification: Primary 15A72, Secondary 15A9.

1. Introduction

In most cases the cross product of two vectors of the three dimensional vector space $\boldsymbol{E} = \mathbb{R}^3$ is considered from the geometric point of view. A typical example is the booklet by Hague ([3], p. 18), where the cross product is introduced as follows: "The vector product of two non-zero vectors $\boldsymbol{A}$ and $\boldsymbol{B}$ in that order is defined as the vector $A\ B \mid sin\ \theta \mid \boldsymbol{n}$, where θ is an angle between $\boldsymbol{A}$ and $\boldsymbol{B}$ and $\boldsymbol{n}$ is that unit normal to the plane determined by $\boldsymbol{A}$ and $\boldsymbol{B}$ which is directed so that $\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{n}$ form a right-handed system, i.e. the rotation needed to move $\boldsymbol{A}$ to the position of $\boldsymbol{B}$ and the positive

*Support by Deutsche Forschungsgemeinschaft, Grant No. TR 253/2-3 is gratefully acknowledged.

direction of $\boldsymbol{n}$ are related in the same way as the rotation and translation of a right-handed screw."

On the other hand, there is a purely algebraic approach to the cross product. Since for a fixed vector $\boldsymbol{a} \in \boldsymbol{E}$ the cross product $\boldsymbol{a} \times \boldsymbol{x}$ is linear in the second component there exists a unique matrix $\boldsymbol{T}_{\mathrm{a}}$ such that

$$\boldsymbol{T}_{\mathrm{a}}\boldsymbol{x} = \boldsymbol{a} \times \boldsymbol{x} \tag{1.1}$$

for all $\boldsymbol{x} \in \boldsymbol{E}$. It is readily seen that for $\boldsymbol{a} = (a_1, a_2, a_3)'$ the matrix $\boldsymbol{T}_{\mathrm{a}}$ is of the form

$$\boldsymbol{T}_{\mathrm{a}} = \begin{pmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{pmatrix} \tag{1.2}$$

(see Rao and Mitra [6], p. 40, or Room [7]).

In Section 2, we shall derive the main properties of the cross product using exclusively (1.1). This operator-theoretic point of view quite naturally invites use of the Moore-Penrose inverse of $\boldsymbol{T}_{\mathrm{a}}$ and related matrices, determined together with eigenvalues and eigenspaces in Section 3. In Section 4 this will allow more elegantly solving some questions of Geometry and Mechanics.

Let $\boldsymbol{A}$ be an $m \times n$ real matrix. Then an $n \times m$ matrix $\boldsymbol{A}^-$ is said to be a *generalized inverse* (short: *g-inverse*) of $\boldsymbol{A}$ if $\boldsymbol{A}\boldsymbol{A}^-\boldsymbol{A} = \boldsymbol{A}$.

The *Moore-Penrose inverse* (short: *MP-inverse*) of $\boldsymbol{A}$ is the unique $n \times m$ matrix $\boldsymbol{A}^+$ satisfying simultaneously the conditions:

(a) $\boldsymbol{A}\boldsymbol{A}^+\boldsymbol{A} = \boldsymbol{A}$,

(b) $\boldsymbol{A}^+\boldsymbol{A}\boldsymbol{A}^+ = \boldsymbol{A}^+$,

(c) $(\boldsymbol{A}^+\boldsymbol{A})' = \boldsymbol{A}^+\boldsymbol{A}$,

(d) $(\boldsymbol{A}\boldsymbol{A}^+)' = \boldsymbol{A}\boldsymbol{A}^+$

(see Ben-Israel and Greville [1], ch. 1).

From (a) it is readily seen that the *MP*-inverse is also a *g*-inverse. Given a linear equation $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$, *g*-inverses of $\boldsymbol{A}$ are useful in deciding if the equation is consistent. A necessary and sufficient condition is

$$\boldsymbol{A}\boldsymbol{A}^-\boldsymbol{b} = \boldsymbol{b}. \tag{1.3}$$

When this happens all solutions are

$$\boldsymbol{x} = \boldsymbol{A}^{-}\boldsymbol{b} + (\boldsymbol{I} - \boldsymbol{A}^{-}\boldsymbol{A})\boldsymbol{z}, \tag{1.4}$$

where $\boldsymbol{z}$ is an arbitrary vector from $\boldsymbol{E}$ (see Rao and Mitra [6], ch. 2).

2. Basic properties

Let $\boldsymbol{e}_i$ be the i-th unit vector of $\boldsymbol{E}$, $i = 1, 2, 3$. For any $\boldsymbol{a} = (a_1, a_2, a_3)' \in \boldsymbol{E}$ the matrix $\boldsymbol{T}_{\mathrm{a}}$ defining the cross product can be written in the form

$$\begin{aligned} \boldsymbol{T}_{\mathrm{a}} &= a_3(\boldsymbol{e}_2\boldsymbol{e}_1' - \boldsymbol{e}_1\boldsymbol{e}_2') + a_2(\boldsymbol{e}_1\boldsymbol{e}_3' - \boldsymbol{e}_3\boldsymbol{e}_1') \\ &\quad + a_1(\boldsymbol{e}_3\boldsymbol{e}_2' - \boldsymbol{e}_2\boldsymbol{e}_3') = \sum_{i=1}^{3} a_i \boldsymbol{T}_{\mathrm{e}_i}. \end{aligned} \tag{2.1}$$

We now list some properties of $\boldsymbol{T}_{\mathrm{a}}$ which can be seen by straightforward calculations:

(i) For $\boldsymbol{a}, \boldsymbol{b} \in \boldsymbol{E}$ and $\alpha, \beta \in \mathbb{R}$ we have $\boldsymbol{T}_{\alpha\mathrm{a}+\beta\mathrm{b}} = \alpha\boldsymbol{T}_{\mathrm{a}} + \beta\boldsymbol{T}_{\mathrm{b}}$,

(ii) $\boldsymbol{T}_{\mathrm{a}}\boldsymbol{b} = -\boldsymbol{T}_{\mathrm{b}}\boldsymbol{a}$,

(iii) $\boldsymbol{T}_{\mathrm{a}} = -\boldsymbol{T}_{\mathrm{a}}'$,

(iv) $\boldsymbol{T}_{\mathrm{a}}\boldsymbol{a} = \boldsymbol{0}$.

These properties are well-known for the cross product. For instance, (iv) expresses the fact that always $\boldsymbol{a} \times \boldsymbol{a} = \boldsymbol{0}$. Condition (iv) also shows that $\boldsymbol{T}_{\mathrm{a}}$ is singular. For, if $\boldsymbol{a} \neq \boldsymbol{0}$, then by $\boldsymbol{T}_{\mathrm{a}}\boldsymbol{a} = \boldsymbol{0}$ the matrix $\boldsymbol{T}_{\mathrm{a}}$ cannot be nonsingular, and if $\boldsymbol{a} = \boldsymbol{0}$, we have $\boldsymbol{T}_{\mathrm{a}} = \boldsymbol{0}$.

Direct calculation also shows the next identity:

(v) $\boldsymbol{T}_{\mathrm{a}}\boldsymbol{T}_{\mathrm{b}} = \boldsymbol{b}\boldsymbol{a}' - \boldsymbol{a}'\boldsymbol{b}\boldsymbol{I}$, where $\boldsymbol{I}$ is the 3×3 identity matrix, and $\boldsymbol{a}, \boldsymbol{b}$ are arbitrary vectors from $\boldsymbol{E}$.

By (v), it is readily established that for vectors $\boldsymbol{a}, \boldsymbol{b}$ and $\boldsymbol{c}$ the following identity holds:

$$\boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c}) + \boldsymbol{b} \times (\boldsymbol{c} \times \boldsymbol{a}) + \boldsymbol{c} \times (\boldsymbol{a} \times \boldsymbol{b}) = \boldsymbol{0}, \text{ i.e.}$$

$$\boldsymbol{T}_{\mathrm{a}}\boldsymbol{T}_{\mathrm{b}}\boldsymbol{c} + \boldsymbol{T}_{\mathrm{b}}\boldsymbol{T}_{\mathrm{c}}\boldsymbol{a} + \boldsymbol{T}_{\mathrm{c}}\boldsymbol{T}_{\mathrm{a}}\boldsymbol{b} = \boldsymbol{0}.$$

From this we immediately get

(vi) $(\boldsymbol{T}_{\mathrm{a}}\boldsymbol{T}_{\mathrm{b}})' = \boldsymbol{T}_{\mathrm{b}}\boldsymbol{T}_{\mathrm{a}}$.

Condition (v) implies for $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c} \in \boldsymbol{E}$:

(vii) $\boldsymbol{T}_{\mathrm{a}}\boldsymbol{T}_{\mathrm{b}}\boldsymbol{T}_{\mathrm{c}} = \boldsymbol{T}_{\mathrm{a}}\boldsymbol{c}\boldsymbol{b}' - (\boldsymbol{b}'\boldsymbol{c})\boldsymbol{T}_{\mathrm{a}}$.

Choosing $\boldsymbol{c} = \boldsymbol{a}$ yields

(viii) $\boldsymbol{T}_{\mathrm{a}}\boldsymbol{T}_{\mathrm{b}}\boldsymbol{T}_{\mathrm{a}} = -(\boldsymbol{a}'\boldsymbol{b})\boldsymbol{T}_{\mathrm{a}}$,

and from (viii), by letting $\boldsymbol{b} = \boldsymbol{a}$, we get

(ix) $\boldsymbol{T}_{\mathrm{a}}\boldsymbol{T}_{\mathrm{a}}\boldsymbol{T}_{\mathrm{a}} = -(\boldsymbol{a}'\boldsymbol{a})\boldsymbol{T}_{\mathrm{a}}$.

Direct calculations show that the following chain of identities holds for vectors $\boldsymbol{a}, \boldsymbol{b}$ and $\boldsymbol{c}$:

(x) $\boldsymbol{c}'\boldsymbol{T}_{\mathrm{a}}\boldsymbol{b} = \boldsymbol{a}'\boldsymbol{T}_{\mathrm{b}}\boldsymbol{c} = \boldsymbol{b}'\boldsymbol{T}_{\mathrm{c}}\boldsymbol{a} = -\boldsymbol{c}'\boldsymbol{T}_{\mathrm{b}}\boldsymbol{a} = -\boldsymbol{b}'\boldsymbol{T}_{\mathrm{a}}\boldsymbol{c} = -\boldsymbol{a}'\boldsymbol{T}_{\mathrm{c}}\boldsymbol{b}$.

Observe that $\boldsymbol{c}'\boldsymbol{T}_{\mathrm{a}}\boldsymbol{b}$ is just the scalar triple product $(\boldsymbol{abc}) = (\boldsymbol{a} \times \boldsymbol{b})'\boldsymbol{c}$.

Our next result deals with the so-called continued vector product. Milne ([5], pp. 18-31), offers three different proofs. Let $\boldsymbol{a}, \boldsymbol{b}$ and $\boldsymbol{c}$ be vectors from $\boldsymbol{E}$. Then by (v) it follows that

(xi) $\boldsymbol{T}_{\mathrm{a}}\boldsymbol{T}_{\mathrm{b}}\boldsymbol{c} = (\boldsymbol{a}'\boldsymbol{c})\boldsymbol{b} - (\boldsymbol{a}'\boldsymbol{b})\boldsymbol{c}$ (*Grassmann's identity*).

Grassmann's identity can be used to demonstrate that the cross product is not associative in general. For, if $\boldsymbol{a}, \boldsymbol{b}$ and $\boldsymbol{c}$ are vectors such that $(\boldsymbol{a}\times\boldsymbol{b})\times\boldsymbol{c} = \boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c})$, as an equivalent condition we may state: $\boldsymbol{a}$ and $\boldsymbol{c}$ are linearly dependent or $\boldsymbol{b}'\boldsymbol{c} = 0 = \boldsymbol{a}'\boldsymbol{b}$. To see this, put $\boldsymbol{e} = \boldsymbol{a} \times \boldsymbol{b} = \boldsymbol{T}_{\mathrm{a}}\boldsymbol{b}$ and $\boldsymbol{f} = \boldsymbol{b}\times\boldsymbol{c} = \boldsymbol{T}_{\mathrm{b}}\boldsymbol{c}$. Then $\boldsymbol{T}_{\mathrm{e}}\boldsymbol{c} = \boldsymbol{T}_{\mathrm{a}}\boldsymbol{f}$ may be written as $-\boldsymbol{T}_{\mathrm{c}}\boldsymbol{T}_{\mathrm{a}}\boldsymbol{b} = \boldsymbol{T}_{\mathrm{a}}\boldsymbol{T}_{\mathrm{b}}\boldsymbol{c}$, or by (xi) as $(\boldsymbol{b}'\boldsymbol{c})\boldsymbol{a} - (\boldsymbol{a}'\boldsymbol{b})\boldsymbol{c} = \boldsymbol{0}$, which is equivalent to the asserted condition.

From (xi) we also get

(xii) $(\boldsymbol{T}_{\mathrm{a}}\boldsymbol{b})'(\boldsymbol{T}_{\mathrm{c}}\boldsymbol{d}) = (\boldsymbol{a}'\boldsymbol{c})(\boldsymbol{b}'\boldsymbol{d}) - (\boldsymbol{a}'\boldsymbol{d})(\boldsymbol{b}'\boldsymbol{c})$ (*Lagrange's identity*),

$\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$ and $\boldsymbol{d}$ are vectors from $\boldsymbol{E}$.

When setting $\boldsymbol{c} = \boldsymbol{a}$ and $\boldsymbol{d} = \boldsymbol{b}$, as a special case of (xii) we obtain

$$||\boldsymbol{T}_{\mathrm{a}}\boldsymbol{b}||^2 = (\boldsymbol{a}'\boldsymbol{a})(\boldsymbol{b}'\boldsymbol{b}) - (\boldsymbol{a}'\boldsymbol{b})^2, \tag{2.2}$$

where $||\cdot||$ is the Euclidean norm in $\boldsymbol{E}$.

Since $\boldsymbol{a}'\boldsymbol{b} = ||\boldsymbol{a}||||\boldsymbol{b}||cos\varphi$, where φ is the angle between the vectors $\boldsymbol{a}$ and $\boldsymbol{b}$, $0 \leq \varphi \leq \pi$, from (2.2) it easily follows that

(xiii) $||\boldsymbol{T}_{\mathrm{a}}\boldsymbol{b}|| = ||\boldsymbol{a}||||\boldsymbol{b}||sin\varphi$.

Another consequence of (xi) is

(xiv) $\boldsymbol{T}_{\mathrm{e}}\boldsymbol{f} = (\boldsymbol{abd})\boldsymbol{c} - (\boldsymbol{abc})\boldsymbol{d} = (\boldsymbol{acd})\boldsymbol{b} - (\boldsymbol{bcd})\boldsymbol{a}$,

where $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}, \boldsymbol{d}$ are vectors and $\boldsymbol{e} = \boldsymbol{T}_{\mathrm{a}}\boldsymbol{b}$, $\boldsymbol{f} = \boldsymbol{T}_{\mathrm{c}}\boldsymbol{d}$.

Recall that $(\boldsymbol{abc}) = \boldsymbol{c}'\boldsymbol{T}_{\mathrm{a}}\boldsymbol{b}$.

Condition (xiv) implies for arbitrary vectors $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$ and $\boldsymbol{d}$:

(xv) $(\boldsymbol{bcd})\boldsymbol{a} - (\boldsymbol{acd})\boldsymbol{b} + (\boldsymbol{abd})\boldsymbol{c} - (\boldsymbol{abc})\boldsymbol{d} = \boldsymbol{0}$.

The last identity shows how the $\boldsymbol{T}$-matrix acts on a cross product.

(xvi) $\boldsymbol{T}_{\mathrm{T_a b}} = \boldsymbol{ba}' - \boldsymbol{ab}'$.

To show this choose a vector $\boldsymbol{x}$ from $\boldsymbol{E}$. Then we have

$$\begin{aligned}
\boldsymbol{T}_{\mathrm{T_a b}}\boldsymbol{x} &= -\boldsymbol{T}_{\mathrm{x}}\boldsymbol{T}_{\mathrm{a}}\boldsymbol{b} \qquad \text{(by (ii))}\\
&= -(\boldsymbol{ax}' - \boldsymbol{a}'\boldsymbol{xI})\boldsymbol{b} \qquad \text{(by (v))}\\
&= (\boldsymbol{a}'\boldsymbol{x})\boldsymbol{b} - (\boldsymbol{b}'\boldsymbol{x})\boldsymbol{a}\\
&= (\boldsymbol{ba}' - \boldsymbol{ab}')\boldsymbol{x}.
\end{aligned}$$

Since $\boldsymbol{x}$ was chosen arbitrarily we get the asserted equality. Note that (xvi) can be rewritten as

$$\boldsymbol{T}_{\mathrm{T_a b}} = \boldsymbol{T}_{\mathrm{a}}\boldsymbol{T}_{\mathrm{b}} - \boldsymbol{T}_{\mathrm{b}}\boldsymbol{T}_{\mathrm{a}}.$$

3. The generalized inverse and the Moore-Penrose inverse

It will now be demonstrated how the concept of the g-inverse and the MP-inverse can be used to facilitate some calculations in connection with the cross product.

Theorem 1. *Let $\boldsymbol{a}$ and $\boldsymbol{b}$ be vectors from $\boldsymbol{E}$.*

1) *If $\boldsymbol{a}'\boldsymbol{b} \neq 0$, then $-\frac{1}{\mathrm{a'b}}\boldsymbol{T}_{\mathrm{b}}$ is a g-inverse of $\boldsymbol{T}_{\mathrm{a}}$.*
2) *If $\boldsymbol{a} \neq \boldsymbol{0}$, then $-\frac{1}{\mathrm{a'a}}\boldsymbol{T}_{\mathrm{a}}$ is the MP-inverse of $\boldsymbol{T}_{\mathrm{a}}$.*
3) *If $\boldsymbol{a}'\boldsymbol{b} = 0$, then $\boldsymbol{T}_{\mathrm{a}}^{+} + \alpha\boldsymbol{T}_{\mathrm{b}}$ is a g-inverse of $\boldsymbol{T}_{\mathrm{a}}$ for any scalar α.*

Proof.

1) Follows directly from (viii).

2) By 1) only properties $(b), (c)$ and (d) of the *MP*-inverse remain to be shown. However, (c) and (d) are obvious since $\boldsymbol{T}_{\mathrm{a}}\boldsymbol{T}_{\mathrm{a}} = \boldsymbol{a}\boldsymbol{a}' - \boldsymbol{a}'\boldsymbol{a}\boldsymbol{I}$ is symmetric. To see (b) observe that

$$\left(-\frac{1}{\boldsymbol{a}'\boldsymbol{a}}\boldsymbol{T}_{\mathrm{a}}\right)\boldsymbol{T}_{\mathrm{a}}\left(-\frac{1}{\boldsymbol{a}'\boldsymbol{a}}\boldsymbol{T}_{\mathrm{a}}\right) = \left(\frac{1}{\boldsymbol{a}'\boldsymbol{a}}\right)^2\boldsymbol{T}_{\mathrm{a}}^3 = -\frac{1}{\boldsymbol{a}'\boldsymbol{a}}\boldsymbol{T}_{\mathrm{a}}, \tag{3.1}$$

the latter identity resulting from property (ix).

3) Follows from property (viii). ■

Note that in Rao and Mitra ([6], p. 40) the matrix $-\frac{1}{\mathrm{a'a}}\boldsymbol{T}_{\mathrm{a}}$ was not identified as the *MP*-inverse of $\boldsymbol{T}_{\mathrm{a}}$, but only as a g-inverse.

Subsequently we use the following facts (cf. Ben-Israel and Greville [1], ch. 2):

(a) If $\boldsymbol{A}$ is a matrix, then $\boldsymbol{A}\boldsymbol{A}^{+}$ is the orthogonal projector on $\mathcal{R}(\boldsymbol{A})$, the column space of $\boldsymbol{A}$.

(b) $\boldsymbol{I} - \boldsymbol{A}\boldsymbol{A}^{+}$ is the orthogonal projector on $\mathcal{R}(\boldsymbol{A})^{\perp}$, the orthogonal complement of $\mathcal{R}(\boldsymbol{A})$.

(c) For any matrix $\boldsymbol{A}$ we have $rk(\boldsymbol{A}) = rk(\boldsymbol{A}\boldsymbol{A}^{+}) = tr(\boldsymbol{A}\boldsymbol{A}^{+})$, where $rk(\cdot)$ and $tr(\cdot)$ denote the rank and the trace of a matrix, respectively.

For instance, we may state for $\boldsymbol{a} \in \boldsymbol{E}$:

$$\boldsymbol{T}_{\mathrm{a}}\boldsymbol{T}_{\mathrm{a}}^{+} = \boldsymbol{I} - \boldsymbol{a}\boldsymbol{a}^{+} \tag{3.2}$$

is the orthogonal projector on $\mathcal{R}(\boldsymbol{a})^{\perp}$, and

$$\boldsymbol{I} - \boldsymbol{T}_{\mathrm{a}}\boldsymbol{T}_{\mathrm{a}}^{+} = \boldsymbol{a}\boldsymbol{a}^{+} \tag{3.3}$$

is the orthogonal projector on $\mathcal{R}(\boldsymbol{a})$. We write

$$\begin{aligned}(3.4)\qquad \boldsymbol{T}_{\mathrm{a}}\boldsymbol{T}_{\mathrm{a}}^{+} &= \boldsymbol{P}_{\mathcal{R}(\mathrm{a})^{\perp}},\\ (3.5)\qquad \boldsymbol{I}-\boldsymbol{T}_{\mathrm{a}}\boldsymbol{T}_{\mathrm{a}}^{+} &= \boldsymbol{P}_{\mathcal{R}(\mathrm{a})}.\end{aligned}$$

From (3.4), it follows that $\boldsymbol{T}_{\mathrm{a}}\boldsymbol{T}_{\mathrm{a}}^{+}\boldsymbol{b}=\boldsymbol{b}$ if and only if $\boldsymbol{a}'\boldsymbol{b}=0$.

If $\mathcal{N}(\cdot)$ denotes the null space of a matrix, we conclude $\mathcal{N}(\boldsymbol{T}_{\mathrm{a}}) = \mathcal{N}(\boldsymbol{T}_{\mathrm{a}}^{+}\boldsymbol{T}_{\mathrm{a}}) = \mathcal{N}(\boldsymbol{T}_{\mathrm{a}}\boldsymbol{T}_{\mathrm{a}}^{+}) = \mathcal{N}(\boldsymbol{I}-\boldsymbol{a}\boldsymbol{a}^{+}) = \mathcal{R}(\boldsymbol{a}\boldsymbol{a}^{+}) = \mathcal{R}(\boldsymbol{a})$. In addition $\mathcal{R}(\boldsymbol{T}_{\mathrm{a}}) = \mathcal{R}(\boldsymbol{T}_{\mathrm{a}}\boldsymbol{T}_{\mathrm{a}}^{+}) = \mathcal{R}(\boldsymbol{I}-\boldsymbol{a}\boldsymbol{a}^{+}) = \mathcal{N}(\boldsymbol{a}\boldsymbol{a}^{+}) = \mathcal{N}(\boldsymbol{a}') = \mathcal{R}(\boldsymbol{a})^{\perp}$. Thus, we may summarize

$$\begin{aligned}(3.6)\qquad \mathcal{N}(\boldsymbol{T}_{\mathrm{a}}) &= \mathcal{R}(\boldsymbol{a}),\\ (3.7)\qquad \mathcal{R}(\boldsymbol{T}_{\mathrm{a}}) &= \mathcal{R}(\boldsymbol{a})^{\perp}.\end{aligned}$$

We should also mention that $\boldsymbol{T}_{\mathrm{a}}$ and $\boldsymbol{T}_{\mathrm{a}}^{+}$ commute, and $\boldsymbol{T}_{\mathrm{a}}$ is normal, i.e. $\boldsymbol{T}_{\mathrm{a}}\boldsymbol{T}_{\mathrm{a}}' = \boldsymbol{T}_{\mathrm{a}}'\boldsymbol{T}_{\mathrm{a}}$. Furthermore, we derive $rk(\boldsymbol{T}_{\mathrm{a}})=2$, if $\boldsymbol{a}\neq\boldsymbol{0}$, for $rk(\boldsymbol{T}_{\mathrm{a}}) = rk(\boldsymbol{T}_{\mathrm{a}}\boldsymbol{T}_{\mathrm{a}}^{+}) = tr(\boldsymbol{T}_{\mathrm{a}}\boldsymbol{T}_{\mathrm{a}}^{+}) = -\frac{1}{\mathrm{a}'\mathrm{a}}tr(\boldsymbol{T}_{\mathrm{a}}\boldsymbol{T}_{\mathrm{a}}) = -\frac{1}{\mathrm{a}'\mathrm{a}}[tr(\boldsymbol{a}\boldsymbol{a}') - \boldsymbol{a}'\boldsymbol{a}tr(\boldsymbol{I})) = -\frac{1}{\mathrm{a}'\mathrm{a}}(-2\boldsymbol{a}'\boldsymbol{a}) = 2$.

Theorem 2. *The eigenvalues of* $\boldsymbol{T}_a$ *are* $0, i||\boldsymbol{a}||$ *and* $-i||\boldsymbol{a}||$, *where* $i=\sqrt{-1}$ *and* $||\boldsymbol{a}||=\sqrt{\boldsymbol{a}'\boldsymbol{a}}$.

Proof. From Theorem 1 and 3.4, we know that $\boldsymbol{T}_{\mathrm{a}}\boldsymbol{T}_{\mathrm{a}} = -\boldsymbol{a}'\boldsymbol{a}\boldsymbol{P}_{\mathcal{R}(\mathrm{a})^{\perp}}$. Since $\boldsymbol{P}_{\mathcal{R}(\mathrm{a})^{\perp}}$ is an orthogonal projector, its eigenvalues are 0 or 1. Hence the eigenvalues of $\boldsymbol{T}_{\mathrm{a}}$ are $0, i||\boldsymbol{a}||$ and $-i||\boldsymbol{a}||$. ∎

Alternatively, this result could have been shown by observing that the characteristic polynomial $P(\lambda)$ of $\boldsymbol{T}_{\mathrm{a}}$ is given by

$$P(\lambda) = det(\boldsymbol{T}_{\mathrm{a}}-\lambda\boldsymbol{I}) = -\lambda^3 - \lambda\boldsymbol{a}'\boldsymbol{a}.$$

From the preceding result, it follows that for all real $\lambda\neq 0$ the matrix $\boldsymbol{T}_{\mathrm{a}}-\lambda\boldsymbol{I}$ is nonsingular. Actually, by straightforward calculation we get

$$(3.8)\qquad (\boldsymbol{T}_{\mathrm{a}}-\lambda\boldsymbol{I})^{-1} = -\frac{1}{\boldsymbol{a}'\boldsymbol{a}+\lambda^2}(\boldsymbol{T}_{\mathrm{a}}+\lambda\boldsymbol{I}+\frac{1}{\lambda}\boldsymbol{a}\boldsymbol{a}').$$

If $\boldsymbol{E}(\boldsymbol{T}_{\mathrm{a}}, \lambda)$ denotes the eigenspace of $\boldsymbol{T}_{\mathrm{a}}$ with respect to the eigenvalue λ, by (3.6) it is clear that

$$\boldsymbol{E}(\boldsymbol{T}_{\mathrm{a}}, 0) = \mathcal{R}(\boldsymbol{a}). \tag{3.9}$$

To calculate the eigenspaces $\boldsymbol{E}(\boldsymbol{T}_{\mathrm{a}}, i||\boldsymbol{a}||)$ and $\boldsymbol{E}(\boldsymbol{T}_{\mathrm{a}}, -i||\boldsymbol{a}||)$ the following results are useful.

Since we now deal with complex matrices the notion of *MP*-inverse has to be modified slightly. If $\boldsymbol{A}$ is an $m \times n$ matrix, $\boldsymbol{A}^+$ is the *MP*-inverse of $\boldsymbol{A}$ if

(a) $\boldsymbol{A}\boldsymbol{A}^+\boldsymbol{A} = \boldsymbol{A}$,

(b) $\boldsymbol{A}^+\boldsymbol{A}\boldsymbol{A}^+ = \boldsymbol{A}^+$,

(c) $(\boldsymbol{A}^+\boldsymbol{A})^* = \boldsymbol{A}^+\boldsymbol{A}$,

(d) $(\boldsymbol{A}\boldsymbol{A}^+)^* = \boldsymbol{A}\boldsymbol{A}^+$,

where $(\cdot)^*$ denotes the conjugate transpose of a matrix.

Theorem 3. *Let $\boldsymbol{a}$ be a vector from $\boldsymbol{E}$ such that $||\boldsymbol{a}|| = 1$. Then*

(i) $(\boldsymbol{T}_{\mathrm{a}} - i\boldsymbol{I})^+ = \frac{1}{4}(-\boldsymbol{T}_{\mathrm{a}} + 3i\boldsymbol{a}\boldsymbol{a}' + i\boldsymbol{I})$,

(ii) $(\boldsymbol{T}_{\mathrm{a}} + i\boldsymbol{I})^+ = -\frac{1}{4}(\boldsymbol{T}_{\mathrm{a}} + 3i\boldsymbol{a}\boldsymbol{a}' + i\boldsymbol{I})$.

Proof.

(i) We assume that $(\boldsymbol{T}_{\mathrm{a}} - i\boldsymbol{I})^+ = \alpha\boldsymbol{T}_{\mathrm{a}} + \beta\boldsymbol{a}\boldsymbol{a}' + \gamma\boldsymbol{I}$ for some complex constants α, β and γ. Then $(\boldsymbol{T}_{\mathrm{a}} - i\boldsymbol{I})(\alpha\boldsymbol{T}_{\mathrm{a}} + \beta\boldsymbol{a}\boldsymbol{a}' + \gamma\boldsymbol{I}) = \varphi\boldsymbol{T}_{\mathrm{a}} + \psi\boldsymbol{a}\boldsymbol{a}' + \varepsilon\boldsymbol{I}$, where $\varphi = \gamma - i\alpha$, $\psi = \alpha - i\beta$ and $\varepsilon = -\alpha - i\gamma$. Property (d) of the *MP*-inverse requires

$$\varphi = -\overline{\varphi},\ \psi = \overline{\psi} \text{ and } \varepsilon = \overline{\varepsilon}, \tag{3.10}$$

where the barred numbers denote the complex conjugate. It follows that

$$\begin{aligned}(\varphi\boldsymbol{T}_{\mathrm{a}} + \psi\boldsymbol{a}\boldsymbol{a}' + \varepsilon\boldsymbol{I})&(\boldsymbol{T}_{\mathrm{a}} - i\boldsymbol{I}) \\ &= (\varepsilon - i\varphi)\boldsymbol{T}_{\mathrm{a}} + (\varphi - i\psi)\boldsymbol{a}\boldsymbol{a}' + (-\varphi - i\varepsilon)\boldsymbol{I}.\end{aligned} \tag{3.11}$$

From property (a) of the *MP*-inverse, we get

$$\varepsilon - i\varphi = 1,\ \varphi - i\psi = 0 \text{ and } -\varphi - i\varepsilon = -i,$$

which can be equivalently expressed as

$$\varphi = i\psi \text{ and } \varepsilon = 1 - \psi. \tag{3.12}$$

Furthermore, it follows that

$$\begin{aligned}&(\alpha \boldsymbol{T}_{\mathrm{a}} + \beta \boldsymbol{a}\boldsymbol{a}' + \gamma \boldsymbol{I})(\varphi \boldsymbol{T}_{\boldsymbol{a}} + \psi \boldsymbol{a}\boldsymbol{a}' + \varepsilon \boldsymbol{I})\\ &\quad = (\varphi\gamma + \varepsilon\alpha)\boldsymbol{T}_{\mathrm{a}} + (\alpha\varphi + \psi\beta + \psi\gamma + \varepsilon\beta)\boldsymbol{a}\boldsymbol{a}' + (\varepsilon\gamma - \alpha\varphi)\boldsymbol{I}.\end{aligned}$$

Property (b) of the *MP*-inverse along with (3.12) leads to the condition

$$\psi(\gamma + \alpha i) = 0. \tag{3.13}$$

Suppose $\psi = 0$. Then by (3.12) we get $\varphi = 0$ and $\varepsilon = 1$ which implies that $\boldsymbol{T}_{\mathrm{a}} - i\boldsymbol{I}$ is nonsingular. This is a contradiction, since i is an eigenvalue of $\boldsymbol{T}_{\mathrm{a}}$. Hence $\gamma + \alpha i = 0$. Moreover, since $\psi + \epsilon = 1$, we have $-i(\beta + \gamma) = 1$. Finally, from $\psi = \alpha - i\beta, \varphi = \gamma - i\alpha$ and $\varphi = i\psi$, we derive $\gamma - \beta = 2i\alpha$. From this it follows that $\alpha = -\frac{1}{4}$, $\beta = \frac{3}{4}i$ and $\gamma = \frac{i}{4}$.

(ii) $(\boldsymbol{T}_{\mathrm{a}} + i\boldsymbol{I})^{+} = -(-\boldsymbol{T}_{\mathrm{a}} - i\boldsymbol{I})^{+} = -(\boldsymbol{T}_{-\mathrm{a}} - i\boldsymbol{I})^{+}$. Since $||-\boldsymbol{a}|| = ||\boldsymbol{a}|| = 1$, from (i) we obtain

$$\begin{aligned}(\boldsymbol{T}_{\mathrm{a}} + i\boldsymbol{I})^{+} &= -\frac{1}{4}(-\boldsymbol{T}_{-\mathrm{a}} + 3i\boldsymbol{a}\boldsymbol{a}' + i\boldsymbol{I})\\ &= -\frac{1}{4}(\boldsymbol{T}_{\mathrm{a}} + 3i\boldsymbol{a}a' + i\boldsymbol{I}).\end{aligned}$$

■

By using the fact that $\boldsymbol{T}_{\mathrm{a}} = ||\boldsymbol{a}||\boldsymbol{T}_{\frac{\mathrm{a}}{||\mathrm{a}||}}$, we immediately get the following result.

Corollary 1. *Let $\boldsymbol{a} \neq \boldsymbol{0}$ be a vector from $\boldsymbol{E}$. Then*

(i) $(\boldsymbol{T}_{\mathrm{a}} - i||\boldsymbol{a}||\boldsymbol{I})^{+} = \frac{1}{4}\frac{1}{||\mathrm{a}||}\left(-\frac{1}{||\mathrm{a}||}\boldsymbol{T}_{\mathrm{a}} + 3i\boldsymbol{a}\boldsymbol{a}^{+} + i\boldsymbol{I}\right)$,

(ii) $(\boldsymbol{T}_{\mathrm{a}} + i||\boldsymbol{a}||\boldsymbol{I})^{+} = -\frac{1}{4}\frac{1}{||\mathrm{a}||}\left(\frac{1}{||\mathrm{a}||}\boldsymbol{T}_{\mathrm{a}} + 3i\boldsymbol{a}\boldsymbol{a}^{+} + i\boldsymbol{I}\right)$. ■

Some straightforward calculations show

$$(3.14)\qquad (\boldsymbol{T}_{\mathrm{a}} - i||\boldsymbol{a}||\boldsymbol{I})^{+}(\boldsymbol{T}_{\mathrm{a}} - i||\boldsymbol{a}||\boldsymbol{I}) = \frac{1}{2}\left(\boldsymbol{a}\boldsymbol{a}^{+} + \boldsymbol{I} + \frac{i}{||\boldsymbol{a}||}\boldsymbol{T}_{\mathrm{a}}\right)$$

and

$$(3.15)\qquad (\boldsymbol{T}_{\mathrm{a}} + i||\boldsymbol{a}||\boldsymbol{I})^{+}(\boldsymbol{T}_{\mathrm{a}} + i||\boldsymbol{a}||\boldsymbol{I}) = \frac{1}{2}\left(\boldsymbol{a}\boldsymbol{a}^{+} + \boldsymbol{I} - \frac{i}{||\boldsymbol{a}||}\boldsymbol{T}_{\mathrm{a}}\right).$$

Consider now the eigenspaces of $\boldsymbol{T}_{\mathrm{a}}$ with respect to $i||\boldsymbol{a}||$ und $-i||\boldsymbol{a}||$. Since

$$(3.16)\qquad \boldsymbol{E}(\boldsymbol{T}_{\mathrm{a}}, i||\boldsymbol{a}||) = \{\boldsymbol{x}|(\boldsymbol{T}_{\mathrm{a}} - i||\boldsymbol{a}||)\boldsymbol{x} = \boldsymbol{0}\},$$

$$(3.17)\qquad \boldsymbol{E}(\boldsymbol{T}_{\mathrm{a}}, -i||\boldsymbol{a}||) = \{\boldsymbol{x}|(\boldsymbol{T}_{\mathrm{a}} + i||\boldsymbol{a}||)\boldsymbol{x} = \boldsymbol{0}\},$$

from (3.14), (3.15) and (1.4), we get in explicit form

$$(3.18)\qquad \boldsymbol{E}(\boldsymbol{T}_{\mathrm{a}}, i||\boldsymbol{a}||) = \mathcal{R}\left(\boldsymbol{I} - \boldsymbol{a}\boldsymbol{a}^{+} - \frac{i}{||\boldsymbol{a}||}\boldsymbol{T}_{\mathrm{a}}\right),$$

$$(3.19)\qquad \boldsymbol{E}(\boldsymbol{T}_{\mathrm{a}}, -i||\boldsymbol{a}||) = \mathcal{R}\left(\boldsymbol{I} - \boldsymbol{a}\boldsymbol{a}^{+} + \frac{i}{||\boldsymbol{a}||}\boldsymbol{T}_{\mathrm{a}}\right).$$

Note that the conditions $\varphi = \gamma - i\alpha$, $\psi = \alpha - i\beta$, $\varepsilon - \alpha - i\gamma$, $\varphi = i\psi$ and $\varepsilon = 1 - \psi$ which occur in the proof of Theorem 3 may be used to construct an infinite class of g-inverses of $\boldsymbol{T}_{\mathrm{a}} - i\boldsymbol{I}$. It is easy to see that these conditions reduce to

$$\begin{aligned} (3.20)\qquad 2i\alpha + \beta - \gamma &= 0, \\ (3.21)\qquad \beta + \gamma &= i \end{aligned}$$

with general solution vector

$$(3.22)\qquad \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} = \mu \begin{pmatrix} i \\ 1 \\ -1 \end{pmatrix} + \begin{pmatrix} -\frac{1}{2} \\ i \\ 0 \end{pmatrix},$$

where μ is arbitrary. For instance, when choosing $\mu = 0$, we see that $-\frac{1}{2}\boldsymbol{T}_{\mathrm{a}} + i\boldsymbol{a}\boldsymbol{a}'$ is a g-inverse of $\boldsymbol{T}_{\mathrm{a}} - i\boldsymbol{I}$, provided $||\boldsymbol{a}|| = 1$. Exploiting the identity $\boldsymbol{T}_{\mathrm{a}} = ||\boldsymbol{a}||\boldsymbol{T}_{\frac{\mathrm{a}}{||\mathrm{a}||}}$, we can also find g-inverses of $\boldsymbol{T}_{\mathrm{a}} - i||\boldsymbol{a}||\boldsymbol{I}$ in the more general case of $\boldsymbol{a} \neq 0$. A similar discussion is now straightforward for the matrix $\boldsymbol{T}_{\mathrm{a}} + i||\boldsymbol{a}||\boldsymbol{I}$.

Theorem 4. *The eigenvalues of* $\boldsymbol{T}_{\mathrm{a}}\boldsymbol{T}_{\mathrm{b}}$ *are* $\lambda = 0$ *or* $\lambda = -\boldsymbol{a}'\boldsymbol{b}$ *for arbitrary vectors* $\boldsymbol{a}, \boldsymbol{b} \in \boldsymbol{E}$.

Proof.

$$\begin{aligned} det(\boldsymbol{T}_{\mathrm{a}}\boldsymbol{T}_{\mathrm{b}} - \lambda\boldsymbol{I}) &= det(\boldsymbol{b}\boldsymbol{a}' - \boldsymbol{a}'\boldsymbol{b}\boldsymbol{I} - \lambda\boldsymbol{I}) \\ &= (-1)^3 det\left((\boldsymbol{a}'\boldsymbol{b} + \lambda)\boldsymbol{I} - \boldsymbol{b}\boldsymbol{a}'\right) \\ &= -det(\gamma\boldsymbol{I} - \boldsymbol{b}\boldsymbol{a}'), \end{aligned}$$

where $\gamma = \boldsymbol{a}'\boldsymbol{b} + \lambda$.

Case 1. $\gamma = 0$. Then $det(\boldsymbol{T}_{\mathrm{a}}\boldsymbol{T}_{\mathrm{b}} - \lambda\boldsymbol{I}) = -det(\boldsymbol{b}\boldsymbol{a}') = 0$, such that $\lambda = -\boldsymbol{a}'\boldsymbol{b}$ is an eigenvalue.

Case 2. $\gamma \neq 0$. Then by a well-known formula we have $-det(\gamma\boldsymbol{I} - \boldsymbol{b}\boldsymbol{a}') = -\left[det(\gamma\boldsymbol{I})\left(1 - \frac{\mathrm{a}'\mathrm{b}}{\gamma}\right)\right]$ implying that $det(\boldsymbol{T}_{\mathrm{a}}\boldsymbol{T}_{\mathrm{b}} - \lambda\boldsymbol{I}) = 0$ if and only if $\gamma = \boldsymbol{a}'\boldsymbol{b}$, i.e. $\lambda = 0$ is an eigenvalue. ∎

If $\lambda \neq 0$ and $\lambda \neq -\boldsymbol{a}'\boldsymbol{b}$, then by direct calculations it is seen that

$$(\boldsymbol{T}_{\mathrm{a}}\boldsymbol{T}_{\mathrm{b}} - \lambda\boldsymbol{I})^{-1} = -\frac{1}{\lambda(\lambda + \boldsymbol{a}'\boldsymbol{b})}(\lambda\boldsymbol{I} + \boldsymbol{b}\boldsymbol{a}'). \tag{3.23}$$

To calculate the corresponding eigenspaces, we need some g-inverses or *MP*-inverses of $\boldsymbol{T}_{\mathrm{a}}\boldsymbol{T}_{\mathrm{b}}$. If $\boldsymbol{a}'\boldsymbol{b} \neq 0$ it is readily seen that a g-inverse of $\boldsymbol{T}_{\mathrm{a}}\boldsymbol{T}_{\mathrm{b}}$ is

$$(\boldsymbol{T}_{\mathrm{a}}\boldsymbol{T}_{\mathrm{b}})^{-} = -\frac{1}{\boldsymbol{a}'\boldsymbol{b}}\boldsymbol{I}. \tag{3.24}$$

The *MP*-inverse of $\boldsymbol{T}_{\mathrm{a}}\boldsymbol{T}_{\mathrm{b}}$ looks more involved.

Theorem 5. *Let* $\boldsymbol{a}, \boldsymbol{b}$ *be vector from* $\boldsymbol{E}$ *such that* $\boldsymbol{a}'\boldsymbol{b} \neq 0$. *Then*

$$(\boldsymbol{T}_{\mathrm{a}}\boldsymbol{T}_{\mathrm{b}})^{+} = -\frac{1}{\boldsymbol{a}'\boldsymbol{b}}\left[\boldsymbol{I} - \boldsymbol{b}\boldsymbol{b}^{+} - \boldsymbol{a}\boldsymbol{a}^{+} + \boldsymbol{a}'\boldsymbol{b}\frac{\boldsymbol{b}\boldsymbol{a}'}{\boldsymbol{a}'\boldsymbol{a}\boldsymbol{b}'\boldsymbol{b}}\right].$$

Proof. As a matter of straightforward calculation one finds that

$$\boldsymbol{T}_{\mathrm{a}}\boldsymbol{T}_{\mathrm{b}}\left[\boldsymbol{I} - \boldsymbol{b}\boldsymbol{b}^{+} - \boldsymbol{a}\boldsymbol{a}^{+} + \boldsymbol{a}'\boldsymbol{b}\frac{\boldsymbol{b}\boldsymbol{a}'}{\boldsymbol{a}'\boldsymbol{a}\boldsymbol{b}'\boldsymbol{b}}\right] = -\boldsymbol{a}'\boldsymbol{b}(\boldsymbol{I} - \boldsymbol{a}\boldsymbol{a}^{+})$$

and

$$\left[\boldsymbol{I} - \boldsymbol{b}\boldsymbol{b}^{+} - \boldsymbol{a}\boldsymbol{a}^{+} + \boldsymbol{a}'\boldsymbol{b}\frac{\boldsymbol{b}\boldsymbol{a}'}{\boldsymbol{a}'\boldsymbol{a}\boldsymbol{b}'\boldsymbol{b}} \right] \boldsymbol{T}_{\mathrm{a}}\boldsymbol{T}_{\mathrm{b}} = -\boldsymbol{a}'\boldsymbol{b}(\boldsymbol{I} - \boldsymbol{b}\boldsymbol{b}^{+}). \tag{3.25}$$

From these two equations all four *MP*-properties follow at once. ∎

If $\boldsymbol{a}'\boldsymbol{b} = 0$, then $\boldsymbol{T}_{\mathrm{a}}\boldsymbol{T}_{\mathrm{b}} = \boldsymbol{b}\boldsymbol{a}'$ is of simple structure, and we get

$$(\boldsymbol{T}_{\mathrm{a}}\boldsymbol{T}_{\mathrm{b}})^{+} = \boldsymbol{a}\boldsymbol{b}'/(\boldsymbol{a}'\boldsymbol{a})(\boldsymbol{b}'\boldsymbol{b}) = (\boldsymbol{b}\boldsymbol{a}')^{+} \tag{3.26}$$

(cf. Ben-Israel and Greville [1], p. 24). Now we have the tools to determine the eigenspaces corresponding to the eigenvalues of $\boldsymbol{T}_{\mathrm{a}}\boldsymbol{T}_{\mathrm{b}}$.

Theorem 6. *Let $\boldsymbol{a}$ and $\boldsymbol{b}$ be vectors from $\boldsymbol{E}$, $\boldsymbol{a} \neq \boldsymbol{0}, \boldsymbol{b} \neq \boldsymbol{0}$. Then*

(i) $\boldsymbol{E}(\boldsymbol{T}_{\mathrm{a}}\boldsymbol{T}_{\mathrm{b}}, -\boldsymbol{a}'\boldsymbol{b}) = \mathcal{N}(\boldsymbol{a}')$.

(ii) *If $\boldsymbol{a}'\boldsymbol{b} \neq 0$, then $\boldsymbol{E}(\boldsymbol{T}_{\mathrm{a}}\boldsymbol{T}_{\mathrm{b}}, 0) = \mathcal{R}(\boldsymbol{b})$.*

Proof.

(i) $\boldsymbol{T}_{\mathrm{a}}\boldsymbol{T}_{\mathrm{b}} - \lambda\boldsymbol{I} = \boldsymbol{T}_{\mathrm{a}}\boldsymbol{T}_{\mathrm{b}} + \boldsymbol{a}'\boldsymbol{b}\boldsymbol{I} = \boldsymbol{b}\boldsymbol{a}'$ if $\lambda = -\boldsymbol{a}'\boldsymbol{b}$. Then we get $\mathcal{N}(\boldsymbol{T}_{\mathrm{a}}\boldsymbol{T}_{\mathrm{b}} - \lambda\boldsymbol{I}) = \mathcal{N}(\boldsymbol{b}\boldsymbol{a}') = \mathcal{N}(\boldsymbol{a}')$.

(ii) Consider $\boldsymbol{T}_{\mathrm{a}}\boldsymbol{T}_{\mathrm{b}}\boldsymbol{x} = \boldsymbol{0}$. Then, according to (1.4) we have $\boldsymbol{x} = (\boldsymbol{I} - (\boldsymbol{T}_{\mathrm{a}}\boldsymbol{T}_{\mathrm{b}})^{-}\boldsymbol{T}_{\mathrm{a}}\boldsymbol{T}_{\mathrm{b}})\boldsymbol{z}$, where $\boldsymbol{z} \in \boldsymbol{E}$. Using the g-inverse of $\boldsymbol{T}_{\mathrm{a}}\boldsymbol{T}_{\mathrm{b}}$ from (3.23) we obtain $\boldsymbol{x} = \left(\boldsymbol{I} + \frac{1}{\mathrm{a}'\mathrm{b}}\boldsymbol{T}_{\mathrm{a}}\boldsymbol{T}_{\mathrm{b}}\right)\boldsymbol{z} = \left(\boldsymbol{I} + \frac{1}{\mathrm{a}'\mathrm{b}}(\boldsymbol{b}\boldsymbol{a}' - \boldsymbol{a}'\boldsymbol{b})\boldsymbol{I}\right)\boldsymbol{z} = \frac{\mathrm{ba}'}{\mathrm{a}'\mathrm{b}}\boldsymbol{z}$. This implies $\boldsymbol{E}(\boldsymbol{T}_{\mathrm{a}}\boldsymbol{T}_{\mathrm{b}}, 0) = \mathcal{R}(\boldsymbol{b}\boldsymbol{a}') = \mathcal{R}(\boldsymbol{b})$, where the latter identity follows from $\mathcal{R}(\boldsymbol{b}\boldsymbol{a}') \subset \mathcal{R}(\boldsymbol{b})$ and $rk(\boldsymbol{b}\boldsymbol{a}') = 1 = rk(\boldsymbol{b})$. ∎

Observe that in case (i) of the preceding theorem we could have replaced $\mathcal{N}(\boldsymbol{a}')$ by $\mathcal{R}(\boldsymbol{a})^{\perp}$.

4. Applications

Subsequently, it will be demonstrated how the results achieved above can be used to facilitate some of the traditional proofs of vector algebra.

Theorem 7. *Let $\boldsymbol{a}$ and $\boldsymbol{b}$ be vectors from $\boldsymbol{E}$, different from the zero vector. Then the following two statements are equivalent:*

(i) *$\boldsymbol{a}$ and $\boldsymbol{b}$ are linearly dependent.*

(ii) $\boldsymbol{T}_{\mathrm{a}}\boldsymbol{b} = \boldsymbol{0}$.

Proof. Suppose that (i) holds. Without loss of generality assume $\boldsymbol{a} = \lambda \boldsymbol{b}$ for some λ. Then $\boldsymbol{T}_{\mathrm{a}}\boldsymbol{b} = -\boldsymbol{T}_{\mathrm{b}}\boldsymbol{a} = -\boldsymbol{T}_{\mathrm{b}}(\lambda \boldsymbol{b}) = -\lambda \boldsymbol{T}_{\mathrm{b}}\boldsymbol{b} = \boldsymbol{0}$.

Vice versa let $\boldsymbol{T}_{\mathrm{a}}\boldsymbol{b} = \boldsymbol{0}$. By (1.4) and (3.6) we get $\boldsymbol{b} = \boldsymbol{a}\boldsymbol{a}^{+}\boldsymbol{z} = \boldsymbol{a}\boldsymbol{a}'\boldsymbol{z}/\boldsymbol{a}'\boldsymbol{a}$. for some vector $\boldsymbol{z}$ so that $\boldsymbol{b} = \lambda \boldsymbol{a}$ for some λ. ■

Clearly, condition $\boldsymbol{T}_{\mathrm{b}}\boldsymbol{a} = \boldsymbol{0}$ is also equivalent to statements (i) and (ii) of the preceding theorem.

The next equivalence deals with the scalar triple product $(\boldsymbol{abc}) = (\boldsymbol{a} \times \boldsymbol{b})'\boldsymbol{c} = \boldsymbol{c}'\boldsymbol{T}_{\mathrm{a}}\boldsymbol{b}$.

Theorem 8. *Let $\boldsymbol{a}, \boldsymbol{b}$ and $\boldsymbol{c}$ be vectors from $\boldsymbol{E}$. Then the following two statements are equivalent:*

(i) *$\boldsymbol{a}, \boldsymbol{b}$ and $\boldsymbol{c}$ are linearly dependent.*

(ii) $\boldsymbol{c}'\boldsymbol{T}_{\mathrm{a}}\boldsymbol{b} = 0$.

Proof. Assume that statement (i) holds. Without loss of generality let $\boldsymbol{a} = \beta \boldsymbol{b} + \gamma \boldsymbol{c}$ for some scalars β and γ. Then $\boldsymbol{c}'\boldsymbol{T}_{\mathrm{b}}\boldsymbol{a} = \gamma \boldsymbol{c}'\boldsymbol{T}_{\mathrm{b}}\boldsymbol{c} = -\gamma \boldsymbol{c}'\boldsymbol{T}_{\mathrm{c}}\boldsymbol{b} = \gamma \boldsymbol{c}'\boldsymbol{T}_{\mathrm{c}}'\boldsymbol{b} = \gamma(\boldsymbol{T}_{\mathrm{c}}\boldsymbol{c})'\boldsymbol{b} = 0$. From property (x) of Section 2, we get $\boldsymbol{c}'\boldsymbol{T}_{\mathrm{a}}\boldsymbol{b} = 0$.

Let now $\boldsymbol{c}'\boldsymbol{T}_{\mathrm{a}}\boldsymbol{b} = 0$. Then we may conclude

$$\begin{pmatrix} \boldsymbol{a}' \\ \boldsymbol{b}' \\ \boldsymbol{c}' \end{pmatrix} \boldsymbol{T}_{\mathrm{a}}\boldsymbol{b} = \begin{pmatrix} \boldsymbol{a}'\boldsymbol{T}_{\mathrm{a}}\boldsymbol{b} \\ \boldsymbol{b}'\boldsymbol{T}_{\mathrm{a}}\boldsymbol{b} \\ \boldsymbol{c}'\boldsymbol{T}_{\mathrm{a}}\boldsymbol{b} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}. \tag{4.1}$$

Case I. $\boldsymbol{a} = \boldsymbol{0}$ or $\boldsymbol{b} = \boldsymbol{0}$. Then $\boldsymbol{a}, \boldsymbol{b}$ and $\boldsymbol{c}$ are linearly dependent.

Case II. $\boldsymbol{a} \neq \boldsymbol{0}$ and $\boldsymbol{b} \neq \boldsymbol{0}$. If $\boldsymbol{T}_{\mathrm{a}}\boldsymbol{b} = \boldsymbol{0}$, by Theorem 7, $\boldsymbol{a}$ and $\boldsymbol{b}$ are linearly dependent, and so $\boldsymbol{a}, \boldsymbol{b}$ and $\boldsymbol{c}$. If $\boldsymbol{T}_{\mathrm{a}}\boldsymbol{b} \neq \boldsymbol{0}$, the matrix $\boldsymbol{a}'$ is singular, where $\boldsymbol{A} = (\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c})$ is the matrix whose columns are $\boldsymbol{a}, \boldsymbol{b}$ and $\boldsymbol{c}$. Thus, $\boldsymbol{a}, \boldsymbol{b}$ and $\boldsymbol{c}$ are linearly dependent. ■

The following result shows how the implicit characterizations of vectors belonging to a plane can be made explicit.

Theorem 9. *Let $\boldsymbol{a} \neq \boldsymbol{0}$ be a vector from $\boldsymbol{E}$* and *α be a scalar. Then the general solution of $\boldsymbol{a}'\boldsymbol{x} = \alpha$ is $\boldsymbol{x} = \frac{\alpha}{\mathrm{a}'\mathrm{a}}\boldsymbol{a} + \boldsymbol{T}_{\mathrm{a}}\boldsymbol{z}$, where $\boldsymbol{z}$ is an arbitrary vector.*

Proof. Obviously the equation $\boldsymbol{a}'\boldsymbol{x} = \alpha$ is consistent. Then by (1.4) we get the general solution as

$$\boldsymbol{x} = \alpha \boldsymbol{a}'^{+} + (\boldsymbol{I} - \boldsymbol{a}'^{+}\boldsymbol{a}')\boldsymbol{y}, \quad \boldsymbol{y} \in \boldsymbol{E}.$$

Now, since $\boldsymbol{a}'^{+} = \boldsymbol{a}^{+'} = \boldsymbol{a}/\boldsymbol{a}'\boldsymbol{a}$ and

$$\boldsymbol{I} - \frac{\boldsymbol{a}\boldsymbol{a}'}{\boldsymbol{a}'\boldsymbol{a}} = -\frac{1}{\boldsymbol{a}'\boldsymbol{a}}\boldsymbol{T}_{\mathrm{a}}^2$$

we arrive at

$$\boldsymbol{x} = \frac{\alpha}{\boldsymbol{a}'\boldsymbol{a}}\boldsymbol{a} - \frac{1}{\boldsymbol{a}'\boldsymbol{a}}\boldsymbol{T}_{\mathrm{a}}^2\boldsymbol{y}.$$

Since $\mathcal{R}\left(-\frac{1}{\mathrm{a}'\mathrm{a}}\boldsymbol{T}_{\mathrm{a}}^2\right) = \mathcal{R}(\boldsymbol{T}_{\mathrm{a}}^2) = \mathcal{R}(-\boldsymbol{T}_{\mathrm{a}}\boldsymbol{T}_{\mathrm{a}}') = \mathcal{R}(\boldsymbol{T}_{\mathrm{a}}\boldsymbol{T}_{\mathrm{a}}') = \mathcal{R}(\boldsymbol{T}_{\mathrm{a}})$, we conclude $\boldsymbol{x} = \frac{\alpha}{\mathrm{a}'\mathrm{a}}\boldsymbol{a} + \boldsymbol{T}_{\mathrm{a}}\boldsymbol{z}$, where $\boldsymbol{z}$ is arbitrary. ■

Observe that we used the identity $\mathcal{R}(\boldsymbol{B}\boldsymbol{B}') = \mathcal{R}(\boldsymbol{B})$, which is valid for any real matrix $\boldsymbol{B}$.

Theorem 9 is also shown in Chambers ([2], p. 52), where however the proof is somewhat incomplete. The next result is taken from the same source.

Theorem 10. *Let $\boldsymbol{a} \neq \boldsymbol{0}$ be a given vector from $\boldsymbol{E}$. Then the equation $\boldsymbol{T}_{\mathrm{a}}\boldsymbol{x} = \boldsymbol{b}$ is consistent if and only if $\boldsymbol{a}'\boldsymbol{b} = 0$, in which case the general solution is $\boldsymbol{x} = -\frac{1}{\mathrm{a}'\mathrm{a}}\boldsymbol{T}_{\mathrm{a}}\boldsymbol{b} + \alpha\boldsymbol{a}$, where α is an arbitrary scalar.*

Proof. From (1.3), we know that $\boldsymbol{T}_{\mathrm{a}}\boldsymbol{x} = \boldsymbol{b}$ is consistent if and only if $\boldsymbol{T}_{\mathrm{a}}\boldsymbol{T}_{\mathrm{a}}^{+}\boldsymbol{b} = \boldsymbol{b}$, which by (3.2) is equivalent to $\boldsymbol{a}\boldsymbol{a}^{+}\boldsymbol{b} = \boldsymbol{0}$, i.e. $\boldsymbol{a}'\boldsymbol{b} = 0$. By (1.4) the general solution is $\boldsymbol{x} = \boldsymbol{T}_{\mathrm{a}}^{+}\boldsymbol{b} + (\boldsymbol{I} - \boldsymbol{T}_{\mathrm{a}}^{+}\boldsymbol{T}_{\mathrm{a}})\boldsymbol{z}$, where $\boldsymbol{z}$ is arbitrary. However $\boldsymbol{T}_{\mathrm{a}}^{+} = -\frac{1}{\mathrm{a}'\mathrm{a}}\boldsymbol{T}_{\mathrm{a}}$ and $\boldsymbol{I} - \boldsymbol{T}_{\mathrm{a}}\boldsymbol{T}_{\mathrm{a}}^{+} = \boldsymbol{a}\boldsymbol{a}^{+}$. Hence $\boldsymbol{x} = -\frac{1}{\mathrm{a}'\mathrm{a}}\boldsymbol{T}_{\mathrm{a}}\boldsymbol{b} + \boldsymbol{a}\boldsymbol{a}^{+}\boldsymbol{z}$. Since $\mathcal{R}(\boldsymbol{a}) = \mathcal{R}(\boldsymbol{a}\boldsymbol{a}^{+})$, we finally obtain the assertion. ■

The preceding theorem shows that every line $\boldsymbol{x} = \boldsymbol{c} + \alpha\boldsymbol{a}$ in $\mathbb{R}^3$ can be equivalently expressed as $\boldsymbol{T}_{\mathrm{a}}\boldsymbol{x} = \boldsymbol{b}$ for some vector $\boldsymbol{b}$.

For the following result we refer to Milne ([5], p. 23). Let $\boldsymbol{a} \neq \boldsymbol{0}$ and $\boldsymbol{b}$ be vectors from $\boldsymbol{E}, \alpha \in \mathbb{R}$. Find $\boldsymbol{x}$ such that $\alpha\boldsymbol{x} - \boldsymbol{T}_{\mathrm{a}}\boldsymbol{x} = \boldsymbol{b}$. Clearly, this equation may be written as

$$(\alpha\boldsymbol{I} - \boldsymbol{T}_{\mathrm{a}})\boldsymbol{x} = \boldsymbol{b}.$$

If $\alpha = 0$, we may consult Theorem 10. If $\alpha \neq 0$, then α is no eigenvalue of $\boldsymbol{T}_{\mathrm{a}}$, and from (3.8) we get

$$\boldsymbol{x} = \frac{1}{\boldsymbol{a}'\boldsymbol{a} + \alpha^2}\left(\boldsymbol{T}_{\mathrm{a}} + \alpha\boldsymbol{I} + \frac{1}{\alpha}\boldsymbol{a}\boldsymbol{a}'\right)\boldsymbol{b}.$$

Our next example is due to Chambers ([2], p. 67). Suppose that a system of forces $\boldsymbol{f}_i$ act at points with position vector $\boldsymbol{r}_i$. Find the locus of points such that the moment about them is parallel to the resultant force.

The resultant force is $\Sigma\boldsymbol{f}_i = \boldsymbol{f}$. The moment of the force system about the point with position vector $\boldsymbol{r}$ is

$$\sum_i (\boldsymbol{r}_i - \boldsymbol{r}) \times \boldsymbol{f}_i = \boldsymbol{g} - \boldsymbol{r} \times \boldsymbol{f}, \tag{4.2}$$

where $\boldsymbol{g} = \sum_i \boldsymbol{r}_i \times \boldsymbol{f}_i$.

It is required to find $\boldsymbol{r}$, if possible, such that

$$\boldsymbol{g} - \boldsymbol{r} \times \boldsymbol{f} = \lambda\boldsymbol{f}. \tag{4.3}$$

In our language, (4.3) is rewritten as

$$\boldsymbol{T}_{\mathrm{f}}\boldsymbol{r} = \lambda\boldsymbol{f} - \boldsymbol{g}. \tag{4.4}$$

From Theorem 10, we know that (4.4) is consistent if and only if $\boldsymbol{f}'(\lambda\boldsymbol{f} - \boldsymbol{g}) = 0$, or equivalently $\lambda = \boldsymbol{f}'\boldsymbol{g}/\boldsymbol{f}'\boldsymbol{f}$. In that case the general solution is

$$\begin{aligned} \boldsymbol{r} &= -\frac{1}{\boldsymbol{f}'\boldsymbol{f}}\boldsymbol{T}_{\mathrm{f}}(\lambda\boldsymbol{f} - \boldsymbol{g}) + \alpha\boldsymbol{f} \\ &= \frac{1}{\boldsymbol{f}'\boldsymbol{f}}\boldsymbol{T}_{\mathrm{f}}\boldsymbol{g} + \alpha\boldsymbol{f}, \end{aligned} \tag{4.5}$$

where α is an arbitrary scalar.

Note that our approach to this problem is more direct than that of Chambers ([2]).

Finally, let us remark that every 3×3 skew-symmetric matrix can be expressed as $\boldsymbol{T}_{\mathrm{a}}$ for some suitable vector $\boldsymbol{a}$. On the other hand, for every real and skew-symmetric matrix we can consider the *Cayley-transform*, which for $\boldsymbol{T}_{\mathrm{a}}$ is given by

$$\boldsymbol{C}(\boldsymbol{T}_{\mathrm{a}}) = (\boldsymbol{I} - \boldsymbol{T}_{\mathrm{a}})(\boldsymbol{I} + \boldsymbol{T}_{\mathrm{a}})^{-1} \tag{4.6}$$

(cf. Lancaster and Tismenetsky [4], p. 219).

Some straightforward calculations yield

$$\boldsymbol{C}(\boldsymbol{T}_{\mathrm{a}}) = \delta(-2\boldsymbol{T}_{\mathrm{a}} + 2\boldsymbol{a}\boldsymbol{a}' + (1 - \boldsymbol{a}'\boldsymbol{a})\boldsymbol{I}), \tag{4.7}$$

where $\delta = 1/(1 + \boldsymbol{a}'\boldsymbol{a})$.

Observe that the Cayley-transform is an orthogonal matrix. Since orthogonal matrices leave lengths and angles unchanged, the matrices $\boldsymbol{C}(\boldsymbol{T}_{\mathrm{a}})$ represent a rotation in the 3-dimensional space $\boldsymbol{E}$. For a nice geometrical discussion we refer to Room [7].

REFERENCES

[1] A. Ben-Israel, and T.N.E. Greville, *Generalized Inverses: Theory and Applications*, John Wiley & Sons, New York 1974.

[2] L.G. Chambers, *A Course in Vector Analysis*, Chapman and Hall, London 1969.

[3] B. Hague, *An Introduction to Vector Analysis for Physicists and Engineers*, (6th edition, revised by D. Martin), Methuen & Science Paperbacks, London 1970.

[4] T. Lancaster, and M. Tismenetsky, *The Theory of Matrices.* Academic Press, New York 1985.

[5] E.A. Milne, *Vectorial Mechanics*, Methuen & Co. Ltd., London 1965.

[6] C.R. Rao, and S.K. Mitra, *Generalized Inverse of Matrices and its Applications*, John Wiley & Sons, New York 1971.

[7] T.G. Room, *The composition of rotations in Euclidean three-space*, Amer. Math. Monthly **59** (1952), 688-692.

Received 26 October 1999
Revised 11 February 2000