

## ON DISTRIBUTIVE TRICES

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### Abstract

A triple-semilattice is an algebra with three binary operations, which is a semilattice in respect of each of them. A trice is a triple-semilattice, satisfying so called roundabout absorption laws. In this paper we investigate distributive trices. We prove that the only subdirectly irreducible distributive trices are the trivial one and a two element one. We also discuss finitely generated free distributive trices and prove that a free distributive trice with two generators has 18 elements.

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### 1. INTRODUCTION

An algebra  $(T; \nearrow_1, \searrow_2, \downarrow_3)$  of a type with three binary operations is a *triple semilattice* if it is a semilattice in respect of each of the operations. We denote orders on  $T$  by

- (1)  $a \leq_1 b$  if and only if  $a \nearrow_1 b = b$ ,
- (2)  $a \leq_2 b$  if and only if  $a \searrow_2 b = b$ ,
- (3)  $a \leq_3 b$  if and only if  $a \downarrow_3 b = b$ .

A triple semilattice  $T$  is a *trice* if it satisfies the *roundabout absorption laws*:

$$(4) \quad ((a \nearrow_1 b) \nwarrow_2 b) \downarrow_3 b = b,$$

$$(5) \quad ((a \nearrow_1 b) \downarrow_3 b) \nwarrow_2 b = b,$$

$$(6) \quad ((a \nwarrow_2 b) \nearrow_1 b) \downarrow_3 b = b,$$

$$(7) \quad ((a \nwarrow_2 b) \downarrow_3 b) \nearrow_1 b = b,$$

$$(8) \quad ((a \downarrow_3 b) \nearrow_1 b) \nwarrow_2 b = b,$$

and

$$(9) \quad ((a \downarrow_3 b) \nwarrow_2 b) \nearrow_1 b = b$$

for all  $a, b \in T$ .

Trices are introduced and investigated in [2] as a generalization of lattices.

A *distributive trice* is a trice satisfying the following six distributive laws:

$$(10) \quad a \nearrow_1 (b \nwarrow_2 c) = (a \nearrow_1 b) \nwarrow_2 (a \nearrow_1 c),$$

$$(11) \quad a \nwarrow_2 (b \nearrow_1 c) = (a \nwarrow_2 b) \nearrow_1 (a \nwarrow_2 c),$$

$$(12) \quad a \nearrow_1 (b \downarrow_3 c) = (a \nearrow_1 b) \downarrow_3 (a \nearrow_1 c),$$

$$(13) \quad a \downarrow_3 (b \nearrow_1 c) = (a \downarrow_3 b) \nearrow_1 (a \downarrow_3 c),$$

$$(14) \quad a \nwarrow_2 (b \downarrow_3 c) = (a \nwarrow_2 b) \downarrow_3 (a \nwarrow_2 c),$$

and

$$(15) \quad a \downarrow_3 (b \nwarrow_2 c) = (a \downarrow_3 b) \nwarrow_2 (a \downarrow_3 c)$$

for all  $a, b, c \in T$ .

## 2. SUBDIRECT DECOMPOSITION OF DISTRIBUTIVE TRICES

**Lemma 1.** *A triple semilattice  $T$  having all three semilattices as chains is a trice if and only if for all  $x, y \in T$ , there are  $\leq_i$  and  $\leq_j$ , for  $i, j \in \{1, 2, 3\}$ , such that  $x \leq_i y$  and  $y \leq_j x$ .*

**Proof.** By contraposition, if for all orderings  $x \leq_i y$   $i \in \{1, 2, 3\}$  is satisfied, than  $x \nearrow_1 (x \nwarrow_2 (x \downarrow_3 y)) = y$ , i.e., roundabout absorption law (9) is not satisfied. On the other hand, if, say,  $x \leq_1 y$  and  $y \leq_2 x$ , then it is easy to prove that all roundabout absorption laws for  $x$  and  $y$  are satisfied. ■

**Lemma 2.** *Let  $(T; \nearrow_1, \searrow_2, \downarrow_3)$  be a distributive trice. Let  $x, y, t \in T$ . If  $x \nearrow_1 t = y \nearrow_1 t$ ,  $x \searrow_2 t = y \searrow_2 t$  and  $x \downarrow_3 t = y \downarrow_3 t$ , then  $x = y$ .*

**Proof.** Using repeatedly the hypotheses, we have

$$\begin{aligned}
x &= x \nearrow_1 (x \searrow_2 (x \downarrow_3 t)) = x \nearrow_1 (x \searrow_2 (y \downarrow_3 t)) \\
&= x \nearrow_1 ((x \searrow_2 y) \downarrow_3 (x \searrow_2 t)) = x \nearrow_1 ((x \searrow_2 y) \downarrow_3 (y \searrow_2 t)) \\
&= x \nearrow_1 (y \searrow_2 (x \downarrow_3 t)) = x \nearrow_1 (y \searrow_2 (y \downarrow_3 t)) \\
&= (x \nearrow_1 y) \searrow_2 (x \nearrow_1 (y \downarrow_3 t)) = (x \nearrow_1 y) \searrow_2 ((x \nearrow_1 y) \downarrow_3 (x \nearrow_1 t)) \\
&= (x \nearrow_1 y) \searrow_2 ((x \nearrow_1 y) \downarrow_3 (y \nearrow_1 t)) = (x \nearrow_1 y) \searrow_2 (y \nearrow_1 (x \downarrow_3 t)) \\
&= (x \nearrow_1 y) \searrow_2 (y \nearrow_1 (y \downarrow_3 t)) = y \nearrow_1 (x \searrow_2 (y \downarrow_3 t)) \\
&= y \nearrow_1 ((x \searrow_2 y) \downarrow_3 (x \searrow_2 t)) = y \nearrow_1 ((x \searrow_2 y) \downarrow_3 (y \searrow_2 t)) \\
&= y \nearrow_1 (y \searrow_2 (x \downarrow_3 t)) = y \nearrow_1 (y \searrow_2 (y \downarrow_3 t)) \\
&= y.
\end{aligned}$$

■

Let  $(T; \nearrow_1, \searrow_2, \downarrow_3)$  be a distributive trice, and let  $p \in T$  be a fixed element. We define relations on  $T$  by

$$(16) \quad x \theta_1 y \text{ if and only if } x \nearrow_1 p = y \nearrow_1 p,$$

$$(17) \quad x \theta_2 y \text{ if and only if } x \searrow_2 p = y \searrow_2 p,$$

and

$$(18) \quad x \theta_3 y \text{ if and only if } x \downarrow_3 p = y \downarrow_3 p.$$

**Lemma 3.** *The relations  $\theta_1$ ,  $\theta_2$  and  $\theta_3$  defined by (16)–(18) are congruences on the distributive trice.*

**Proof.** It is obvious that every  $\theta_i$ , for  $i \in \{1, 2, 3\}$  is an equivalence relation. Moreover, it is compatible with all operations. Let  $x \theta_1 y$  and  $z \theta_1 t$ . Then  $x \nearrow_1 p = y \nearrow_1 p$  and  $z \nearrow_1 p = t \nearrow_1 p$ . And then  $(x \nearrow_1 p) \searrow_2 (z \nearrow_1 p) = (y \nearrow_1 p) \searrow_2 (t \nearrow_1 p)$ . By distributivity,  $(x \searrow_2 z) \nearrow_1 p = (y \searrow_2 t) \nearrow_1 p$ , i.e.,  $(x \searrow_2 z) \theta_1 (y \searrow_2 t)$ . Similarly, we get  $(x \downarrow_3 z) \theta_1 (y \downarrow_3 t)$ . Hence,  $\theta_1$  is a congruence on the trice. For  $\theta_2$  and  $\theta_3$ , we can prove it in a similar way. ■

**Lemma 4.** *The relation  $\theta_i$  is the identity relation if and only if  $p$  is the bottom element in the  $(T; \leq_i)$ , for all  $i \in \{1, 2, 3\}$ .*

**Proof.** If  $p$  is the bottom element in  $(T, \leq_1)$ , then  $p \leq_1 x$  for all  $x \in T$ . Hence,  $x \theta_1 y$  if and only if  $x = x \nearrow_1 p = y \nearrow_1 p = y$ . That is,  $\theta_1 = \Delta$ .

On the other hand, if there exists  $x \in T$  such that  $\neg(p \leq_1 x)$ , then  $(p \nearrow_1 x) \neq x$ . As  $(p \nearrow_1 x) \nearrow_1 p = x \nearrow_1 p$ , we get  $(p \nearrow_1 x) \theta_1 x$ . Then,  $\theta_1 \neq \Delta$ . For  $\theta_2$  and  $\theta_3$ , we can prove the statements in a similar way. ■

**Lemma 5.** *If  $p$  is not the bottom element of any of semilattices of the distributive trice  $T$ , then not all of  $\theta_1$ ,  $\theta_2$  and  $\theta_3$  are equal.*

**Proof.** Suppose that all the congruences are equal. Let  $x <_1 p$ . Then,  $x \theta_1 p$ . As congruences are the same,  $x \theta_2 p$  and  $x \theta_3 p$ . Hence  $x \leq_2 p$  and  $x \leq_3 p$ , and thus  $((x \nearrow_1 p) \nwarrow_2 p) \downarrow_3 p = p$ , and finally from our assumption we obtain  $p = x$ , a contradiction. ■

**Lemma 6.** *There are no subdirectly irreducible distributive trices with more than three elements.*

**Proof.** Suppose that  $T$  is a subdirectly irreducible distributive trice with four or more elements. Then, there is an element, say  $p \in T$ , which is not the bottom element in any of the semilattices. This element determines three congruences  $\theta_1$ ,  $\theta_2$  and  $\theta_3$ , defined by formulas (16) – (18). Those relations are all distinct from the identity relation by Lemma 4, and at least two of them are not equal by Lemma 5. Using Lemma 2 we easily prove that

$$\theta_1 \cap \theta_2 \cap \theta_3 = \Delta.$$

By the well known theorem on congruence lattice of subdirectly irreducible algebras (see e.g. [1], p. 57. Thm. 8.4), we have that  $T$  is not subdirectly irreducible. ■

**Lemma 7.** *There are no subdirectly irreducible distributive trices with three elements.*

**Proof.** There is only one (up to the isomorphism and the order of operations) distributive trice with three elements  $(T; \nearrow_1, \nwarrow_2, \downarrow_3)$ , diagrams of its semilattices given in Figure 1. It is not subdirectly irreducible. Indeed, congruences of this trice, besides  $\Delta$  and  $\nabla$ , are  $\{\{a, b\}, \{c\}\}$ , and  $\{\{a\}, \{b, c\}\}$ , that is, congruence lattice is the four element boolean algebra. Thus, this trice is not subdirectly irreducible. ■

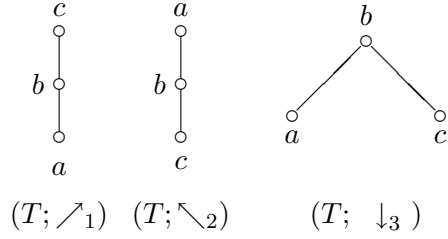


Figure 1

By lemmas 1 – 7, we have:

**Theorem 1.** *The only subdirectly irreducible distributive trices are, up to the isomorphism and the order of operations, the two element one, given in Figure 2, and the trivial one.*

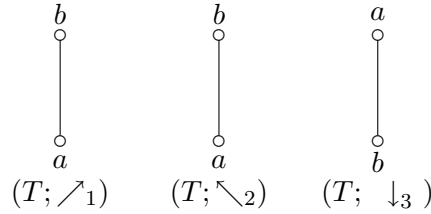


Figure 2

■

**Theorem 2.** *Every non-trivial distributive trice is isomorphic to a subdirect product of two element trices.*

**Proof.** This is a consequence of previous theorem and the Birkhoff theorem on subdirect products. ■

An obvious corollary is that every distributive trice is a subtrice of the direct product of two element trices.

**Example 1.** In the sequel, we give representation of the three element distributive trice in Figure 1, as a subdirect product of two element trices.

**Proof.** Let  $T_1 = \{a, b\}$  and  $T_2 = \{c, d\}$ , with  $a \leq_1 b$ ,  $a \leq_2 b$ ,  $b \leq_3 a$ ,  $c \leq_1 d$ ,  $d \leq_2 c$  and  $d \leq_3 c$ . The direct product has four elements  $\{ac, bc, ad, bd\}$ . The mentioned three element trice is isomorphic with the subtrice  $\{ac, bc, bd\}$ .

## 3. FREE DISTRIBUTIVE TRICES

In the sequel we consider free distributive trices.

Obviously, free distributive trice with one generator is the one element trivial trice. Now, consider  $n$  generators,  $x_1, \dots, x_n$ . Every element of a free distributive trice can be written in the form  $F_1 \downarrow_3 F_2 \downarrow_3 \dots \downarrow_3 F_m$ , where every  $F_i (i \in \{1, 2, \dots, m\})$  is of the form:  $g_1 \searrow_2 g_2 \searrow_2 \dots \searrow_2 g_k$ , and every  $g_j (j \in \{1, 2, \dots, k\})$  is of the form:  $x_{i_1} \nearrow_1 x_{i_2} \nearrow_1 \dots \nearrow_1 x_{i_l}$ , where all  $x_s$  appearing in the mentioned expression, are generators. We can easily prove, by using distributive laws, that every element of a free distributive trice have a representation of that form. And obviously, some elements have several different representations.

By the previous considerations, the following theorem is evident:

**Theorem 3.** *Every free distributive trice with a finite set of generators is finite.*

**Proof.** Let  $n$  be the number of generators. Let  $G$  be the set of all elements of the form:  $x_{i_1} \nearrow_1 x_{i_2} \nearrow_1 \dots \nearrow_1 x_{i_l}$ , where all  $x_s$  appearing in the mentioned expressions are generators. Then, the cardinality of  $G$  is not greater than  $2^n - 1$ . Let  $F$  be the set of all elements of the form:  $g_1 \searrow_2 g_2 \searrow_2 \dots \searrow_2 g_k$ , where  $g_i \in G$ , for all  $i \in \{1, \dots, k\}$ . Then, the cardinality of  $F$  is not greater than  $2^{2^n - 1} - 1$ . As every element of a free distributive trice can be written in the form  $F_1 \downarrow_3 F_2 \downarrow_3 \dots \downarrow_3 F_m$ , with  $F_i \in F$ , the order of free distributive trice with  $n$  generators is not greater than  $2^{2^{2^n - 1} - 1} - 1$ . There is some possibility of overlapping. But, this completes the proof. ■

**Example 2.** Free distributive trice with two generators has 18 elements.

We effectively construct a free distributive trice with two generators  $x$  and  $y$ . The notations in the sequel is taken from the proof of the previous theorem. Now, the set  $G$  is  $\{x, y, x \nearrow_1 y\}$ . From  $x \searrow_2 y \searrow_2 (x \nearrow_1 y) = (x \searrow_2 y \searrow_2 x) \nearrow_1 (x \searrow_2 y \searrow_2 y) = x \searrow_2 y$ , it follows that the set  $F$  is  $\{x, y, x \nearrow_1 y, x \searrow_2 y, x \searrow_2 (x \nearrow_1 y), y \searrow_2 (x \nearrow_1 y)\}$ . In a similar way, we can deduce that the free distributive trice with two generators has 18 elements.



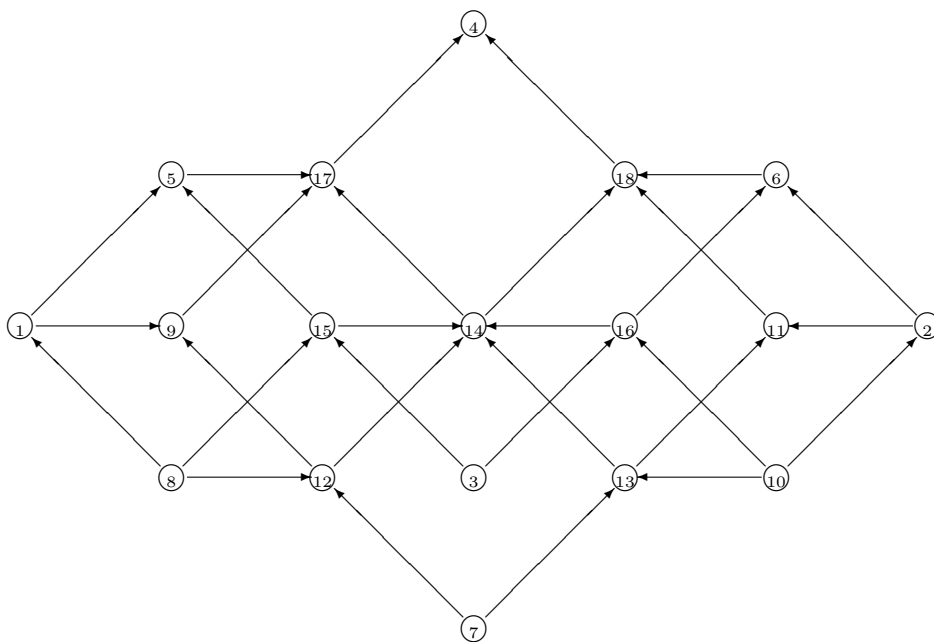


Figure 3 – 2

The order  $\leq_2$

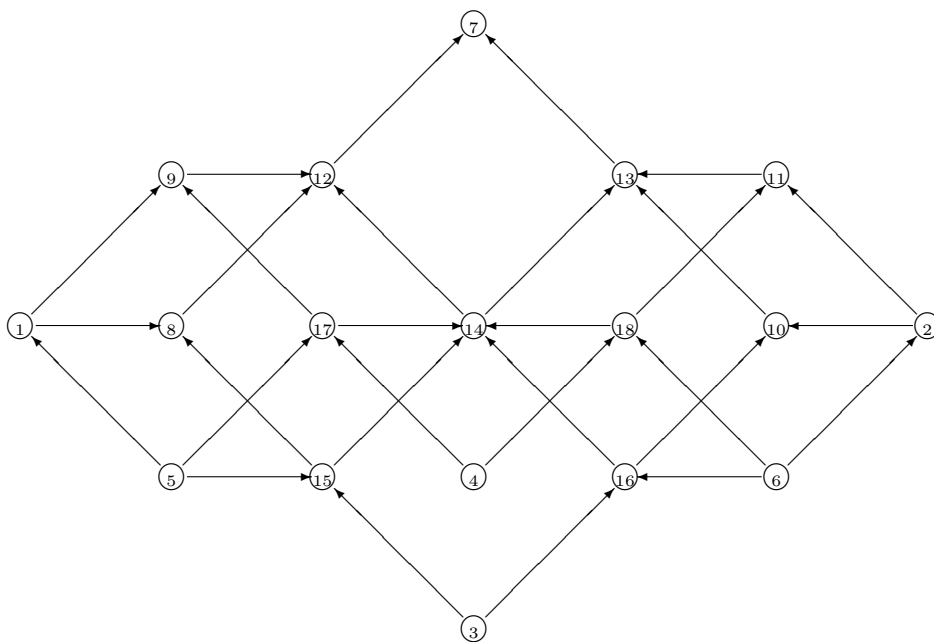


Figure 3 – 3

The order  $\leq_3$



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