Discussiones Mathematicae
General Algebra and Applications 21 (2001) 57–66

SOME PROPERTIES OF CONGRUENCE RELATIONS ON ORTHOMODULAR LATTICES

Gerhard Dorfer*

Institute of Algebra and Computational Mathematics
Vienna University of Technology
Wiedner Hauptstr. 8–10/118, A–1040 Vienna, Austria
e-mail: g.dorfer@tuwien.ac.at

Abstract

In this paper congruences on orthomodular lattices are studied with particular regard to analogies in Boolean algebras. For this reason the lattice of p-ideals (corresponding to the congruence lattice) and the interplay between congruence classes is investigated. From the results adduced there, congruence regularity, uniformity and permutability for orthomodular lattices can be derived easily.

Keywords: orthomodular lattice, congruence relation, congruence-regular, congruence-uniform, congruence-permutable.

2000 Mathematics Subject Classification: 06C15, 08A30, 03G12, 81P10.

1. Basic facts

Orthomodular lattices are a well studied structure, we refer for instance to the monographs [1], [5] and [7]. Of special interest is the occurrence of these algebras in axiomatic quantum mechanics as so-called quantum logics. This has caused and continuously stimulated their investigation (see, e.g., [7]). In notation we follow [5] wherein proofs of the basic facts stated in this section can be found unless an other reference is given.

An *orthomodular lattice* (OML, for short) $\mathcal{L} = \langle L; \wedge, \vee, ', 0, 1\rangle$ is a bounded lattice $\langle L; \wedge, \vee, 0, 1\rangle$ with an *orthocomplementation* $'$, i.e. for all $x, y \in L$ it holds

$$x \wedge x' = 0,\ x \vee x' = 1,\ x'' = x,\ x \leq y \text{ implies } y' \leq x',$$

*The work on this paper was partially supported by the grant 22p2 "Ordered algebraic structures and applications" of the "Aktion Österreich – Tschechische Republik".

and $\mathcal{L}$ satisfies the *orthomodular law*:

$$x \leq y \text{ implies } y = x \vee (y \wedge x').$$

As distinguished from Boolean algebras in orthomodular lattices the distributive law $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$ does not hold in general. The following two relations provide a measure for how far a particular OML $\mathcal{L}$ is from being a Boolean algebra:

1. The *commutativity relation* C: aCb if and only if the subalgebra generated by $\{a, b\}$ in $\mathcal{L}$ is Boolean. For instance, $a \leq b$ or $a \leq b'$ implies aCb.
2. The *perspectivity relation* $\sim$: $a \sim b$ if and only if a and b have a common (algebraic) complement, i.e. there exists an element $c \in L$ such that $a \wedge c = b \wedge c = 0$ and $a \vee c = b \vee c = 1$ holds.

An OML $\mathcal{L}$ is a Boolean algebra if and only if C is the all relation or equivalently, if and only if $\sim$ is the identity.

The most effective application of the orthomodular law is furnished by the Theorem of Foulis and Holland (see, e.g., Theorem I.3.3 in [5]), which ensures distributivity for the elements a, b, c provided one of them commutes with the other two. This theorem will be used extensively in the subsequent calculations without referring to it explicitly.

In the following let $\mathcal{L}$ denote an arbitrary OML. It is a well known fact that there is a bijection between congruence relations of $\mathcal{L}$ and certain ideals of $\mathcal{L}$, so called *p-ideals* (cf. [3]): A (lattice-)ideal I is a p-ideal if it is closed under perspectivity, i.e. $a \in I$ and $b \sim a$ together imply $b \in I$. For every congruence relation θ the class of 0, which we will denote by $[0]\theta$, is a p-ideal, and starting from a p-ideal I, the relation θ defined by

$$a\theta b \text{ if and only if } a\triangle b := (a \vee b) \wedge (a' \vee b') \in I$$

is a congruence relation. Furthermore, these mappings connecting p-ideals and congruence relations are inverse to each other. The following characterization of p-ideals (cf. [1], Theorem V.4.2) will be used frequently:

Theorem 1.1. *Let I be an ideal of $\mathcal{L}$, then the following conditions are equivalent:*

1. *I is a p-ideal,*
2. *$x \wedge (i \vee x') \in I$, for all $i \in I$ and all $x \in L$,*
3. *$(i \vee x) \wedge (i \vee x') \in I$, for all $i \in I$ and all $x \in L$.*

2. The lattice of p-ideals

We firstly deal with the question how the operations of infimum and supremum in the congruence lattice of $\mathcal{L}$, denoted by $\mathcal{C}on(\mathcal{L}) = \langle Con(\mathcal{L}\,); \wedge, \vee\rangle$, can be translated into the language of p-ideals.

Whenever in the following the operations $\wedge$, $\vee$ and $'$ occur in connection with subsets of L, the complex product is meant, for instance

$$I_1 \wedge I_2 = \{i_1 \wedge i_2 \mid i_1 \in I_1, i_2 \in I_2\}, \quad a \vee I' = \{a \vee i' \mid i \in I\}.$$

Proposition 2.1. *The (set–theoretical) intersection of an arbitrary set of p-ideals is again a p-ideal, and for two p-ideals I_1 and I_2 it holds $I_1 \cap I_2 = I_1 \wedge I_2$.*

The proof is evident. ■

The next proposition leads to a simple description of the supremum of two p-ideals:

Proposition 2.2. *Let I_1 and I_2 be p-ideals, then $I_1 \vee I_2$ also is a p-ideal.*

Proof. Firstly, we show that $I_1 \vee I_2$ is an ideal: Let $i_1 \in I_1$, $i_2 \in I_2$ and $x \leq i_1 \vee i_2$, then $x \wedge (i_1 \vee x') \in I_1$ and $x \wedge (i_2 \vee x') \in I_2$ by Theorem 1.1. By forming the join of these elements we get

$$(x \wedge (i_1 \vee x')) \vee (x \wedge (i_2 \vee x')) = x \wedge ((i_1 \vee x') \vee (i_2 \vee x')) = x \wedge (i_1 \vee i_2 \vee x') = x,$$

hence $x \in I_1 \vee I_2$ and $I_1 \vee I_2$ is an ideal.

$I_1 \vee I_2$ is also a p-ideal: For $i_1 \in I_1$, $i_2 \in I_2$ and arbitrary $y \in L$, we have

$$y \wedge (i_1 \vee i_2 \vee y') = y \wedge ((i_1 \vee y') \vee (i_2 \vee y')) = (y \wedge (i_1 \vee y')) \vee (y \wedge (i_2 \vee y')) \in I_1 \vee I_2,$$

and Theorem 1.1 yields that $I_1 \vee I_2$ is a p-ideal. ■

So, if we denote the set of all p-ideals of $\mathcal{L}$ by $I(\mathcal{L})$ and $\mathcal{I}(\mathcal{L}) = \langle I(\mathcal{L}); \wedge, \vee\rangle$, we attain the following result:

Theorem 2.3. *$\mathcal{C}on(\mathcal{L})$ is isomorphic to $\mathcal{I}(\mathcal{L})$ (by the aforementioned correspondence).* ∎

In the following the structure of $\mathcal{I}(\mathcal{L})$ will be studied with respect to complements. For an ideal I let $I^* = \{x \in L \mid \forall i \in I : \; x \le i'\}$, i.e. I^* consists of the lower bounds of I'. It is evident that I^* is an ideal and $I \wedge I^* = \{0\}$.

Lemma 2.4. *For a p-ideal I the following assertions hold:*

1. $I^* = \{x \in L \mid x \wedge I = \{0\}\}$,
2. *I^* is a p-ideal.*

Proof.
Ad 1: Since $x \le i'$ implies $x \wedge i = 0$, the inclusion $\subseteq$ is true for every ideal I. To show the converse relation, let $x \wedge i = 0$ for all $i \in I$. Then also $x \wedge (x \wedge (i \vee x')) = x \wedge (i \vee x') = 0$, and forming the join with i' in the last equation leads to $x \vee i' = i'$, hence $x \in I^*$ and the proof of this part is complete.

Ad 2: For $a \in I^*$, $i \in I$, $x \in L$ we have $x \wedge (i \vee x') \in I$, and as a consequence $a \le x' \vee (i' \wedge x) \in I'$. This implies $x \wedge (a \vee x') \le i' \wedge x \le i'$, hence $x \wedge (a \vee x') \in I^*$, and I^* is a p-ideal by Theorem 1.1. ∎

It can be seen easily that for p-ideals I, J it holds $(I \vee J)^* = I^* \wedge J^*$, whereas $(I \wedge J)^* = I^* \vee J^*$ and $I \vee I^* = L$ are not true in general (not even for (non-complete) Boolean algebras $\mathcal{L}$).

Let $C(\mathcal{L})$ denote the center of $\mathcal{L}$ consisting of those elements which commute with every element of L. The following characterization for principal p-ideals is well-known:

Proposition 2.5. *The interval $[0, c]$ is a p-ideal if and only if c is a central element.*

It is a trivial observation that $\sup I$ exists if and only if $\max I^*$ does. So we infer:

Proposition 2.6. *If the supremem of a p-ideal exists it is a central element.*

A lattice $\langle V; \wedge, \vee \rangle$ with least element 0 is called *pseudocomplemented*, if for every $x \in L$ there exists $x^* \in L$ such that $x \wedge x^* = 0$ and $x \wedge y = 0$ implies $y \le x^*$. x^* then is called the *pseudocomplement of* x. If for every pair $(x, y) \in V^2$ there is an element $x * y$ such that $x \wedge (x * y) \le y$ and $x \wedge z \le y$ implies $z \le x * y$, then $x * y$ is called the *relative pseudocomplement of x with respect to y* and the lattice is called *Brouwerian.* It is well known that the congruence lattice of an OML is Brouwerian. The next proposition gives an explicit description of the pseudocomplement and relative pseudocomplement in $\mathcal{C}on(\mathcal{L})$.

Proposition 2.7. *For p-ideals I and J, the ideal I^* is the pseudocomplement of I and $\{x \in L \mid x \wedge I \subseteq J\}$ is the relative pseudocomplement of I with respect to J.*

Proof. Firstly we prove that I^* is the pseudocomplement of I: Let K be a p-ideal with $I \wedge K = \{0\}$. We have to show that $K \subseteq I^*$. If $k \in K$, $i \in I$, then $i \wedge (k \vee i') \in K$, hence $i \wedge (i \wedge (k \vee i')) = i \wedge (k \vee i') = 0$. Joining both sides with i' this implies $k \leq i'$, thus $k \in I^*$ and we are done.

The relative pseudocomplement $I*J$ is the pseudocomplement of I in the interval $[I \wedge J, L]$ (cf. 2.9 in [6]). Let θ denote the congruence corresponding to $I \wedge J$. The interval $[I \wedge J, L]$ is the congruence lattice of the quotient $\mathcal{L}/\theta$ and $x \wedge I \subseteq J$ is equivalent to $[x]\theta \wedge [i]\theta = [0]\theta$ for all $i \in I$. By applying 1. of Lemma 2.4 we obtain $I * J = \{x \in L \mid x \wedge I \subseteq J\}$. ■

If I has a complement J in $\mathcal{I}(\mathcal{L})$, then Proposition 2.7 implies $J = I^*$. However, not every I has a complement:

Theorem 2.8. *In the congruence lattice of an OML a congruence has a complement if and only if it stems from a principal p-ideal.*

Proof. A principal p-ideal $I = [0, c]$, $c \in C(\mathcal{L})$, has the complement $I^* = [0, c']$. Conversely, if $I \vee I^* = L$ then $i \vee j = 1$ for suitable $i \in I$, $j \in I^*$. Because $j \leq i'$, this yields $j = i'$, thus $I = [0, i]$. ■

Since OMLs are relatively complemented, this result can also be derived by combining Theorem 4.1 and Theorem 4.2 in [2].

3. Representation of congruence classes

In this section the interplay of congruence classes of OMLs will be investigated. For principal p-ideals the congruence classes are connected rather simply:

Proposition 3.1. *Let $c \in C(\mathcal{L})$ and let θ denote the congruence relation corresponding to the p-ideal $I = [0, c]$. Then all blocks of θ are intervals, in* *particular* $[a]\theta = [a \wedge c', a \vee c]$. *With respect to the orthomodular structure induced by* $\mathcal{L}$ *all these intervals are isomorphic, in particular*

$$\varphi_a : \begin{cases} I & \to & [a]\theta \\ i & \mapsto & (a \wedge c') \vee i \end{cases}$$

is an isomorphism.

Proof. Let a, b be arbitrary elements of L. $b\theta a$ is equivalent to $(a \vee b) \wedge (a' \vee b') \leq c$. Joining both sides with a leads to $a \vee b \leq a \vee c$, and since $a \vee c \in [a]\theta$, this implies $a \vee c = \max [a]\theta$.
$b\theta a$ is also equivalent to $(a \wedge b) \vee (a' \wedge b') \geq c'$, and similar arguments (intersecting with a) yield $a \wedge c' = \min [a]\theta$. Since congruence classes in lattices are always convex subsets, we obtain $[a]\theta = [a \wedge c', a \vee c]$.

Now we show that φ_a is an isomorphism:

1. φ_a is well defined: $((a \wedge c') \vee i)\theta((a \wedge 1) \vee 0) = a$.
2. φ_a is onto: $b\theta a$ implies $a \wedge c' = \min [a]\theta = \min [b]\theta = b \wedge c'$. Because $c \in C(\mathcal{L})$, we obtain $b = (b \wedge c') \vee (b \wedge c) = (a \wedge c') \vee i$ with $i = b \wedge c \in I$.
3. φ_a is one-to-one: From $b = (a \wedge c') \vee i$ with $i \in I$, we derive (by intersecting both sides with c and using $i \leq c$) $b \wedge c = i$, thus i is uniquely determined.
4. φ_a is compatible with the operations: Let * denote the complementation within $[a \wedge c', a \vee c]$ and $^\perp$ that within $[0, c]$, i.e. $z^\star = (a \wedge c') \vee (z' \wedge (a \vee c))$ and $z^\perp = z' \wedge c$, respectively.
   Then $(\varphi_a(i))^\star = (a \wedge c') \vee (i' \wedge (a' \vee c) \wedge (a \vee c)) = (a \wedge c') \vee (i' \wedge c) = \varphi_a(i^\perp)$, for all $i \in I$.
   Obviously $\varphi_a(x) \vee \varphi_a(y) = \varphi_a(x \vee y)$, and together with the compatibilty of φ_a with complementation this implies $\varphi_a(x) \wedge \varphi_a(y) = \varphi_a(x \wedge y)$.

■

Remarks.

- The mapping $\tilde{\varphi}_a : \begin{cases} I & \to & [a]\theta \\ i & \mapsto & (a \vee c) \wedge i' \end{cases}$ is a dual isomorphism.
- The inverse of φ_a, $\varphi_a^{-1} : [a]\theta \to [0, c]$ is given by $\varphi_a^{-1}(x) = x \wedge c$.
- The congruence θ corresponding to the p-ideal $[0, c]$ can be characterized as follows: $a\theta b$ if and only if $a \vee c = b \vee c$, or equivalently, if and only if $a \wedge c' = b \wedge c'$.

Next we study general p-ideals. The representation of congruence classes turns out to be slightly more complicated in this case.

Proposition 3.2. *Let θ be a congruence relation on $\mathcal{L}$ and I the corresponding p-ideal. Then for all $a \in L$*

$$\begin{aligned}[a]\theta &= (a \vee I) \wedge I' \\ &= (a \wedge I') \vee I.\end{aligned}$$

Proof. We work out the proof for the representation $[a]\theta = (a \vee I) \wedge I'$ only – the other can be proven similarly.

Ad $\supseteq$: For $i, j \in I$ we have $((a \vee i) \wedge j')\theta((a \vee 0) \wedge 1) = a$.

Ad $\subseteq$: Let $b\theta a$, by applying the orthomodular law twice we obtain

$$b = (a \vee b) \wedge (b \vee (a' \wedge b')) = (a \vee ((a \vee b) \wedge a')) \wedge (b \vee (a' \wedge b')) \in (a \vee I) \wedge I',$$

since $((a \vee b) \wedge a')\theta((a \vee a) \wedge a') = 0$ and $(b \vee (a' \wedge b'))\theta(b \vee (b' \wedge b')) = 1$. ■

Remark. In Boolean algebras the simple representation $[a]\theta = a \triangle I$ is possible. For orthomodular lattices only the inclusion $[a]\theta \supseteq a \triangle I$ remains valid as can be seen from the following example: Let $\mathcal{L} = \mathrm{MO}_2$, the smallest OML which is not a Boolean algebra, with generating elements a and b, and let θ denote the all relation, i.e. $I = L$. Then $a \triangle I = \{0, a, a', 1\} \neq [a]\theta$. So, even for principal congruences the assertion is not true.

The next proposition describes the p-ideal starting with an arbitrary block:

Proposition 3.3. *Let θ be a congruence relation and I the corresponding p-ideal, then for all $a \in L$*

$$I = [a]\theta \triangle [a]\theta$$

holds.

Proof.

Ad $\supseteq$: $a\theta b$ implies $a \triangle b \in I$ by definition of θ via I (cf. section 1).

Ad $\subseteq$: Let $i \in I$, then $b = a \vee i \in [a]\theta$ and $(b \triangle i)\theta(b \triangle 0) = b$, so $b \triangle i \in [a]\theta$. Now from $b \triangle (b \triangle i) = b \triangle (b \wedge i') = b \wedge (i \vee b') = i$, it follows that $i \in [a]\theta \triangle [a]\theta$. ■

As an application of these representations, we characterize those (closed) intervals which are blocks of some congruence.

Proposition 3.4. *Let $a, b \in L$ with $a \leq b$. The interval $[a, b]$ is a block of a congruence if and only if $b \wedge a' \in C(\mathcal{L})$.*

Proof. If $[a, b]$ is a block of θ, then due to Proposition 3.3 the corresponding p-ideal is $[a, b] \triangle [a, b] = [0, b \wedge a']$, and $b \wedge a' \in C(\mathcal{L})$ by Proposition 2.5.

If $b \wedge a' = c \in C(\mathcal{L})$, then we derive $a \wedge c' = a$ and $a \vee c = b$, hence $[a, b] = [a \wedge c', a \vee c]$ is a block (cf. Proposition 3.1). ∎

4. Congruence regularity, uniformity and permutability

In universal algebra a variety $\mathcal{V}$ is called *congruence regular* if for every algebra $\mathcal{A} = \langle A, \Omega \rangle \in \mathcal{V}$ and $\theta, \psi \in Con(\mathcal{A})$ the equation $[a]\theta = [b]\psi$ for some $a, b \in A$ implies $\theta = \psi$, i.e. every congruence class determines the congruence uniquely. A variety $\mathcal{V}$ is called *congruence uniform* if for every congruence of an algebra of $\mathcal{V}$ all congruence classes have the same cardinality. Moreover, $\mathcal{V}$ is called *arithmetical* if the congruence lattice of every algebra of $\mathcal{V}$ is distributive and all congruences *permute*, i.e. $\theta \circ \psi = \psi \circ \theta$ holds for all congruences θ and ψ, where $\circ$ denotes the relational composite.

Theorem 3.6 in [4] shows that OMLs are congruence regular and congruence permutable (cf. also [2], Theorem 4.2). These results and congruence uniformity can also be obtained easily by using the representations of the congruence classes derived in section 3.

Proposition 3.3 delivers a formula for the p-ideal (and hence the whole congruence is determined) by means of an arbitrary congruence class. From this we infer:

Theorem 4.1. *The variety of orthomodular lattices is congruence regular.*

∎

In the following we denote the cardinality of a set M by $|M|$.

Theorem 4.2. *The variety of orthomodular lattices is congruence uniform.*

Proof. Let θ be a congruence of an OML $\mathcal{L}$ and let I be the corresponding p-ideal. We distinguish two cases:

1. I is finite: Then I is a principal ideal $[0, c]$ with $c \in C(\mathcal{L})$ and according to Proposition 3.1 all congruence classes are isomorphic in this case and hence have the same cardinality.
2. I is infinite: For $a \in L$ we have $[a]\theta = (a \vee I) \wedge I'$, hence $|[a]\theta| \leq |I|^2 = |I|$. Conversely, because of $I = [a]\theta \triangle [a]\theta$ we obtain $|I| \leq |[a]\theta|^2 = |[a]\theta|$, which all together implies $|[a]\theta| = |I|$. ∎

Theorem 4.3. *The variety of orthomodular lattices is arithmetical.*

Proof. Lattices are congruence distributive, therefore only congruence permutability has to be shown. This is equivalent to $\theta \vee \psi = \theta \circ \psi$ for arbitrary congruences θ and ψ of an OML $\mathcal{L}$. Let in general I_ϕ be the p-ideal corresponding to the congruence ϕ.
Since $\theta \vee \psi \supseteq \theta \circ \psi$ always holds, the reverse relation has to be established: If $(x, y) \in \theta \vee \psi$, i.e. $y \in [x](\theta \vee \psi)$, then, by Proposition 3.2, there exist $i, j \in I_{\theta\vee\psi}$ such that $y = (x \vee i) \wedge j'$. Because $I_{\theta\vee\psi} = I_\theta \vee I_\psi$ holds (Proposition 2.2), $i = i_1 \vee i_2,\ j = j_1 \vee j_2$ for suitable $i_1, j_1 \in I_\theta,\ i_2, j_2 \in I_\psi$. Substituting this for i and j and simplifying modulo ψ and θ we obtain

$$y = (x \vee i_1 \vee i_2) \wedge (j_1' \wedge j_2') \psi (x \vee i_1 \vee 0) \wedge (j_1' \wedge 1) = (x \vee i_1) \wedge j_1' \theta (x \vee 0) \wedge 1 = x,$$

thus $(x, y) \in \theta \circ \psi$ and $\theta \vee \psi \subseteq \theta \circ \psi$. ∎

As far as congruences are concerned, astonishingly enough, OMLs behave very similar to Boolean algebras.

We want to emphasize that without assuming the validity of the orthomodular law the results adduced above are not true. Algebras satisfying all axioms mentioned in section 1 except the orthomodular law are called ortholattices.

For instance O_6, the smallest ortholattice which is not orthomodular, is neither congruence regular nor congruence uniform: Let a, b with $a < b$ be generating elements of O_6. If we consider the principal congruence $\theta = \theta(a, b)$, then $[0]\theta = \{0\} = [0]\omega$ (ω the identity), and $|[a]\theta| = |\{a, b\}| > |[0]\theta|$.

Finally we give an example of an ortholattice whose congruences do not permute: Let L denote the eight–element ortholattice $\{0, a, b, c, a', b', c', 1\}$ with $a < b < c$, then $(a, c) \in \theta(a, b) \circ \theta(b, c)$ but $(a, c) \notin \theta(b, c) \circ \theta(a, b)$.

References

[1] L. Beran, *Orthomodular Lattices*, D. Reidel Publishing Company, Dordrecht-Boston 1985.

[2] R.P. Dilworth, *The structure of relatively complemented lattices*, Ann. of Math. **51** (1950), 348–359.

[3] P.D. Finch, *Congruence relations on orthomodular lattices*, J. Austral. Math. Soc. **6** (1966), 46–54.

[4] J. Hashimoto, *Congruence relations and congruence classes in lattices*, Osaka Math. J. **15** (1963), 71–86.

[5] G. Kalmbach, *Orthomodular Lattices*, Academic Press, London 1983.

[6] T. Katriňák, *Die Kennzeichnung der distributiven pseudokomplementären Halbverbände*, J. Reine Angew. Math. **241** (1970), 160–179.

[7] P. Pták and S. Pulmannová, *Orthomodular Structures as Quantum Logics*, Kluwer Academic Publishers, Dordrecht 1991.

Received 30 August 1999
Revised 19 April 2001