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GENERALIZED MORPHISMS OF ABELIAN m-ARY GROUPS

Alexander M. Gal'mak

Department of Mathematics, Technological Institute
Shmidt ave. 3, 212027 Mogilev, Belarus

Abstract

We prove that the set of all n-ary endomorphisms of an abelian m-ary group forms an (m, n) - ring.

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The terminology and notation used in this paper is standard (see, for example, [7] and [5]). The bibliography of m-ary groups (till 1982) is given in the survey [3] prepared by K. Głazek.

Let $\{A_1, A_2, ..., A_{n-1}, A_n\}$ be the sequence of m-ary groups, where $m, n \geq 2$ are fixed. The sequence $f = \{f_1, f_2, ..., f_{n-1}\}$ of homomorphisms

$$A_1 \xrightarrow{f_1} A_2 \xrightarrow{f_2} ... \xrightarrow{f_{n-2}} A_{n-1} \xrightarrow{f_{n-1}} A_n$$

is called an *n-ary homomorphism* (cf. [2]).

If $A_n = A_1$, then this homomorphism is called an *n-ary endomorphism*. By $\mathrm{End}(A_1, A_2, ..., A_{n-1})$ we denote the set of all n-ary endomorphisms of the sequences $\{A_1, A_2, ..., A_{n-1}, A_1\}$ of m-ary groups. It is clear that f defined in such a way is an n-ary isomorphism iff all f_i are isomorphisms.

Let $f_i = \{f_{i1}, f_{i2}, ..., f_{i(n-1)}\}, i = 1, ..., n$, be an n-ary homomorphism which corresponds to the sequence

$$f_i : B_i \xrightarrow{f_{i1}} A_1 \xrightarrow{f_{i2}} ... \xrightarrow{f_{i(n-2)}} A_{n-2} \xrightarrow{f_{i(n-1)}} B_{i+1},$$

where $B_1, ..., B_{n+1}, A_1, ..., A_{n-2}$ are m-ary groups. The *n-ary product* of such n-ary homomorphisms is defined in the same way as E.L. Post defines the composition of m-ary permutations (cf. [5], p. 249 and [6]).
Namely:

$$\begin{aligned} g = [f_1 f_2 ... f_{n-1} f_n] = \\ &= \{f_{11} f_{22} ... f_{(n-2)(n-2)} f_{(n-1)(n-1)} f_{n1}, \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ &f_{1k} f_{2(k+1)} \cdots f_{(n-k)(n-1)} f_{(n-k+1)1} ... f_{(n-1)(k-1)} f_{nk}, \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ &f_{1(n-1)} f_{21} ... f_{(n-2)(n-3)} f_{(n-1)(n-2)} f_{n(n-1)}\} = \{g_1, g_2, ... g_{n-1}\}, \end{aligned}$$

i.e., as the *skew product* in the matrix $[f_{ij}]_{m\times(n-1)}$.

Such defined a product is an n-ary homomorphism of the sequence $\{B_1, A_1, ..., A_{n-2}, B_{n+1}\}$ because

$$g : B_1 \xrightarrow{g_1} A_1 \xrightarrow{g_2} A_2 \xrightarrow{g_3} ... \xrightarrow{g_{n-2}} A_{n-2} \xrightarrow{g_{n-1}} B_{n+1}.$$

In [2] is proved that $< \mathrm{End}(A_1, A_2, ..., A_{n-1}); [\ ] >$ is an n-ary semigroup. Remark that some results on m-ary transformations of commutative n-ary groups are also contained in [7].

Now, let $A_1, A_2, ..., A_{n-1}$ be abelian m-ary groups and let φ_j be the mapping defined by the formula

$$a^{\varphi_j} = (a^{f_{1j}} a^{f_{2j}} ... a^{f_{mj}}),$$

where $\{f_{i1}, ..., f_{i(n-1)}\} = f_i \in \mathrm{End}(A_1, A_2, ..., A_{n-1})$, $i = 1, ..., m$, $a \in A_j$, $j = 1, ..., n-1$. Since such defined φ_j are homomorphisms (cf. [2]), we have

$$\{\varphi_1, \varphi_2, ..., \varphi_{n-1}\} = \varphi \in \mathrm{End}(A_1, A_2, ..., A_{n-1}).$$

This means that in $\mathrm{End}(A_1, A_2, ..., A_{n-1})$ is defined an m-ary operation ( ) by the formula

$$(f_1 f_2 ... f_m) = \varphi.$$

Recall (cf. for example [1]) that a non-empty set A with two operations $(\ ) : A^m \to A$ and $[\ ] : A^n \to A$ is said to be an (m, n)-*ring* if

1) $< A; (\ ) >$ is an abelian m-ary group;
2) $< A; [\ ] >$ is an n-ary semigroup;
3) $[a_1^{i-1}(b_1^m)a_{i+1}^n] = ([a_1^{i-1}b_1 a_{i+1}^n]...[a_1^{i-1}b_m a_{i+1}^n])$ for all $i = 1, ..., n$ and $a_1, ..., a_n, b_1, ..., b_m \in A$.

Theorem. *If all m-ary groups $A_1, ..., A_{n-1}$ are abelian, then*

$$< \mathrm{End}(A_1, ..., A_{n-1}); (\ ), [\ ] >$$

is an (m, n)-ring.

In the proof of this theorem we use properties of elements formulated in two easily verified lemmas given below.

Recall that two sequences α and β of elements from an m-ary group $< A; [\ ] >$ are *equivalent* if there are sequences δ and γ of elements from A such that $[\gamma, \alpha, \delta] = [\gamma, \beta, \delta]$.

Lemma 1. *Let $\varphi : A \to B$ be a homomorphism of an m-ary groups. If a_1^i and $b_1^{i+k(m-1)}$ are equivalent in A, then $a_1^\varphi \ldots a_i^\varphi$ and $b_1^\varphi ... b_{i+k(m-1)}^\varphi$ are equivalent in B.*

Lemma 2. *Let $\varphi : A \to B$ be a homomorphism of an m-ary groups. If a_1^k is the inverse sequence for $a \in A$, then $a^\varphi a_1^\varphi \ldots a_k^\varphi$ and $a_1^\varphi \ldots a_k^\varphi a^\varphi$ are neutral sequences in B.*

Proof of Theorem. In [2] it is proved that $< \mathrm{End}\ (A_1, ..., A_{n-1}); [\ ] >$ is an n-ary semigroup.

Now, we prove that $< \mathrm{End}\ (A_1, ..., A_{n-1}); () >$ is an m-ary group.

Let

$$((f_1f_2...f_m)f_{m+1}...f_{2m-1}) = g = \{g_1, g_2, ..., g_{n-1}\};$$

$$(f_1...f_i(f_{i+1}...f_{i+m})f_{i+m+1}...f_{2m-1}) = h$$

$$= \{h_1, h_2, ..., h_{n-1}\},\ i = 1, 2, ..., m-1;$$

$$(f_1f_2...f_m) = \varphi = \{\varphi_1, \varphi_2, ..., \varphi_{n-1}\};$$

and

$$(f_{i+1}...f_{i+m}) = \psi = \{\psi_1, \psi_2, ..., \psi_{n-1}\},$$

where $f_j = \{f_{j1}, f_{j2}, ..., f_{j(n-1)}\},\ j = 1, 2, ..., 2m-1.$

Moreover the brackets ( ) will be also denoted the derived (extended) operation.

At first, we prove the associativity of the m-ary operation ( ). Observe that $a^{\varphi_j} = (a^{f_{1j}}a^{f_{2j}}\dots a^{f_{mj}})$, where $a \in A_j, j = 1, ...n-1$, implies

$$a^{g_j} = \left(a^{\varphi_j}a^{f_{(m+1)j}}\dots a^{f_{(2m-1)j}}\right) =$$

$$= \left(\left(a^{f_{1j}}a^{f_{2j}}\dots a^{f_{mj}}\right)a^{f_{(m+1)j}}\dots a^{f_{(2m-1)j}}\right) =$$

$$= \left(a^{f_{1j}}a^{f_{2j}}\dots a^{f_{(2m-1)j}}\right).$$

Hence,

$$a^{g_j} = \left(a^{f_{1j}}a^{f_{2j}}\dots a^{f_{(2m-1)j}}\right). \tag{1}$$

Similarly, $a^{\psi_j} = (a^{f_{(i+1)j}}\dots a^{f_{(i+m)j}})$ implies

$$a^{h_j} = \left(a^{f_{1j}}\dots a^{f_{ij}}a^{\psi_j}a^{f_{(i+m+1)j}}\dots a^{f_{(2m-1)j}}\right) =$$

$$= \left(a^{f_{1j}}\dots a^{f_{ij}}\left(a^{f_{(i+1)j}}\dots a^{f_{(i+m)j}}\right)a^{f_{(i+m+1)j}}\dots a^{f_{(2m-1)j}}\right) =$$

$$= \left(a^{f_{1j}}a^{f_{2j}}\dots a^{f_{(2m-1)j}}\right),$$

i. e.,

(2) $$a^{h_j} = \left(a^{f_{1j}} a^{f_{2j}} \dots a^{f_{(2m-1)j}}\right).$$

From (1) and (2), we get $g_j = h_j$, for all $j = 1, ..., n-1$. Therefore $g = h$ and, in the consequence,

$$\left(\left(f_1^m\right) f_{m+1}^{2m-1}\right) = \left(f_1^i \left(f_{i+1}^{i+m}\right) f_{i+m+1}^{2m-1}\right)$$

for all $i = 1, ..., m-1$, which proves that $<\mathrm{End}(A_1, ..., A_{n-1}); (\ )>$ is an m-ary semigroup. It is an abelian m-ary semigroup, because all m-ary groups $A_1, ..., A_{n-1}$ are abelian.

Now we prove that the equation

(3) $$(f_1 f_2 ... f_{m-1} u) = \varphi,$$

where

$$f_1, f_2, ..., f_{m-1}, \varphi \in \mathrm{End}(A_1, ..., A_{n-1}),$$

$$f_i = \{f_{i1}, f_{i2}, ..., f_{i(n-1)}\},\ i = 1, 2, ..., m-1,$$

$$\varphi = \{\varphi_1, \varphi_2, ..., \varphi_{n-1}\},$$

has a solution $u \in \mathrm{End}(A_1, ..., A_{n-1})$.

Note that $a_1, ..., a_k$ is the inverse sequence for $a_j \in A_j$, then the mapping

$$u_j : a \to \left(a_1^{f_{(m-1)j}} \dots a_k^{f_{(m-1)j}} \dots a_1^{f_{1j}} \dots a_k^{f_{1j}} a^{\varphi_j}\right)$$

is a homomorphism.

Indeed, if $b_{i1}, ..., b_{ik} \in A_j$ is the inverse sequence for $b_i \in A_j$ $(i = 1, 2, ..., m)$ and $d_1, ..., d_k \in A_j$ is the inverse sequence for $(b_1 b_2 ... b_m) \in A_j$, then

(4) $$b_{m1}, ..., b_{mk}, ..., b_{21}, ..., b_{2k}, b_{11}, ..., b_{1k}$$

is an inverse sequence for $(b_1 b_2 ... b_m)$. Thus $d_1, ..., d_k$ and (4) are equivalent. By Lemma 1,

$$b_{m1}^{f_{ij}},\dots,b_{mk}^{f_{ij}},\dots,b_{21}^{f_{ij}},\dots,b_{2k}^{f_{ij}},b_{11}^{f_{ij}},\dots,b_{1k}^{f_{ij}} \text{ and } d_1^{f_{ij}},\dots,d_k^{f_{ij}}$$

are also equivalent sequences.

Using this fact and the abelianity of all m-groups $A_1,...,A_{n-1}$, we get

$$(b_1b_2\dots b_m)^{u_j} = \left(d_1^{f_{(m-1)j}}\dots d_k^{f_{(m-1)j}}\dots d_1^{f_{1j}}\dots d_k^{f_{1j}}(b_1b_2\dots b_m)^{\varphi_j}\right) =$$

$$=\left(b_{m1}^{f_{(m-1)j}}\dots b_{mk}^{f_{(m-1)j}}\dots b_{11}^{f_{(m-1)j}}\dots b_{1k}^{f_{(m-1)j}}\dots b_{m1}^{f_{1j}}\dots b_{mk}^{f_{1j}}\dots b_{11}^{f_{1j}}\dots b_{1k}^{f_{1j}}b_1^{\varphi_j}b_2^{\varphi_j}\dots b_m^{\varphi_j}\right)$$

$$=\left(\left(b_{11}^{f_{(m-1)j}}\dots b_{1k}^{f_{(m-1)j}}\dots b_{11}^{f_{1j}}\dots b_{1k}^{f_{1j}}b_1^{\varphi_j}\right)\dots\left(b_{m1}^{f_{(m-1)j}}\dots b_{mk}^{f_{(m-1)j}}\dots b_{m1}^{f_{1j}}\dots b_{mk}^{f_{1j}}b_m^{\varphi_j}\right)\right)$$

$$=\left(b_1^{u_j}\dots b_m^{u_j}\right).$$

This proves that u_j is a homomorphism for every $j = 1,...,n-1$. Hence $u = \{u_1,...,u_{n-1}\} \in \mathrm{End}(A_1,...,A_{n-1})$. Moreover, by Lemma 2, we get

$$\left(a^{f_{1j}}\dots a^{f_{(m-1)j}}a^{u_j}\right) =$$

$$\left(a^{f_{1j}}\dots a^{f_{(m-1)j}}\left(a_1^{f_{(m-1)j}}\dots a_k^{f_{(m-1)j}}\dots a_1^{f_{1j}}\dots a_k^{f_{1j}}a^{\varphi_j}\right)\right) =$$

$$= \left(\underbrace{a^{f_{1j}}a_1^{f_{1j}}\dots a_k^{f_{1j}}}_{neutral}\dots\underbrace{a^{f_{(m-1)j}}a_1^{f_{(m-1)j}}\dots a_k^{f_{(m-1)j}}}_{neutral}a^{\varphi_j}\right) = a^{\varphi_j}$$

Therefore, we have (3). Since, the operation ( ) defined on $\mathrm{End}(A_1,...,A_{n-1})$, is abelian, we have that $< \mathrm{End}(A_1,...,A_{n-1}); () >$ is an abelian m-ary group.

Now, we prove the identity

$$\left[f_1^{i-1}(g_1^m)f_{i+1}^n\right] = \left(\left[f_1^{i-1}g_1f_{i+1}^n\right]\dots\left[f_1^{i-1}g_mf_{i+1}^n\right]\right), \tag{5}$$

where $i = 1,...,n$.

Let

$$\left[f_1^{i-1}\,(g_1^m)\,f_{i+1}^n\right] = \{s_1, s_2, ..., s_{n-1}\};$$

$$\left(\left[f_1^{i-1} g_1 f_{i+1}^n\right] \cdots \left[f_1^{i-1} g_m f_{i+1}^n\right]\right) = \{r_1, r_2, ..., r_{n-1}\};$$

$$f_k = \{f_{k1}, f_{k2}, ..., f_{k(n-1)}\},\ k = 1, ..., n;$$

$$g_j = \{g_{j1}, g_{j2}, ..., g_{j(n-1)}\},\ j = 1, ..., m;$$

$$\left[f_1^{i-1} g_j f_{i+1}^n\right] = \{t_{j1}, t_{j2}, ..., t_{j(n-1)}\},\ j = 1, ..., m;$$

and

$$(g_1^m) = \{\varphi_1, \varphi_2, ..., \varphi_{n-1}\}.$$

It is clear that identity (5) is satisfied only in the case when $s_k = r_k$ for all $k = 1, ..., n-1$.

For $1 \le i \le n-k$, we have

$$a^{s_k} =$$

$$a^{f_{1k} \cdots f_{(i-1)(i+k-2)} \varphi_i f_{(i+1)(i+k)} \cdots f_{(n-k)(n-1)} f_{(n-k+1)1} \cdots f_{(n-1)(k-1)} f_{nk}}$$

$$= \left(a^{f_{1k} \cdots f_{(i-1)(i+k-2)} g_{1i}} \ldots a^{f_{1k} \cdots f_{(i-1)(i+k-2)} g_{mi}}\right)^{f_{(i+1)(i+k)} \cdots f_{(n-k)(n-1)} f_{(n-k+1)1} \cdots f_{(n-k)(k-1)} f_{nk}}$$

$$= \Big(a^{f_{1k} \cdots f_{(i-1)(n+k-2)} g_{1i} f_{(i+1)(i+k)} \cdots f_{(n-k)(n-1)} f_{(n-k+1)1} \cdots f_{(n-1)(k-1)} f_{nk}} \ldots$$

$$\ldots a^{f_{1k} \cdots f_{(i-1)(i+k-2)} g_{mi} f_{(i+1)(i+k)} \cdots f_{(n-k)(n-1)} f_{(n-k+1)1} \cdots f_{(n-1)(k-1)} f_{nk}}\Big)$$

and

$$a^{r_k} = \left(a^{t_{1k}} \ldots a^{t_{mk}}\right) =$$

$$= \Big(a^{f_{1k} \cdots f_{(i-1)(i+k-2)} g_{1i} f_{(i+1)(i+k)} \cdots f_{(n-k)(n-1)} f_{(n-k+1)1} \cdots f_{(n-1)(k-1)} f_{nk}} \ldots$$

$$\ldots a^{f_{1k} \cdots f_{(i-1)(i+k-2)} g_{mi} f_{(i+1)(i+k)} \cdots f_{(n-k)(n-1)} f_{(n-k+1)1} \cdots f_{(n-1)(k-1)} f_{nk}}\Big).$$

Thus, $a^{s_k} = a^{r_k}$ and, in the consequence, $s_k = r_k$.

If $n - k < i \leq n$, then

$$a^{s_k} = a^{f_{1k}\cdots f_{(n-k)(n-1)} f_{(n-k+1)1}\cdots f_{(i-1)(i+k-n-1)}\varphi_i f_{(i+1)(i+k-n+1)}\cdots f_{(n-1)(k-1)} f_{nk}} =$$

$$= \Big(a^{f_{1k}\cdots f_{(n-k)(n-1)} f_{(n-k+1)1}\cdots f_{(i-1)(i+k-n-1)} g_{1i}} \ldots$$

$$a^{f_{1k}\cdots f_{(n-1)(k-1)} f_{(n-k+1)1}\cdots f_{(i-1)(i+k-n-1)} g_{mi}}\Big)^{f_{(i+1)(i+k-n+1)}\cdots f_{(n-1)(k-1)} f_{nk}} =$$

$$= \Big(a^{f_{1k}\cdots f_{(n-k)(n-1)} f_{(n-k+1)1}\cdots f_{(i-1)(i+k-n-1)} g_{1i} f_{(i+1)(i+k-n+1)}\cdots f_{(n-1)(k-1)} f_{nk}} \ldots$$

$$\ldots a^{f_{1k}\cdots f_{(n-k)(n-1)} f_{(n-k+1)1}\cdots f_{(i-1)(i+k-n-1)} g_{mi} f_{(i+1)(i+k-n+1)}\cdots f_{(n-1)(k-1)} f_{nk}}\Big)$$

and

$$a^{r_k} = \Big(a^{t_{1k}} \ldots a^{t_{mk}}\Big) =$$

$$= \Big(a^{f_{1k}\cdots f_{(n-k)(n-1)} f_{(n-k+1)1}\cdots f_{(i-1)(i+k-n-1)} g_{1i} f_{(i+1)(i+k-n+1)}\cdots f_{(n-1)(k-1)} f_{nk}} \ldots$$

$$\ldots a^{f_{1k}\cdots f_{(n-k)(n-1)} f_{(n-k+1)1}\cdots f_{(i-1)(i+k-n-1)} g_{mi} f_{(i+1)(i+k-n+1)}\cdots f_{(n-1)(k-1)} f_{nk}}\Big),$$

which – similarly as in the previous case – give, $s_k = r_k$.

This completes the proof. ■

Corollary 1. *If* $< A_1; +, -, 0 >, ..., < A_{n-1}; \{+, -, 0\} >$ *are abelian groups, then* $< \mathrm{End}(A_1, ..., A_{n-1}); \{+, -, \Theta, [\,]\} >$ *is the multiring, where* $\Theta = (0, ..., 0)$.

Corollary 2 ([4]). *The set of all endomorphisms of an abelian m-ary group forms an $(m, 2)$-ring.*

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