Discussiones Mathematicae
General Algebra and Applications 21 (2001) 31–46

CANTOR EXTENSION OF A HALF LINEARLY CYCLICALLY ORDERED GROUP

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Abstract

Convergent and fundamental sequences are studied in a half linearly cyclically ordered group G with the abelian increasing part. The main result is the construction of the Cantor extension of G.

Keywords: convergent sequence, fundamental sequence, C-complete half lc-group, Cantor extension of a half lc-group.

2000 Mathematics Subject Classification: Primary 06F15; Secondary 20F60.

M. Giraudet and F. Lucas [3] introduced and investigated the notion of a half linearly ordered group (cf. also D.R. Ton [14], J. Jakubík [6], [7]). J. Jakubík [8] defined and studied the notion of a half linearly cyclically ordered group (lc-group) generalizing the notion of a half linearly ordered group.

The author [1] investigated the Cantor extension of an abelian lc-group. We remark that the Cantor extension of lattice ordered groups was studied by C.J. Everett [2].

Let G be a half lc-group such that its increasing part is abelian and its decreasing part is nonempty (thus G fails to be an lc-group). The notions of a convergent sequence and a fundamental sequence are defined in a natural way. If every fundamental sequence in G is convergent in G, then G is said to be C-complete.

In the present paper necessary and sufficient conditions are found under which G is C-complete. Further, we define the notion of a Cantor extension and we prove that every half lc-group has a Cantor extension which is uniquely determined up to isomorphisms leaving all elements of G fixed.

1. l-cyclically ordered sets and groups

We recall the definitions and some results concerning l-cyclically ordered sets (cf. Novák and Novotný [10], Novák [9], Quilot [11]) and l-cyclically ordered groups (cf. Rieger [12], Świerczkowski [13], Jakubík and Pringerová [4], [5]).

Definition 1.1. Let M be a nonempty set and T a ternary relation on M such that the following conditions are satisfied:

(I) if $[x, y, z] \in T$ then $[y, x, z] \notin T$.

(II) $[x, y, z] \in T$ implies $[y, z, x] \in T$.

(III) $[x, y, z] \in T, [y, u, z] \in T$ imply $[x, u, z] \in T$.

Then T is said to be a *cyclic order* on M and (M, T) is called a *cyclically ordered set.*

Let T be a cyclic order on M satisfying the condition:

(IV) if $x, y, z \in M, x \neq y \neq z \neq x$, then either $[x, y, z] \in T$ or $[z, y, x] \in T$.

Then T is said to be an *l-cyclic order* on M and (M, T) is called an *l-cyclically ordered set.*

Several terms are used in papers for the term l-cyclic order. For instance "l-cyclic order" is called "linear cyclic order" in [9], "complete cyclic order" in [11] and simply "cyclic order" in [12] and [13].

Definition 1.2. Let $(H; +)$ be a group and $(H; T)$ an l-cyclically ordered set such that the following condition is fulfilled:

(V) if $[x, y, z] \in T, u, v \in H$, then $[u + x + v, u + y + v,\ u + z + v] \in T$.

Then $(H; +, T)$ is said to be an *l-cyclically ordered group* or *lc-group (linearly cyclically ordered group).*

We often write H or $(H; T)$ instead of $(H; +, T)$.

Every subgroup of an lc-group is considered as an lc-group under the induced l-cyclic order.

Example 1.3. Let $(L; \leq)$ be a linearly ordered group $x, y, z \in L$. Define the ternary relation T_L on L by putting

$$[x, y, z] \in T_L \text{ if } x < y < z \text{ or } y < z < x \text{ or } z < x < y.$$

Then $(L; T_L)$ is an lc-group. T_L is called the l-cyclic order generated by the linear order $\leq$ on L. Hence every linearly ordered group is an lc-group (under the l-cyclic order generated by its linear order).

Example 1.4. Let K be the group of all reals k such that $0 \leq k < 1$ with the group operation defined as the addition mod 1. Consider the natural linear order $\leq$ and the ternary relation T_1 on K defined in the same way as T_L in 1.3. Then $(K; T_1)$ is an lc-group.

Define the ternary relation T on the direct product $L \times K$ of groups L and K as follows: for elements $u_1, u_2, u_3 \in L \times K$, $u_1 = (x, k_1)$, $u_2 = (y, k_2)$, $u_3 = (z, k_3)$ we put $[u_1, u_2, u_3] \in T$ if some of the following conditions is valid:

(i) $[k_1, k_2, k_3] \in T_1$;

(ii) $k_1 = k_2 \neq k_3$ and $x < y$;

(iii) $k_2 = k_3 \neq k_1$ and $y < z$;

(iv) $k_3 = k_1 \neq k_2$ and $z < x$;

(v) $k_1 = k_2 = k_3$ and $[x, y, z] \in T_L$.

Then $(L \times K; T)$ is an lc-group which will be denoted by $L \otimes K$.

The notion of an isomorphism of lc-groups is defined in a natural way.

Theorem 1.5 (Świerczkowski [13]). *Let H be an lc-group. Then there exists a linearly ordered group L such that H is isomorphic to a subgroup of $L \otimes K$.*

Assume that $(H; T)$ is an lc-group. By 1.5, there exists an isomorphism f of H into $L \otimes K$. Let H_o be the set of all $h \in H$ such that there exists $x \in L$ with the property $f(h) = (x, 0)$. Then H_o is a subgroup of $H, H_o = \{0\}$ or $H_o \neq \{0\}$. Let $H_o \neq \{0\}$, $h \in H_o$, $h \neq 0$. There exists $x \in L$ such that $f(h) = (x, 0)$. H_o turns out to be a linearly ordered grup if we put $h > 0$ if $x > 0$. The l-cyclic order T_{H_o} on H_o coincides with the l-cyclic order induced by T.

2. Cantor extension of an abelian lc-group

Let $(H; T)$ be an abelian lc-group. A construction of a Cantor extension of H will be described (cf. [1]) and some results from [1] will be presented.

Definition 2.1. Let (x_n) be a sequence in H and $x \in H$.

a) We say that (x_n) *converges* to x (or x is a limit of (x_n)) in H and we write $x_n \to x$ (or $lim x_n = x$)

(i) if card $H = 2$ and there exists $n_o \in N$ such that $x_n = x$ for each $n \in N, n \geq n_o$,

or

(ii) if card $H > 2$ and for each $\varepsilon \in H, \varepsilon \neq 0$ with the property

$[-\varepsilon, 0, \varepsilon] \in T$ there exists $n_o \in N$ such that $[-\varepsilon, x_n - x, \varepsilon] \in T$ for each $n \in N, n \geq n_o$.

b) The sequence (x_n) is called *fundamental* in H if for each $\varepsilon \in H, \varepsilon \neq 0$ with the property $[-\varepsilon, 0, \varepsilon] \in T$ there exists $n_o \in N$ such that $[-\varepsilon, x_n - x_m, \varepsilon] \in T$ for each $m, n \in N, m, n \geq n_o$.

c) By a *zero sequence* we understand a sequence (x_n) such that $x_n \to 0$.

d) H is called *C-complete* if each fundamental sequence in H is convergent in H.

The set of all fundamental (zero) sequences in H will be denoted by $F_H(E_H)$.

Definition 2.2. Let H_1 be an abelian lc-group satisfying the following conditions:

(a) H_1 is C-complete.

(b) H is a subgroup of H_1.

(c) Every element of H_1 is a limit of some fundamental sequence in H.

(d) Let (x_n) be a sequence in H such that $x_n \to 0$ in H. Then $x_n \to 0$ in H_1.

Then H_1 is said to be a *Cantor extension* of H.

Now we consider two cases: $H_0 \neq \{0\}$ and $H_0 = \{0\}$.

1) Assume that $H_o \neq \{0\}$. Let $(x_n), (y_n) \in F_H$. Under the natural definition of the operation $+$ on $F_H, (x_n) + (y_n) = (x_n + y_n), F_H$ is a group and E_H is a subgroup of F_H. We form the factor group $H^* = F_H/E_H$. Symbol $(x_n)^*$ will denote the coset of H^* containing the sequence $(x_n) \in F_H$.

Suppose that $(x_n)^*, (y_n)^*, (z_n)^*$ are mutually distinct elements of H^*. Let T^* be the set of all triples $[(x_n)^*, (y_n)^*, (z_n)^*]$ of elements of H^* such that there exists $n_o \in N$ with $[x_n, y_n, z_n] \in T$ for each $n \in N, n \geq n_o$. Then (H^*, T^*) is an lc-group.

Let φ be the mapping from H into H^* defined by $\varphi(x) = (x, x, \ldots)^*$ for each $x \in H$. Then φ is an isomorphism of the lc-group H into H^*. We identify x and $\varphi(x)$ for each $x \in H$. Then H is a subgroup of H^* and H^* is a Cantor extension of H.

If we denote $(x_n)^* = X$ and $(x_n, x_n, \ldots)^* = X_n$, then we have (cf. [1], the proof of Lemma 3.12)

$$X_n \to X \ \text{ in } \ H^*. \tag{A}$$

Lemma 2.3 ([1], Lemma 3.9). *H is C-complete if and only if H_o is C-complete.*

2) Now assume that $H_o = \{0\}$. Then H can be considered as a subgroup of K.

Lemma 2.4 ([1], Lemma 4.2). *If H is a finite subgroup of K, then H is C-complete.*

Lemma 2.5 ([1], Lemma 4.5). *If H is an infinite subgroup of K, then K is a Cantor extension of H.*

The following result is valid in both cases 1) and 2).

Theorem 2.6 ([1], Theorem 4.9). *Let H be an abelian lc-group. Then*

(i) *there exists a Cantor extension of H,*

(ii) *if H_1 and H_2 are Cantor extensions of H, then there exists an isomorphism Φ from the lc-group H_1 onto H_2 such that $\Phi(x) = x$ for each $x \in H$.*

3. Half lc-groups

The notion of a half lc-group was introduced by Jakubík [8]. Now we recall the definitions and results that will be applied in the next sections.

Let $(G; +, T)$ be a system such that $(G; +)$ is a group and $(G; T)$ is a cyclically ordered set. Assume that $x, y, z \in G$. Denote

$$G\!\uparrow = \{u \in G : [x, y, z] \in T \Rightarrow [u + x, u + y, u + z] \in T\},$$

$$G\!\downarrow = \{u \in G : [x, y, z] \in T \Rightarrow [u + z, u + y, u + x] \in T\}.$$

Definition 3.1. Let $(G;+,T)$ be as above. Assume that the following conditions are fulfilled:

(1) The system T is nonempty.

(2) If $[x,y,z]\in T$, then $[x+u,y+u,z+u]\in T$ for each $u\in G$.

(3) $G=G{\uparrow}\cup G{\downarrow}$.

(4) If $[x,y,z]\in T$, then either $\{x,y,z\}\subseteq G{\uparrow}$ or $\{x,y,z\}\subseteqq G{\downarrow}$.

Then $(G;+,T)$ is said to be a *half cyclically ordered group.*

Let $(G;+,T)$ be a half cyclically ordered group. The definition implies that $G{\uparrow}$ is a cyclically ordered group. If $G{\uparrow}$ is an lc-group then $(G;+,T)$ is called a *half lc-group* (half linearly cyclically ordered group).
There are elements $x,y,z\in G$ with $[x,y,z]\in T$. This is an immediate consequence of (1).

Again, we often write G or $(G;T)$ instead of $(G;+,T)$.
In the next, let G be a half lc-group. $G\uparrow$ $(G\downarrow)$ is called the *increasing (decreasing, resp.)* part of G.

A subgroup G' of G is said to be a *half lc-subgroup* of G if the *induced* l-cyclic order on G' is nonempty.

Each lc-group G with card $G\geq 3$ is a half lc-group (with $G{\uparrow}=G$ and $G{\downarrow}=\emptyset$). Every linearly ordered group is an lc-group. Hence every half linearly ordered group (for the definition cf. [3]) is a half lc-group.

The notion of an isomorphism of half lc-groups is defined in a natural way.

From the definition 3.1 it follows (cf. [8]):

(i) If $x,y\in G{\downarrow}$, then $x+y\in G{\uparrow}$;

(ii) If $x\in G{\uparrow}, y\in G{\downarrow}$, then $x+y\in G{\downarrow}$ and $y+x\in G{\downarrow}$.

4. Cantor extension of a half lc-group

In what follows, we assume that (G,T) is a half lc-group such that $G\uparrow$ is abelian and $G{\downarrow}\neq\emptyset$. Hence G is neither abelian group nor lc-group.

We will use the notation $G{\uparrow}=H$ and $G{\downarrow}=H'$.

Definition 4.1. Let (x_n) be a sequence in G and $x \in G$.

- **a)** We say that (x_n) *converges* to x (or x is a *limit* of (x_n)) in G and we write $x_n \to x$ (or $lim x_n = x$) if for each $\varepsilon \in G, \varepsilon \neq 0$ with the property $[-\varepsilon, 0, \varepsilon] \in T$ there exists $n_o \in N$ such that $[-\varepsilon, x_n - x, \varepsilon] \in T$ and $[-\varepsilon, -x + x_n, \varepsilon] \in T$ for each $n \in N, n \geq n_o$.
- **b)** The sequence (x_n) is said to be *fundamental* if for each $\varepsilon \in G, \varepsilon \neq 0$ with $[-\varepsilon, 0, \varepsilon] \in T$ there exists $n_o \in N$ such that $[-\varepsilon, x_n - x_m, \varepsilon] \in T$ and $[-\varepsilon, -x_m + x_n, \varepsilon] \in T$ for each $m, n \in N, m, n \geq n_o$.
- **c)** If $x_n \to 0$ in G, then (x_n) is called a *zero sequence* in G.
- **d)** G is said to be *C-complete* if every fundamental sequence in G is convergent in G.

Definition 4.2. Let G_1 be a half *lc*-group with the following properties:

- (α) G_1 is C-complete;
- (β) G is a half *lc*-subgroup of G_1;
- (γ) Every element of G_1 is a limit of some fundamental sequence in G;
- (δ) Let (x_n) be a sequence in G such that $x_n \to 0$ in G. Then $x_n \to 0$ in G_1.

Then G_1 is said to be a *Cantor extension* of G.

We prove that G has a Cantor extension and this is uniquely determined up to isomorphisms leaving all elements of G fixed.

Denote by $F(E)$ the set of all fundamental (zero) sequences in G. Symbols F_H and E_H have the same meaning as in the section 2.

The following two lemmas are easy to prove.

Lemma 4.3. *Let (x_n) be a sequence in G. Then $x_n \to x$ in G if and only if $x_n - x \to 0$ and $-x + x_n \to 0$ in G.* ∎

For a fixed element $n_o \in N$ and a sequence (x_n) in G we apply the notation $x_n^o = x_{no+n-1}$ for each $n \in N$.

Lemma 4.4. *Let (x_n) be a sequence in G.*

(i) $(x_n) \in E$ *if and only if there exists* $n_o \in N$ *such that* (x_n^o) *is a sequence in* H *and* $(x_n^o) \in E_H$.

(ii) *Let* $x \in G$ *such that* $x_n \to x$ *in* G. *Then there exists* $n_o \in N$ *such that either* (x_n^o) *is a sequence in* H *(and then* $x \in H$*) or* (x_n^o) *is a sequence in* H' *(and then* $x \in H'$*).*

(iii) *Let* $(x_n) \in F$. *Then there exists* $n_o \in N$ *such that either* (x_n^o) *is a sequence in* H *(and then* $(x_n^o) \in F_H$*) or* (x_n^o) *is a sequence in* H'.

Let (x_n) *be a sequence in* $H, x \in H$. *Then*

(iv) $x_n \to x$ *in* H *if and only if* $x_n \to x$ *in* G. ■

Let $\varepsilon \in G, \varepsilon \neq 0$. If $[-\varepsilon, 0, \varepsilon] \in T$, then $\varepsilon \in H$. Thus we have:

Lemma 4.5. $E_H \subseteq E$ *and* $F_H \subseteq F$. ■

Let a be a fixed element of H'. Every element of H' can be expressed in the form $a + x$ for some $x \in H$.

Lemma 4.6. *Let* (x_n) *be a sequence in* $H, x \in H$. *Then*

(i) $x_n \to x$ *in* H *if and only if* $a + x_n \to a + x$ *in* G.

(ii) $x_n \to x$ *in* H *if and only if* $a + x_n + a \to a + x + a$ *in* H.

(iii) $(x_n) \in F_H$ *if and only if some of the following conditions is satisfied* $(a + x_n) \in F, (a + x_n + a) \in F_H, (-a + x_n + a) \in F_H$.

Proof. (i) and (ii) are easy to verification.

(iii): Let $(x_n) \in F_H$. We intend to show that $(a + x_n) \in F$. Assume that $\varepsilon \in G, \varepsilon \neq 0, [-\varepsilon, 0, \varepsilon] \in T$. Then $\varepsilon \in H$ and so $-a - \varepsilon + a \in H$. Since $(x_n) \in F_H, [-a + \varepsilon + a, 0, -a - \varepsilon + a] \in T$ implies that there exists $n_o \in N$ such that $[-a+\varepsilon+a, x_n - x_m, -a-\varepsilon+a] \in T$ for each $m, n \in N, m, n \geqslant n_o$. Therefore $[-\varepsilon, a + x_n - (a + x_m), \varepsilon] \in T$. From $[-\varepsilon, -x_m + x_n, \varepsilon] \in T$ it follows that $[-\varepsilon, -(a+x_m)+a+x_n, \varepsilon] \in T$. We conclude that $(a+x_n) \in F$.

The converse and remaining cases are similar. ■

Lemma 4.7. *G is C-complete if and only if H is C-complete.*

Proof. Let G be C-complete and let $(x_n) \in F_H$. In view of Lemma 4.5, we get $(x_n) \in F$. Hence there exists $x \in G$ with $x_n \to x$ in G. Applying Lemma 4.4 (ii) and Lemma 4.4 (iv), we obtain $x \in H$ and $x_n \to x$ in H. Hence H is C-complete.

Let H be C-complete and let $(x_n) \in F$. From Lemma 4.4 (iii), we infer that there exists $n_o \in N$ such that either $(x_n^o) \in F_H$ or $(x_n^o) \in H'$. Assume that $(x_n^o) \in F_H$. Then $x_n^o \to x$ in H. With respect to Lemma 4.4 (iv), $x_n^o \to x$ in G. This yields that $x_n \to x$ in G. Assume that $(x_n^o) \in H'$. There exists $(h_n^o) \in H$ with $x_n^o = a + h_n^o$ for each $n \in N$. Since $(a + h_n^o) \in F$, Lemma 4.6 (iii) implies that $(h_n^o) \in F_H$. Hence, $h_n^o \to h$ in H and by Lemma 4.6 (i) $a + h_n^o \to a + h$ in G. We conclude now that $x_n \to a + h$ in G and the proof is complete. ∎

The following result is an immediate consequence of Lemmas 4.6 and 4.7.

Lemma 4.8. *Let G be a subgroup of a half lc-group G_1. Then G_1 is a Cantor extension of G if and only if $G_1 \uparrow$ is a Cantor extension of H.* ∎

Investigating a Cantor extension of G, two cases are distinguished: $H_o \neq \{0\}$ and $H_o = \{0\}$ (H_o is as in the section 2).

5. THE CASE $H_o \neq \{0\}$

In the whole section we suppose that $H_o \neq \{0\}$. Since H_o is infinite, G is infinite as well.

We form the sets

$$\text{(B)} \qquad \begin{aligned} &a + H^* = \{a + (x_n)^* : (x_n)^* \in H^*\}, \\ &C_h(G) = H^* \cup (a + H^*). \end{aligned}$$

Assume that $(x_n) \in F_H$. With respect to Lemma 4.6 (iii), we get $(a + x_n + a) \in F_H$ and $(-a + x_n + a) \in F_H$.

We intend to define a group operation $+$ and a ternary relation T^h on $C_h(G)$. Let $(x_n)^*, (y_n)^*, (z_n)^* \in H^*$.

The operation $+$ on $C_h(G)$ is defined to coincide with the operation $+$ on H^* defined in the section 2, i.e., we put

$$(x_n)^* + (y_n)^* = (x_n + y_n)^*.$$

Further, we put

$$(a + (x_n)^*) + (a + (y_n)^*) = (a + x_n + a + y_n)^*,$$
$$(x_n)^* + (a + (y_n)^*) = a + (-a + x_n + a + y_n)^*,$$
$$(a + (x_n)^*) + (y_n)^* = a + (x_n + y_n)^*.$$

We define the ternary relation T^h on $C_h(G)$ in such a way that T^h coincides with T^* on H^*.

Further, we put

$$[a + (x_n)^*,\ a + (y_n)^*,\ a + (z_n)^*] \in T^h \ if\ [(z_n)^*, (y_n)^*, (x_n)^*] \in T^*.$$

If p, q and r are distinct elements of $C_h(G)$ such that $[p, q, r] \in T^h$, then either $\{p, q, r\} \subseteq H^*$ or $\{p, q, r\} \subseteq a + H^*$.

Lemma 5.1. $(C_h(G); +)$ *is a group.*

Proof. First, we verify that the operation + is associative. Only three cases are considered. The remaining cases are similar.
$((a+(x_n)^*)+(a+(y_n)^*))+(a+(z_n)^*) = (a+x_n+a+y_n)^*+(a+(z_n)^*) = a+(-a+a+x_n+a+y_n+a+z_n)^* = a+(x_n+a+y_n+a+z_n)^*$,

$(a+(x_n)^*)+((a+(y_n)^*)+(a+(z_n)^*)) = (a+(x_n^*))+(a+y_n+a+z_n)^* = a+(x_n+a+y_n+a+z_n)^*$.

Hence,
$((a+(x_n)^*)+(a+(y_n)^*))+(a+z_n)^*) = (a+(x_n)^*)+((a+(y_n)^*)+(a+(z_n)^*))$.
$((a+(x_n)^*)+(a+(y_n)^*))+(z_n)^* = (a+x_n+a+y_n)^*+(z_n)^* = (a+x_n+a+y_n+z_n)^*, (a+(x_n^*)+((a+(y_n)^*)+(z_n)^*) = (a+(x_n)^*)+(a+(y_n+z_n)^*) = (a+x_n+a+y_n+z_n)^*$.

Thus, $((a+(x_n)^*)+(a+(y_n)^*))+(z_n)^* = (a+(x_n)^*)+((a+(y_n)^*)+(z_n)^*)$.
$((x_n)^*+(y_n)^*)+(a+(z_n)^*) = (x_n+y_n)^*+(a+(z_n)^*) = a+(-a+x_n+y_n+a+z_n)^*, (x_n)^*+((y_n)^*+(a+(z_n)^*)) = (x_n^*+(a+(-a+y_n+a+z_n)^*) = a+(-a+x_n+a-a+y_n+a+z_n)^* = a+(-a+x_n+y_n+a+z_n)^*$.

Therefore, $((x_n)^*+(y_n)^*)+(a+(z_n)^*) = (x_n)^*+((y_n)^*+(a+(z_n)^*))$.

Now, we show that every element of $C_h(G)$ has an inverse in $C_h(G)$. It suffices to consider elements of $a + H^*$. Assume that $a + (x_n)^* \in a + H^*$. Then $a+(-a-x_n-a)^* \in a+H^*$ and it is the inverse to $a+(x_n)^*$ in $C_h(G)$. ■

Lemma 5.2. *Let* $(x_n)^*, (y_n)^*, (z_n)^* \in H^*$. *Then* $[(x_n)^*, (y_n)^*, (z_n)^*] \in T^*$ *if and only if some of the following conditions is satisfied:*

(i) $[(-a+z_n+a)^*, (-a+y_n+a)^*, (-a+x_n+a)^*] \in T^*$,

(ii) $[(a+z_n+a)^*, (a+y_n+a)^*, (a+x_n+a)^*] \in T^*$.

Proof. (i): Assume that $[(x_n)^*, (y_n)^*, (z_n)^*] \in T^*$. Hence there exists $n_o \in N$ such that $[x_n, y_n, z_n] \in T$ for each $n \in N, n \geq n_o$. This yields that $[-a+z_n+a, -a+y_n+a, -a+x_n+a] \in T$ for each $n \in N, n \geq n_o$. According to Lemma 4.6 (iii), we have $(-a+z_n+a), (-a+y_n+a), (-a+x_n+a) \in F_H$. We conclude that $[(-a+z_n+a)^*, (-a+y_n+a)^*, (-a+x_n+a)^*] \in T^*$.

The converse and (ii) are similar. ■

Lemma 5.3. *Let* $(x_n)^*,\ (y_n)^*,\ (z_n)^*,\ (u_n)^* \in H^*$.

(i) *If* $[(x_n)^*, (y_n)^*, z_n)^*] \in T^*$, *then* $[(x_n)^* + (u_n)^*,\ (y_n)^* + (u_n)^*,\ (z_n)^* + (u_n)^*] \in T^*$ *and* $[(x_n)^* + (a + (u_n)^*),\ (y_n)^* + (a + (u_n)^*),\ (z_n)^* + (a + (u_n)^*)] \in T^h$.

(ii) *If* $[a + (x_n)^*,\ a + (y_n)^*,\ a + (z_n)^*] \in T^h$, *then* $[(a + (x_n)^*) + (u_n)^*,\ (a+(y_n)^*)+(u_n)^*,\ (a+(z_n)^*)+(u_n)^*] \in T^h$ *and* $[(a+(x_n)^*)+(a+(u_n)^*),\ (a + (y_n)^*) + (a + (u_n)^*),\ (a + (z_n)^*) + (a + (u_n)^*)] \in T^*$.

Proof. (i): Assume that $[(x_n)^*, (y_n)^*, (z_n)^*] \in T^*$. The first part of the assertion follows from the fact that H^* is an lc-group. Now, we prove the second part. From Lemma 5.2 (i), we infer that $[(-a+z_n+a)^*,\ (-a+y_n+a)^*,\ (-a+x_n+a)^*] \in T^*$. Then $[(-a+z_n+a)^* + (u_n)^*,\ (-a+y_n+a)^* + (u_n)^*,\ (-a+x_n+a)^* + (u_n)^*) \in T^*$, $[(-a+z_n+a+u_n)^*,\ (-a+y_n+a+u_n)^*,\ (-a+x_n+a+u_n)^*] \in T^*$. Hence $[a+(-a+x_n+a+u_n)^*,\ a+(-a+y_n+a+u_n)^*,\ a+(-a+z_n+a+u_n)^*] \in T^h$, i.e., $[(x_n)^*+(a+(u_n)^*),\ (y_n)^*+(a+(u_n)^*),\ (z_n)^*+(a+(u_n)^*) \in T^h$.

The proof of (ii) is analogous. ■

Lemma 5.4. *Let* $(x_n)^*, (y_n)^*, (z_n)^*, (u_n)^* \in H^*$.

(i) *If* $[(x_n)^*, (y_n)^*, (z_n)^*] \in T^*$, *then* $[(u_n)^* + (x_n)^*, (u_n)^* + (y_n)^*, (u_n)^* + (z_n)^*] \in T^*$ *and* $[(a + (u_n)^*) + (z_n)^*, (a + (u_n)^*) + (y_n)^*, (a + (u_n)^*) + (x_n)^*] \in T^h$.

(ii) *If* $[a+(x_n)^*, a+(y_n)^*, a+(z_n)^*] \in T^h$, *then* $[(u_n)^*+(a+(x_n)^*), (u_n)^*+(a+(y_n)^*), (u_n)^*+(a+(z_n)^*)] \in T^h$ *and* $[(a+(u_n)^*)+(a+(z_n)^*), (a+(u_n)^*) + (a + (y_n)^*), (a + (u_n)^*) + (a + (x_n)^*)] \in T^*$.

Proof. (i) Assume that $[(x_n)^*, (y_n)^*, (z_n)^*] \in T^*$. The first assertion holds because of the fact that H^* is an lc-group. Now, we prove the second assertion. The assumption implies that $[(u_n)^* + (x_n)^*, (u_n)^* + (y_n)^*, (u_n)^* + (z_n)^*] \in T^*$ and so $[(u_n + x_n)^*, (u_n + y_n)^*, (u_n + z_n)^*] \in T^*$. Whence $[a + (u_n + z_n)^*, a + (u_n + y_n)^*, a + (u_n + x_n)^*] \in T^h$. Thus $[(a + (u_n)^*) + (z_n)^*, (a + (u_n)^*) + (y_n)^*, (a + (u_n)^*) + (x_n)^*] \in T^h$.

To prove (ii), we proceed in a similar way. ∎

From Lemma 5.4 and (B), we infer the validity of the following result:

Lemma 5.5. $C_h(G)\uparrow = H^*, C_h(G)\downarrow = a + H^*$ *and* $C_h(G) = C_h(G)\uparrow \cup C_h(G)\downarrow$. ∎

Since T is nonempty, T^h is nonempty as well. Then Lemmas 5.1, 5.3 and 5.5 yield:

Lemma 5.6. $(C_h(G), +, T^h)$ *is a half lc-group.* ∎

Let $x \in H$. Define the mapping ψ from G into $C_h(G)$ by

$$\psi(x) = (x, x, \ldots)^*, \ \psi(a + x) = a + \psi(x).$$

Then ψ is an isomorphism of the half lc-group G into $C_h(G)$. In the next, we identify x and $\psi(x)$ for each $x \in H$. Then G is a half lc-subgroup of $C_h(G)$. Since H^* is a Cantor extension of H, from Lemma 4.8, we conclude.

Theorem 5.7. $C_h(G)$ *is a Cantor extension of* G. ∎

Remark that it is easy to verify that (A) implies $X_n \to X$ and $a + X_n \to a + X$ in G.

Theorem 5.8. *Let* G_1 *and* G_2 *be Cantor extensions of* G. *Then there exists an isomorphism* f *from the half lc-group* G_1, *onto* G_2 *such that* $f(x) = x$ *for each* $x \in G$.

Proof. With respect to 4.8, $G_1\uparrow$ and $G_2\uparrow$ are Cantor extension of H. By Theorem 2.6, there exists an isomorphism ϕ from $G_1\uparrow$ onto $G_2\uparrow$ with $\phi(x) = x$ for any $x \in H$.

Choose an arbitrary element $z \in G_1\uparrow$. The mapping $f : G_1 \to G_2$ defined by $f(z) = \phi(z)$ and $f(a + z) = a + \phi(z)$ is an isomorphism of the half lc-group G_1 onto G_2 and $f(a + x) = a + \phi(x) = a + x$ for each $x \in H$. ∎

A half lc-group $C_h(G)$ corresponds to an element $a \in H'$. Let $a' \in H'$, $a' \neq a$. Then the half lc-group $(C'_h(G); +', T')$ corresponding to a' can be constructed formally in the same way $(+, T_h$ and a are replaced by $+', T'$ and a', respectively). Therefore, the operations $+$ and $+'$ (relations T^h and T') coincide on G and H^*. From Theorems 5.7 and 5.8, it follows that $C_h(G)$ and $C'_h(G)$ are isomorphic half lc-groups. Moreover, we have:

Lemma 5.9. *A half lc-group* $C_h(G) = C'_h(G)$.

Proof. For each $(x_n)^* \in H^*$ we get $a + (x_n)^* = a' +' (-a' + a + x_n)^*$. Hence, $a + H^* \subseteq a' +' H^*$. Analogously, we get $a' +' H^* \subseteq a + H'$. Therefore, the set $C_h(G) = C'_h(G)$.

Evidently, that relations T^h and T' coincide. Now we show that group operations $+$ on $C_h(G)$ and $+'$ on $C'_h(G)$ coincide.

Let $(x_n)^*, (y_n)^* \in H^*$. Then
$(a+(x_n)^*)+(a+y_n)^*) = (a+x_n+a+y_n)^* = (a'-a'+a+x_n+a'-a'+a+y_n)^* = (a' +' (-a' + a + x_n)^*) +' (a' +' (-a' + a + y_n)^*)$;
$(x_n)^*+(a+(y_n)^*) = a+(-a+x_n+a+y_n)^* = a'+'(-a'+a-a+x_n+a+y_n)^* = a' +' (-a' + x_n + a' - a' + a + y_n)^* = (x_n)^* +' (a' +' (-a' + a + y_n)^*)$;
$(a + (x_n)^*) + (y_n)^* = a + (x_n + y_n)^* = a' +' (-a' + a + x_n + y_n)^* = a' +' ((-a' + a + x_n)^*) + (y_n)^*)$. ∎

6. The case $H_o = \{0\}$

In this section, we assume that $H_o = \{0\}$. Then H can be considered as a subgroup of K.

Assume that G is a finite half lc-group. Then H is a finite lc-group. With respect to Lemmas 2.4 and 4.7, we obtain:

Lemma 6.1. *Let G be a finite half lc-group. Then G is C-complete.*

∎

Now, assume that G is an infinite half lc-group. Then H is an infinite lc-group.

Let a be a fixed element of H'. We denote

$$a + K = \{a + x : x \in K\};$$

$$C_h(G) = K \cup (a + K).$$

We will define a group operation $+$ and a ternary relation T^h on $C_h(G)$.

Let $x, y, z \in K$. From Lemma 2.5, we infer that there are fundamental sequences (x_n) and (y_n) in G such that $lim\ x_n = x, lim\ y_n = y$ in K.

The operation $x + y$ on $C_h(G)$ coincides with $x + y$ on K.

Further, we put

$$(a + x) + (a + y) = lim(a + x_n + a + y_n),$$

$$x + (a + y) = a + lim(-a + x_n + a + y_n),$$

$$(a + x) + y = a + lim(x_n + y_n).$$

Limits are taken into account in K. The operation $+$ is correctly defined.

The ternary relation T^h on $C_h(G)$ is defined in the following way:

$$T^h \text{ coincides with } T_1 \text{ on } K.$$

Further, we put

$$[a + x, a + y, a + z] \in T^h \text{if} [x, y, z] \in T_1,$$

if $p, q, r \in C_h(G)$, $[p, q, r] \in T^h$, then either $\{p, q, r\} \subseteq K$ or $\{p, q, r\} \subseteq a + K$.

It is a routine to verify that the following assertion is true:

Lemma 6.2. $(C_h(G), +, T^h)$ *is a half lc-group,* $C_h(G) \uparrow = K$, $C_h(G) \downarrow = a + K$. ∎

From Lemmas 2.5 and 4.7, it follows:

Lemma 6.3. *Let G be an infinite half lc-group.Then $C_h(G)$ is a Cantor extension of G.* ∎

Let a' and $C'_h(G)$ be as in the Section 5.

Remark 6.4. It is easy to verify that Theorem 5.8 and Lemma 5.9 are valid also in the case $H_o = \{0\}$.

From Lemmas 4.7, 2.5 and Theorem 2.6, we obtain:

Lemma 6.5. *Let G be an infinite half lc-group. Then G is C-complete if and only if H is isomorphic to K.* ∎

Let G be an arbitrary *lc*-group as in the section 4 (neither $H_o \neq \{0\}$ nor $H_o = \{0\}$ is supposed). From Lemmas 6.1, 6.5 and 2.3, we conclude:

Theorem 6.6. *Let G be a half lc-group such that H is abelian and $H' \neq \emptyset$. Then G is C-complete if and only if some of the following conditions is fulfilled:*

(i) *G is finite;*

(ii) *H is isomorphic to K;*

(iii) *$H_o \neq \{0\}$ and H_o is C-complete.* ■

By summarizing Theorem 5.7, Lemma 6.3, Theorem 5.8, and Remark 6.4 we get:

Theorem 6.7. *Let G be a half lc-group such that H is abelian and $H' \neq \emptyset$. Then*

(i) *There exists a Cantor extension of G.*

(ii) *If G_1 and G_2 are Cantor extensions of G, then there exists an isomorphism from the half lc-group G_1 onto G_2 leaving all elements of G fixed.*

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Received 7 June 1999
Revised 30 April 1996