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CANTOR EXTENSION OF A HALF LINEARLY CYCLICALLY ORDERED GROUP

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Abstract

Convergent and fundamental sequences are studied in a half linearly cyclically ordered group G with the abelian increasing part. The main result is the construction of the Cantor extension of G.

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M. Giraudet and F. Lucas [3] introduced and investigated the notion of a half linearly ordered group (cf. also D.R. Ton [14], J. Jakubík [6], [7]). J. Jakubík [8] defined and studied the notion of a half linearly cyclically ordered group (*lc*-group) generalizing the notion of a half linearly ordered group.

The author [1] investigated the Cantor extension of an abelian lc-group. We remark that the Cantor extension of lattice ordered groups was studied by C.J. Everett [2].

Let G be a half lc-group such that its increasing part is abelian and its decreasing part is nonempty (thus G fails to be an lc-group). The notions of a convergent sequence and a fundamental sequence are defined in a natural way. If every fundamental sequence in G is convergent in G, then G is said to be C-complete.

In the present paper necessary and sufficient conditions are found under which G is C-complete. Further, we define the notion of a Cantor extension and we prove that every half lc-group has a Cantor extension which is uniquely determined up to isomorphisms leaving all elements of G fixed.

1. *l*-cyclically ordered sets and groups

We recall the definitions and some results concerning *l*-cyclically ordered sets (cf. Novák and Novotný [10], Novák [9], Quilot [11]) and *l*-cyclically ordered groups (cf. Rieger [12], Świerczkowski [13], Jakubík and Pringerová [4], [5]).

Definition 1.1. Let M be a nonempty set and T a ternary relation on M such that the following conditions are satisfied:

- (I) if $[x, y, z] \in T$ then $[y, x, z] \notin T$.
- (II) $[x, y, z] \in T$ implies $[y, z, x] \in T$.
- (III) $[x, y, z] \in T, [y, u, z] \in T$ imply $[x, u, z] \in T$.

Then T is said to be a cyclic order on M and (M,T) is called a cyclically ordered set.

Let T be a cyclic order on M satisfying the condition:

(IV) if $x, y, z \in M, x \neq y \neq z \neq x$, then either $[x, y, z] \in T$ or $[z, y, x] \in T$. Then T is said to be an *l*-cyclic order on M and (M, T) is called an *l*-cyclically ordered set.

Several terms are used in papers for the term *l*-cyclic order. For instance "*l*-cyclic order" is called "linear cyclic order" in [9], "complete cyclic order" in [11] and simply "cyclic order" in [12] and [13].

Definition 1.2. Let (H; +) be a group and (H; T) an *l*-cyclically ordered set such that the following condition is fulfilled:

(V) if $[x, y, z] \in T, u, v \in H$, then $[u + x + v, u + y + v, u + z + v] \in T$.

Then (H; +, T) is said to be an *l*-cyclically ordered group or *l*c-group (linearly cyclically ordered group).

We often write H or (H;T) instead of (H;+,T).

Every subgroup of an lc-group is considered as an lc-group under the induced l-cyclic order.

Example 1.3. Let $(L; \leq)$ be a linearly ordered group $x, y, z \in L$. Define the ternary relation T_L on L by putting

$$[x, y, z] \in T_L$$
 if $x < y < z$ or $y < z < x$ or $z < x < y$.

Then $(L; T_L)$ is an *lc*-group. T_L is called the *l*-cyclic order generated by the linear order \leq on L. Hence every linearly ordered group is an *lc*-group (under the *l*-cyclic order generated by its linear order).

Example 1.4. Let K be the group of all reals k such that $0 \le k < 1$ with the group operation defined as the addition mod 1. Consider the natural linear order \le and the ternary relation T_1 on K defined in the same way as T_L in 1.3. Then $(K; T_1)$ is an *lc*-group.

Define the ternary relation T on the direct product $L \times K$ of groups L and K as follows: for elements $u_1, u_2, u_3 \in L \times K$, $u_1 = (x, k_1)$, $u_2 = (y, k_2)$, $u_3 = (z, k_3)$ we put $[u_1, u_2, u_3] \in T$ if some of the following conditions is valid:

- (i) $[k_1, k_2, k_3] \in T_1;$
- (ii) $k_1 = k_2 \neq k_3$ and x < y;
- (iii) $k_2 = k_3 \neq k_1$ and y < z;
- (iv) $k_3 = k_1 \neq k_2$ and z < x;
- (v) $k_1 = k_2 = k_3$ and $[x, y, z] \in T_L$.

Then $(L \times K; T)$ is an *lc*-group which will be denoted by $L \otimes K$.

The notion of an isomorphism of *lc*-groups is defined in a natural way.

Theorem 1.5 (Świerczkowski [13]). Let H be an lc-group. Then there exists a linearly ordered group L such that H is isomorphic to a subgroup of $L \otimes K$.

Assume that (H;T) is an lc-group. By 1.5, there exists an isomorphism f of H into $L \otimes K$. Let H_o be the set of all $h \in H$ such that there exists $x \in L$ with the property f(h) = (x, 0). Then H_o is a subgroup of $H, H_o = \{0\}$ or $H_o \neq \{0\}$. Let $H_o \neq \{0\}$, $h \in H_o$, $h \neq 0$. There exists $x \in L$ such that f(h) = (x, 0). H_o turns out to be a linearly ordered grup if we put h > 0 if x > 0. The l-cyclic order T_{H_o} on H_o coincides with the l-cyclic order induced by T.

2. CANTOR EXTENSION OF AN ABELIAN lc-GROUP

Let (H;T) be an abelian *lc*-group. A construction of a Cantor extension of H will be described (cf. [1]) and some results from [1] will be presented. **Definition 2.1.** Let (x_n) be a sequence in H and $x \in H$.

a) We say that (x_n) converges to x (or x is a limit of (x_n)) in H and we write $x_n \to x$ (or $\lim x_n = x$)

(i) if card H = 2 and there exists $n_o \in N$ such that $x_n = x$ for each $n \in N, n \ge n_o$,

- or
 - (ii) if card H > 2 and for each $\varepsilon \in H, \varepsilon \neq 0$ with the property

 $[-\varepsilon, 0, \varepsilon] \in T$ there exists $n_o \in N$ such that $[-\varepsilon, x_n - x, \varepsilon] \in T$ for each $n \in N, n \ge n_o$.

- **b)** The sequence (x_n) is called *fundamental* in H if for each $\varepsilon \in H, \varepsilon \neq 0$ with the property $[-\varepsilon, 0, \varepsilon] \in T$ there exists $n_o \in N$ such that $[-\varepsilon, x_n x_m, \varepsilon] \in T$ for each $m, n \in N, m, n \ge n_o$.
- c) By a zero sequence we understand a sequence (x_n) such that $x_n \to 0$.
- d) H is called *C*-complete if each fundamental sequence in H is convergent in H.

The set of all fundamental (zero) sequences in H will be denoted by $F_H(E_H)$.

Definition 2.2. Let H_1 be an abelian *lc*-group satisfying the following conditions:

- (a) H_1 is C-complete.
- (b) H is a subgroup of H_1 .
- (c) Every element of H_1 is a limit of some fundamental sequence in H.
- (d) Let (x_n) be a sequence in H such that $x_n \to 0$ in H. Then $x_n \to 0$ in H_1 .

Then H_1 is said to be a *Cantor extension* of H.

Now we consider two cases: $H_0 \neq \{0\}$ and $H_0 = \{0\}$.

1) Assume that $H_o \neq \{0\}$. Let $(x_n), (y_n) \in F_H$. Under the natural definition of the operation + on $F_H, (x_n) + (y_n) = (x_n + y_n), F_H$ is a group and E_H is a subgroup of F_H . We form the factor group $H^* = F_H/E_H$. Symbol $(x_n)^*$ will denote the coset of H^* containing the sequence $(x_n) \in F_H$.

Suppose that $(x_n)^*, (y_n)^*, (z_n)^*$ are mutually distinct elements of H^* . Let T^* be the set of all triples $[(x_n)^*, (y_n)^*, (z_n)^*]$ of elements of H^* such that there exists $n_o \in N$ with $[x_n, y_n, z_n] \in T$ for each $n \in N, n \ge n_o$. Then (H^*, T^*) is an *lc*-group. Let φ be the mapping from H into H^* defined by $\varphi(x) = (x, x, ...)^*$ for each $x \in H$. Then φ is an isomorphism of the *lc*-group H into H^* . We identify x and $\varphi(x)$ for each $x \in H$. Then H is a subgroup of H^* and H^* is a Cantor extension of H.

If we denote $(x_n)^* = X$ and $(x_n, x_n, \ldots)^* = X_n$, then we have (cf. [1], the proof of Lemma 3.12)

(A)
$$X_n \to X$$
 in H^* .

Lemma 2.3 ([1], Lemma 3.9). *H* is *C*-complete if and only if H_o is *C*-complete.

2) Now assume that $H_o = \{0\}$. Then *H* can be considered as a subgroup of *K*.

Lemma 2.4 ([1], Lemma 4.2). If H is a finite subgroup of K, then H is *C*-complete.

Lemma 2.5 ([1], Lemma 4.5). If H is an infinite subgroup of K, then K is a Cantor extension of H.

The following result is valid in both cases 1) and 2).

Theorem 2.6 ([1], Theorem 4.9). Let H be an abelian lc-group. Then

- (i) there exists a Cantor extension of H,
- (ii) if H_1 and H_2 are Cantor extensions of H, then there exists an isomorphism Φ from the lc-group H_1 onto H_2 such that $\Phi(x) = x$ for each $x \in H$.
 - 3. Half *lc*-groups

The notion of a half lc-group was introduced by Jakubík [8]. Now we recall the definitions and results that will be applied in the next sections.

Let (G; +, T) be a system such that (G; +) is a group and (G; T) is a cyclically ordered set. Assume that $x, y, z \in G$. Denote

 $G \uparrow = \{ u \in G : [x, y, z] \in T \Rightarrow [u + x, u + y, u + z] \in T \},\$ $G \downarrow = \{ u \in G : [x, y, z] \in T \Rightarrow [u + z, u + y, u + x] \in T \}.$

Definition 3.1. Let (G; +, T) be as above. Assume that the following conditions are fulfilled:

- (1) The system T is nonempty.
- (2) If $[x, y, z] \in T$, then $[x + u, y + u, z + u] \in T$ for each $u \in G$.
- (3) $G = G \uparrow \cup G \downarrow$.
- (4) If $[x, y, z] \in T$, then either $\{x, y, z\} \subseteq G \uparrow$ or $\{x, y, z\} \subseteq G \downarrow$.

Then (G; +, T) is said to be a half cyclically ordered group.

Let (G; +, T) be a half cyclically ordered group. The definition implies that $G \uparrow$ is a cyclically ordered group. If $G \uparrow$ is an *lc*-group then (G; +, T) is called a *half lc-group* (half linearly cyclically ordered group).

There are elements $x, y, z \in G$ with $[x, y, z] \in T$. This is an immediate consequence of (1).

Again, we often write G or (G;T) instead of (G;+,T).

In the next, let G be a half *lc*-group. $G \uparrow (G \downarrow)$ is called the *increasing* (decreasing, resp.) part of G.

A subgroup G' of G is said to be a *half lc-subgroup* of G if the *induced* l-cyclic order on G' is nonempty.

Each lc-group G with card $G \geq 3$ is a half lc-group (with $G \uparrow = G$ and $G \downarrow = \emptyset$). Every linearly ordered group is an lc-group. Hence every half linearly ordered group (for the definition cf. [3]) is a half lc-group.

The notion of an isomorphism of half lc-groups is defined in a natural way.

From the definition 3.1 it follows (cf. [8]):

- (i) If $x, y \in G \downarrow$, then $x + y \in G \uparrow$;
- (ii) If $x \in G \uparrow, y \in G \downarrow$, then $x + y \in G \downarrow$ and $y + x \in G \downarrow$.

4. Cantor extension of a half lc-group

In what follows, we assume that (G, T) is a half *lc*-group such that $G \uparrow$ is abelian and $G \downarrow \neq \emptyset$. Hence G is neither abelian group nor *lc*-group.

We will use the notation $G \uparrow = H$ and $G \downarrow = H'$.

Definition 4.1. Let (x_n) be a sequence in G and $x \in G$.

- **a)** We say that (x_n) converges to x (or x is a limit of (x_n)) in G and we write $x_n \to x$ (or $\lim x_n = x$) if for each $\varepsilon \in G, \varepsilon \neq 0$ with the property $[-\varepsilon, 0, \varepsilon] \in T$ there exists $n_o \in N$ such that $[-\varepsilon, x_n x, \varepsilon] \in T$ and $[-\varepsilon, -x + x_n, \varepsilon] \in T$ for each $n \in N, n \ge n_o$.
- **b)** The sequence (x_n) is said to be *fundamental* if for each $\varepsilon \in G, \varepsilon \neq 0$ with $[-\varepsilon, 0, \varepsilon] \in T$ there exists $n_o \in N$ such that $[-\varepsilon, x_n - x_m, \varepsilon] \in T$ and $[-\varepsilon, -x_m + x_n, \varepsilon] \in T$ for each $m, n \in N, m, n \ge n_o$.
- c) If $x_n \to 0$ in G, then (x_n) is called a zero sequence in G.
- d) G is said to be *C*-complete if every fundamental sequence in G is convergent in G.

Definition 4.2. Let G_1 be a half *lc*-group with the following properties:

- (α) G_1 is C-complete;
- (β) G is a half *lc*-subgroup of G_1 ;
- (γ) Every element of G_1 is a limit of some fundamental sequence in G;
- (δ) Let (x_n) be a sequence in G such that $x_n \to 0$ in G. Then $x_n \to 0$ in G_1 .

Then G_1 is said to be a *Cantor extension* of G.

We prove that G has a Cantor extension and this is uniquely determined up to isomorphisms leaving all elements of G fixed.

Denote by F(E) the set of all fundamental (zero) sequences in G. Symbols F_H and E_H have the same meaning as in the section 2.

The following two lemmas are easy to prove.

Lemma 4.3. Let (x_n) be a sequence in G. Then $x_n \to x$ in G if and only if $x_n - x \to 0$ and $-x + x_n \to 0$ in G.

For a fixed element $n_o \in N$ and a sequence (x_n) in G we apply the notation $x_n^o = x_{no+n-1}$ for each $n \in N$.

Lemma 4.4. Let (x_n) be a sequence in G.

- (i) $(x_n) \in E$ if and only if there exists $n_o \in N$ such that (x_n^o) is a sequence in H and $(x_n^o) \in E_H$.
- (ii) Let $x \in G$ such that $x_n \to x$ in G. Then there exists $n_o \in N$ such that either (x_n^o) is a sequence in H (and then $x \in H$) or (x_n^o) is a sequence in H' (and then $x \in H'$).
- (iii) Let $(x_n) \in F$. Then there exists $n_o \in N$ such that either (x_n^o) is a sequence in H (and then $(x_n^o) \in F_H$) or (x_n^o) is a sequence in H'.
- Let (x_n) be a sequence in $H, x \in H$. Then
- (iv) $x_n \to x$ in H if and only if $x_n \to x$ in G.

Let $\varepsilon \in G, \varepsilon \neq 0$. If $[-\varepsilon, 0, \varepsilon] \in T$, then $\varepsilon \in H$. Thus we have:

Lemma 4.5. $E_H \subseteq E$ and $F_H \subseteq F$.

Let a be a fixed element of H'. Every element of H' can be expressed in the form a + x for some $x \in H$.

Lemma 4.6. Let (x_n) be a sequence in $H, x \in H$. Then

- (i) $x_n \to x$ in H if and only if $a + x_n \to a + x$ in G.
- (ii) $x_n \to x$ in H if and only if $a + x_n + a \to a + x + a$ in H.
- (iii) $(x_n) \in F_H$ if and only if some of the following conditions is satisfied $(a + x_n) \in F, (a + x_n + a) \in F_H, (-a + x_n + a) \in F_H.$

Proof. (i) and (ii) are easy to verification.

(iii): Let $(x_n) \in F_H$. We intend to show that $(a + x_n) \in F$. Assume that $\varepsilon \in G, \varepsilon \neq 0, [-\varepsilon, 0, \varepsilon] \in T$. Then $\varepsilon \in H$ and so $-a - \varepsilon + a \in H$. Since $(x_n) \in F_H, [-a + \varepsilon + a, 0, -a - \varepsilon + a] \in T$ implies that there exists $n_o \in N$ such that $[-a + \varepsilon + a, x_n - x_m, -a - \varepsilon + a] \in T$ for each $m, n \in N, m, n \ge n_o$. Therefore $[-\varepsilon, a + x_n - (a + x_m), \varepsilon] \in T$. From $[-\varepsilon, -x_m + x_n, \varepsilon] \in T$ it follows that $[-\varepsilon, -(a + x_m) + a + x_n, \varepsilon] \in T$. We conclude that $(a + x_n) \in F$.

The converse and remaining cases are similar.

Lemma 4.7. G is C-complete if and only if H is C-complete.

Proof. Let G be C-complete and let $(x_n) \in F_H$. In view of Lemma 4.5, we get $(x_n) \in F$. Hence there exists $x \in G$ with $x_n \to x$ in G. Applying Lemma 4.4 (ii) and Lemma 4.4 (iv), we obtain $x \in H$ and $x_n \to x$ in H. Hence H is C-complete.

Let H be C-complete and let $(x_n) \in F$. From Lemma 4.4 (iii), we infer that there exists $n_o \in N$ such that either $(x_n^o) \in F_H$ or $(x_n^o) \in H'$. Assume that $(x_n^o) \in F_H$. Then $x_n^o \to x$ in H. With respect to Lemma 4.4 (iv), $x_n^o \to x$ in G. This yields that $x_n \to x$ in G. Assume that $(x_n^o) \in H'$. There exists $(h_n^o) \in H$ with $x_n^o = a + h_n^o$ for each $n \in N$. Since $(a + h_n^o) \in F$, Lemma 4.6 (iii) implies that $(h_n^o) \in F_H$. Hence, $h_n^o \to h$ in H and by Lemma 4.6 (i) $a + h_n^o \to a + h$ in G. We conclude now that $x_n \to a + h$ in G and the proof is complete.

The following result is an immediate consequence of Lemmas 4.6 and 4.7.

Lemma 4.8. Let G be a subgroup of a half lc-group G_1 . Then G_1 is a Cantor extension of G if and only if $G_1 \uparrow$ is a Cantor extension of H.

Investigating a Cantor extension of G, two cases are distinguished: $H_o \neq \{0\}$ and $H_o = \{0\}$ (H_o is as in the section 2).

5. The case
$$H_o \neq \{0\}$$

In the whole section we suppose that $H_o \neq \{0\}$. Since H_o is infinite, G is infinite as well.

We form the sets

(B)
$$a + H^* = \{a + (x_n)^* : (x_n)^* \in H^*\}, C_h(G) = H^* \cup (a + H^*).$$

Assume that $(x_n) \in F_H$. With respect to Lemma 4.6 (iii), we get $(a + x_n + a) \in F_H$ and $(-a + x_n + a) \in F_H$.

We intend to define a group operation + and a ternary relation T^h on $C_h(G)$. Let $(x_n)^*, (y_n)^*, (z_n)^* \in H^*$.

The operation + on $C_h(G)$ is defined to coincide with the operation + on H^* defined in the section 2, i.e., we put

$$(x_n)^* + (y_n)^* = (x_n + y_n)^*.$$

Further, we put

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$$(a + (x_n)^*) + (a + (y_n)^*) = (a + x_n + a + y_n)^*,$$

$$(x_n)^* + (a + (y_n)^*) = a + (-a + x_n + a + y_n)^*,$$

$$(a + (x_n)^*) + (y_n)^* = a + (x_n + y_n)^*.$$

We define the ternary relation T^h on $C_h(G)$ in such a way that T^h coincides with T^* on H^* .

Further, we put

$$[a + (x_n)^*, a + (y_n)^*, a + (z_n)^*] \in T^h \text{ if } [(z_n)^*, (y_n)^*, (x_n)^*] \in T^*.$$

If p,q and r are distinct elements of $C_h(G)$ such that $[p,q,r] \in T^h$, then either $\{p,q,r\} \subseteq H^*$ or $\{p,q,r\} \subseteq a + H^*$.

Lemma 5.1. $(C_h(G); +)$ is a group.

Proof. First, we verify that the operation + is associative. Only three cases are considered. The remaining cases are similar.

 $((a + (x_n)^*) + (a + (y_n)^*)) + (a + (z_n)^*) = (a + x_n + a + y_n)^* + (a + (z_n)^*) = a + (-a + a + x_n + a + y_n + a + z_n)^* = a + (x_n + a + y_n + a + z_n)^*,$ $(a + (x_n)^*) + ((a + (y_n)^*) + (a + (z_n)^*)) = (a + (x_n^*)) + (a + y_n + a + z_n)^* = a + (x_n + a + y_n + a + z_n)^*.$ Hence, $((a + (x_n)^*) + ((a + (y_n)^*)) + ((a + y_n)^*) + ((a + (y_n)^*)) +$

 $\begin{aligned} &((a+(x_n)^*)+(a+(y_n)^*))+(a+z_n)^*)=(a+(x_n)^*)+((a+(y_n)^*)+(a+(z_n)^*)).\\ &((a+(x_n)^*)+(a+(y_n)^*))+(z_n)^*=(a+x_n+a+y_n)^*+(z_n)^*=(a+x_n+a+y_n+z_n)^*,\\ &(a+(x_n^*)+((a+(y_n)^*)+(z_n)^*)=(a+(x_n)^*)+(a+(y_n+z_n)^*)=(a+x_n+a+y_n+z_n)^*.\end{aligned}$

Thus, $((a+(x_n)^*)+(a+(y_n)^*))+(z_n)^* = (a+(x_n)^*)+((a+(y_n)^*)+(z_n)^*)$. $((x_n)^*+(y_n)^*)+(a+(z_n)^*) = (x_n+y_n)^*+(a+(z_n)^*) = a+(-a+x_n+y_n+a+z_n)^*$, $(x_n)^*+((y_n)^*+(a+(z_n)^*)) = (x_n^*+(a+(-a+y_n+a+z_n)^*) = a+(-a+x_n+a-a+y_n+a+z_n)^* = a+(-a+x_n+y_n+a+z_n)^*$.

Therefore, $((x_n)^* + (y_n)^*) + (a + (z_n)^*) = (x_n)^* + ((y_n)^* + (a + (z_n)^*)).$

Now, we show that every element of $C_h(G)$ has an inverse in $C_h(G)$. It suffices to consider elements of $a + H^*$. Assume that $a + (x_n)^* \in a + H^*$. Then $a + (-a - x_n - a)^* \in a + H^*$ and it is the inverse to $a + (x_n)^*$ in $C_h(G)$.

Lemma 5.2. Let $(x_n)^*, (y_n)^*, (z_n)^* \in H^*$. Then $[(x_n)^*, (y_n)^*, (z_n)^*] \in T^*$ if and only if some of the following conditions is satisfied:

- (i) $[(-a+z_n+a)^*, (-a+y_n+a)^*, (-a+x_n+a)^*] \in T^*,$
- (ii) $[(a+z_n+a)^*, (a+y_n+a)^*, (a+x_n+a)^*] \in T^*.$

Proof. (i): Assume that $[(x_n)^*, (y_n)^*, (z_n)^*] \in T^*$. Hence there exists $n_o \in N$ such that $[x_n, y_n, z_n] \in T$ for each $n \in N, n \ge n_o$. This yields that $[-a+z_n+a, -a+y_n+a, -a+x_n+a] \in T$ for each $n \in N, n \ge n_o$. According to Lemma 4.6 (iii), we have $(-a+z_n+a), (-a+y_n+a), (-a+x_n+a) \in F_H$. We conclude that $[(-a+z_n+a)^*, (-a+y_n+a)^*, (-a+x_n+a)^*] \in T^*$.

The converse and (ii) are similar.

Lemma 5.3. Let $(x_n)^*$, $(y_n)^*$, $(z_n)^*$, $(u_n)^* \in H^*$.

- (i) If $[(x_n)^*, (y_n)^*, z_n)^*] \in T^*$, then $[(x_n)^* + (u_n)^*, (y_n)^* + (u_n)^*, (z_n)^* + (u_n)^*] \in T^*$ and $[(x_n)^* + (a + (u_n)^*), (y_n)^* + (a + (u_n)^*), (z_n)^* + (a + (u_n)^*)] \in T^h$.
- (ii) If $[a + (x_n)^*, a + (y_n)^*, a + (z_n)^*] \in T^h$, then $[(a + (x_n)^*) + (u_n)^*, (a + (y_n)^*) + (u_n)^*, (a + (z_n)^*) + (u_n)^*] \in T^h$ and $[(a + (x_n)^*) + (a + (u_n)^*), (a + (y_n)^*) + (a + (u_n)^*)] \in T^*$.

Proof. (i): Assume that $[(x_n)^*, (y_n)^*, (z_n)^*] \in T^*$. The first part of the assertion follows from the fact that H^* is an *lc*-group. Now, we prove the second part. From Lemma 5.2 (i), we infer that $[(-a+z_n+a)^*, (-a+y_n+a)^*, (-a+x_n+a)^*] \in T^*$. Then $[(-a+z_n+a)^* + (u_n)^*, (-a+y_n+a)^* + (u_n)^*, (-a+x_n+a)^* + (u_n)^*) \in T^*$, $[(-a+z_n+a+u_n)^*, (-a+y_n+a+u_n)^*, (-a+x_n+a+u_n)^*] \in T^*$. Hence $[a+(-a+x_n+a+u_n)^*, a+(-a+y_n+a+u_n)^*, (a+(u_n)^*), (y_n)^* + (a+(u_n)^*), (z_n)^* + (a+(u_n)^*) \in T^h$.

The proof of (ii) is analogous.

Lemma 5.4. Let $(x_n)^*, (y_n)^*, (z_n)^*, (u_n)^* \in H^*$.

- (i) If $[(x_n)^*, (y_n)^*, (z_n)^*] \in T^*$, then $[(u_n)^* + (x_n)^*, (u_n)^* + (y_n)^*, (u_n)^* + (z_n)^*] \in T^*$ and $[(a + (u_n)^*) + (z_n)^*, (a + (u_n)^*) + (y_n)^*, (a + (u_n)^*) + (x_n)^*] \in T^h$.
- (ii) If $[a+(x_n)^*, a+(y_n)^*, a+(z_n)^*] \in T^h$, then $[(u_n)^*+(a+(x_n)^*), (u_n)^*+(a+(y_n)^*), (u_n)^*+(a+(z_n)^*)] \in T^h$ and $[(a+(u_n)^*)+(a+(z_n)^*), (a+(u_n)^*)+(a+(x_n)^*)] \in T^*$.

Proof. (i) Assume that $[(x_n)^*, (y_n)^*, (z_n)^*] \in T^*$. The first assertion holds because of the fact that H^* is an *lc*-group. Now, we prove the second assertion. The assumption implies that $[(u_n)^* + (x_n)^*, (u_n)^* + (y_n)^*, (u_n)^* + (z_n)^*] \in T^*$ and so $[(u_n + x_n)^*, (u_n + y_n)^*, (u_n + z_n)^*] \in T^*$. Whence $[a + (u_n + z_n)^*, a + (u_n + y_n)^*, a + (u_n + x_n)^*] \in T^h$. Thus $[(a + (u_n)^*) + (z_n)^*, (a + (u_n)^*) + (x_n)^*] \in T^h$.

To prove (ii), we proceed in a similar way.

From Lemma 5.4 and (B), we infer the validity of the following result:

Lemma 5.5.
$$C_h(G) \uparrow = H^*, C_h(G) \downarrow = a + H^* and C_h(G) = C_h(G) \uparrow \cup C_h(G) \downarrow$$

Since T is nonempty, T^h is nonempty as well. Then Lemmas 5.1, 5.3 and 5.5 yield:

Lemma 5.6. $(C_h(G), +, T^h)$ is a half lc-group.

Let $x \in H$. Define the mapping ψ from G into $C_h(G)$ by

$$\psi(x) = (x, x, \ldots)^*, \ \psi(a+x) = a + \psi(x).$$

Then ψ is an isomorphism of the half *lc*-group *G* into $C_h(G)$. In the next, we identify x and $\psi(x)$ for each $x \in H$. Then *G* is a half *lc*-subgroup of $C_h(G)$. Since H^* is a Cantor extension of *H*, from Lemma 4.8, we conclude.

Theorem 5.7. $C_h(G)$ is a Cantor extension of G.

Remark that it is easy to verify that (A) implies $X_n \to X$ and $a + X_n \to a + X$ in G.

Theorem 5.8. Let G_1 and G_2 be Cantor extensions of G. Then there exists an isomorphism f from the half lc-group G_1 , onto G_2 such that f(x) = xfor each $x \in G$.

Proof. With respect to 4.8, $G_1 \uparrow$ and $G_2 \uparrow$ are Cantor extension of H. By Theorem 2.6, there exists an isomorphism ϕ from $G_1 \uparrow$ onto $G_2 \uparrow$ with $\phi(x) = x$ for any $x \in H$.

Choose an arbitrary element $z \in G_1 \uparrow$. The mapping $f : G_1 \to G_2$ defined by $f(z) = \phi(z)$ and $f(a + z) = a + \phi(z)$ is an isomorphism of the half *lc*-group G_1 onto G_2 and $f(a + x) = a + \phi(x) = a + x$ for each $x \in H$.

A half lc-group $C_h(G)$ corresponds to an element $a \in H'$. Let $a' \in H'$, $a' \neq a$. Then the half lc-group $(C'_h(G); +', T')$ corresponding to a' can be constructed formally in the same way $(+, T_h \text{ and } a \text{ are replaced by } +', T'$ and a', respectively). Therefore, the operations + and +' (relations T^h and T') coincide on G and H^* . From Theorems 5.7 and 5.8, it follows that $C_h(G)$ and $C'_h(G)$ are isomorphic half lc-groups. Moreover, we have:

Lemma 5.9. A half lc-group $C_h(G) = C'_h(G)$.

Proof. For each $(x_n)^* \in H^*$ we get $a + (x_n)^* = a' + (-a' + a + x_n)^*$. Hence, $a + H^* \subseteq a' + H^*$. Analogously, we get $a' + H^* \subseteq a + H'$. Therefore, the set $C_h(G) = C'_h(G)$.

Evidently, that relations T^h and T' coincide. Now we show that group operations + on $C_h(G)$ and +' on $C'_h(G)$ coincide.

Let $(x_n)^*, (y_n)^* \in H^*$. Then

 $\begin{aligned} (a+(x_n)^*)+(a+y_n)^*) &= (a+x_n+a+y_n)^* = (a'-a'+a+x_n+a'-a'+a+y_n)^* = \\ (a'+'(-a'+a+x_n)^*)+'(a'+'(-a'+a+y_n)^*); \\ (x_n)^*+(a+(y_n)^*) &= a+(-a+x_n+a+y_n)^* = a'+'(-a'+a-a+x_n+a+y_n)^* = \\ a'+'(-a'+x_n+a'-a'+a+y_n)^* &= (x_n)^*+'(a'+'(-a'+a+y_n)^*); \\ (a+(x_n)^*)+(y_n)^* &= a+(x_n+y_n)^* = a'+'(-a'+a+x_n+y_n)^* = \\ a'+'((-a'+a+x_n)^*)+(y_n)^*). \end{aligned}$

6. The case $H_o = \{0\}$

In this section, we assume that $H_o = \{0\}$. Then H can be considered as a subgroup of K.

Assume that G is a finite half lc-group. Then H is a finite lc-group. With respect to Lemmas 2.4 and 4.7, we obtain:

Lemma 6.1. Let G be a finite half lc-group. Then G is C-complete.

Now, assume that G is an infinite half lc-group. Then H is an infinite lc-group.

Let a be a fixed element of H'. We denote

$$a + K = \{a + x : x \in K\};$$
$$C_h(G) = K \cup (a + K).$$

We will define a group operation + and a ternary relation T^h on $C_h(G)$.

Let $x, y, z \in K$. From Lemma 2.5, we infer that there are fundamental sequences (x_n) and (y_n) in G such that $\lim x_n = x, \lim y_n = y$ in K.

The operation x + y on $C_h(G)$ coincides with x + y on K.

Further, we put

$$(a + x) + (a + y) = lim(a + x_n + a + y_n),$$

$$x + (a + y) = a + lim(-a + x_n + a + y_n),$$

$$(a + x) + y = a + lim(x_n + y_n).$$

Limits are taken into account in K. The operation + is correctly defined.

The ternary relation T^h on $C_h(G)$ is defined in the following way:

 T^h coincides with T_1 on K.

Further, we put

$$[a+x, a+y, a+z] \in T^{h} \text{if}[x, y, z] \in T_{1},$$

if $p,q,r \in C_h(G)$, $[p,q,r] \in T^h$, then either $\{p,q,r\} \subseteq K$ or $\{p,q,r\} \subseteq a+K$.

It is a routine to verify that the following assertion is true:

Lemma 6.2. $(C_h(G), +, T^h)$ is a half lc-group, $C_h(G) \uparrow = K$, $C_h(G) \downarrow = a + K$.

From Lemmas 2.5 and 4.7, it follows:

Lemma 6.3. Let G be an infinite half lc-group. Then $C_h(G)$ is a Cantor extension of G.

Let a' and $C'_h(G)$ be as in the Section 5.

Remark 6.4. It is easy to verify that Theorem 5.8 and Lemma 5.9 are valid also in the case $H_o = \{0\}$.

From Lemmas 4.7, 2.5 and Theorem 2.6, we obtain:

Lemma 6.5. Let G be an infinite half lc-group. Then G is C-complete if and only if H is isomorphic to K.

Let G be an arbitrary *lc*-group as in the section 4 (neither $H_o \neq \{0\}$ nor $H_o = \{0\}$ is supposed). From Lemmas 6.1, 6.5 and 2.3, we conclude:

Theorem 6.6. Let G be a half lc-group such that H is abelian and $H' \neq \emptyset$. Then G is C-complete if and only if some of the following conditions is fulfilled:

- (i) G is finite;
- (ii) H is isomorphic to K;
- (iii) $H_o \neq \{0\}$ and H_o is C-complete.

By summarizing Theorem 5.7, Lemma 6.3, Theorem 5.8, and Remark 6.4 we get:

Theorem 6.7. Let G be a half lc-group such that H is abelian and $H' \neq \emptyset$. Then

- (i) There exists a Cantor extension of G.
- (ii) If G_1 and G_2 are Cantor extensions of G, then there exists an isomorphism from the half lc-group G_1 onto G_2 leaving all elements of G fixed.

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