Discussiones Mathematicae General Algebra and Applications 21 (2001) 21–29

## **ON DISTRIBUTIVE TRICES**

KIYOMITSU HORIUCHI

Department of Information Science and Systems Engineering, Faculty of Science and Engineering, Konan University Okamoto, Higashinada, Kobe 658-8501, Japan e-mail: horiuchi@konan-u.ac.jp

AND

Andreja Tepavčević

Institute of Mathematics Fac. of Sci., University of Novi Sad Trg D. Obradovića 4, 21000 Novi Sad Yugoslavia e-mail: etepavce@EUnet.yu

## Abstract

A triple-semilattice is an algebra with three binary operations, which is a semilattice in respect of each of them. A trice is a triple-semilattice, satisfying so called roundabout absorption laws. In this paper we investigate distributive trices. We prove that the only subdirectly irreducible distributive trices are the trivial one and a two element one. We also discuss finitely generated free distributive trices and prove that a free distributive trice with two generators has 18 elements.

Keywords and phrases: triple semilattice, trice, distributive trice.2000 Mathematics Subject Classification: 06A12, 08B20, 08B26.

# 1. INTRODUCTION

An algebra  $(T; \nearrow_1, \searrow_2, \downarrow_3)$  of a type with three binary operations is a *triple semilattice* if it is a semilattice in respect of each of the operations. We denote orders on T by

- (1)  $a \leq_1 b$  if and only if  $a \nearrow_1 b = b$ ,
- (2)  $a \leq_2 b$  if and only if  $a \searrow_2 b = b$ ,
- (3)  $a \leq_3 b$  if and only if  $a \downarrow_3 b = b$ .

A triple semilattice T is a *trice* if it satisfies the *roundabout absorption laws*:

- (4)  $((a \nearrow_1 b) \searrow_2 b) \downarrow_3 b = b,$
- (5)  $((a \nearrow_1 b) \downarrow_3 b) \searrow_2 b = b,$
- (6)  $((a \searrow_2 b) \nearrow_1 b) \downarrow_3 b = b,$
- (7)  $((a \searrow_2 b) \downarrow_3 b) \nearrow_1 b = b,$
- (8)  $((a \downarrow_3 b) \nearrow_1 b) \searrow_2 b = b,$

and

(9)  $((a \downarrow_3 b) \searrow_2 b) \nearrow_1 b = b$ 

for all  $a, b \in T$ .

Trices are introduced and investigated in [2] as a generalization of lattices. A *distributive trice* is a trice satisfying the following six distributive laws:

- (10)  $a \nearrow_1 (b \searrow_2 c) = (a \nearrow_1 b) \searrow_2 (a \nearrow_1 c),$
- (11)  $a \searrow_2 (b \nearrow_1 c) = (a \searrow_2 b) \nearrow_1 (a \searrow_2 c),$
- (12)  $a \nearrow_1 (b \downarrow_3 c) = (a \nearrow_1 b) \downarrow_3 (a \nearrow_1 c),$

(13) 
$$a \downarrow_3 (b \nearrow_1 c) = (a \downarrow_3 b) \nearrow_1 (a \downarrow_3 c),$$

(14) 
$$a \searrow_2 (b \downarrow_3 c) = (a \searrow_2 b) \downarrow_3 (a \searrow_2 c),$$

and

(15) 
$$a \downarrow_3 (b \searrow_2 c) = (a \downarrow_3 b) \searrow_2 (a \downarrow_3 c)$$

for all  $a, b, c \in T$ .

## 2. Subdirect decomposition of distributive trices

**Lemma 1.** A triple semilattice T having all three semilattices as chains is a trice if and only if for all  $x, y \in T$ , there are  $\leq_i$  and  $\leq_j$ , for  $i, j \in \{1, 2, 3\}$ , such that  $x \leq_i y$  and  $y \leq_j x$ .

**Proof.** By contraposition, if for all orderings  $x \leq_i y$   $i \in \{1, 2, 3\}$  is satisfied, than  $x \nearrow_1 (x \searrow_2 (x \downarrow_3 y))) = y$ , i.e., roundabout absorption law (9) is not satisfied. On the other hand, if, say,  $x \leq_1 y$  and  $y \leq_2 x$ , then it is easy to prove that all roundabout absorption laws for x and y are satisfied.

ON DISTRIBUTIVE TRICES

**Lemma 2.** Let  $(T; \nearrow_1, \searrow_2, \downarrow_3)$  be a distributive trice. Let  $x, y, t \in T$ . If  $x \nearrow_1 t = y \nearrow_1 t, x \searrow_2 t = y \searrow_2 t$  and  $x \downarrow_3 t = y \downarrow_3 t$ , then x = y.

**Proof.** Using repeatedly the hypotheses, we have

$$\begin{aligned} x &= x \nearrow_1 (x \swarrow_2 (x \downarrow_3 t))) = x \nearrow_1 (x \searrow_2 (y \downarrow_3 t))) \\ &= x \nearrow_1 ((x \searrow_2 y) \downarrow_3 (x \searrow_2 t)) = x \nearrow_1 ((x \searrow_2 y) \downarrow_3 (y \searrow_2 t)) \\ &= x \nearrow_1 (y \diagdown_2 (x \downarrow_3 t)) = x \nearrow_1 (y \searrow_2 (y \downarrow_3 t)) \\ &= (x \nearrow_1 y) \searrow_2 (x \nearrow_1 (y \downarrow_3 t)) = (x \nearrow_1 y) \searrow_2 ((x \nearrow_1 y) \downarrow_3 (x \nearrow_1 t)) \\ &= (x \nearrow_1 y) \searrow_2 ((x \swarrow_1 y) \downarrow_3 (y \nearrow_1 t)) = (x \swarrow_1 y) \searrow_2 (y \nearrow_1 (x \downarrow_3 t)) \\ &= (x \nearrow_1 y) \searrow_2 (y \swarrow_1 (y \downarrow_3 t)) = y \nearrow_1 (x \searrow_2 (y \downarrow_3 t)) \\ &= y \nearrow_1 ((x \searrow_2 y) \downarrow_3 (x \searrow_2 t)) = y \nearrow_1 ((x \searrow_2 y) \downarrow_3 (y \searrow_2 t)) \\ &= y \nearrow_1 (y \searrow_2 (x \downarrow_3 t)) = y \nearrow_1 (y \searrow_2 (y \downarrow_3 t)) \\ &= y.\end{aligned}$$

Let  $(T; \nearrow_1, \searrow_2, \downarrow_3)$  be a distributive trice, and let  $p \in T$  be a fixed element. We define relations on T by

(16)  $x \theta_1 y$  if and only if  $x \nearrow_1 p = y \nearrow_1 p$ ,

(17)  $x \theta_2 y$  if and only if  $x \searrow_2 p = y \searrow_2 p$ ,

and

(18) 
$$x \theta_3 y$$
 if and only if  $x \downarrow_3 p = y \downarrow_3 p$ .

**Lemma 3.** The relations  $\theta_1$ ,  $\theta_2$  and  $\theta_3$  defined by (16)–(18) are congruences on the distributive trice.

**Proof.** It is obvious that every  $\theta_i$ , for  $i \in \{1, 2, 3\}$  is an equivalence relation. Moreover, it is compatible with all operations. Let  $x \ \theta_1 y$  and  $z \ \theta_1 t$ . Then  $x \ \nearrow_1 p = y \ \swarrow_1 p$  and  $z \ \swarrow_1 p = t \ \swarrow_1 p$ . And then  $(x \ \swarrow_1 p) \ \searrow_2 (z \ \swarrow_1 p) = (y \ \swarrow_1 p) \ \searrow_2 (t \ \swarrow_1 p)$ . By distributivity,  $(x \ \searrow_2 z) \ \swarrow_1 p = (y \ \searrow_2 t) \ \swarrow_1 p$ , i.e.,  $(x \ \searrow_2 z) \ \theta_1 (y \ \searrow_2 t)$ . Similary, we get  $(x \ \downarrow_3 z) \ \theta_1 (y \ \downarrow_3 t)$ . Hence,  $\theta_1$  is a congruence on the trice. For  $\theta_2$  and  $\theta_3$ , we can prove it in a similar way. **Lemma 4.** The relation  $\theta_i$  is the identity relation if and only if p is the bottom element in the  $(T; \leq_i)$ , for all  $i \in \{1, 2, 3\}$ .

**Proof.** If p is the bottom element in  $(T, \leq_1)$ , then  $p \leq_1 x$  for all  $x \in T$ . Hence,  $x \theta_1 y$  if and only if  $x = x \nearrow_1 p = y \nearrow_1 p = y$ . That is,  $\theta_1 = \Delta$ .

On the other hand, if there exists  $x \in T$  such that  $\neg (p \leq_1 x)$ , then  $(p \nearrow_1 x) \neq x$ . As  $(p \nearrow_1 x) \nearrow_1 p = x \nearrow_1 p$ , we get  $(p \nearrow_1 x) \theta_1 x$ . Then,  $\theta_1 \neq \Delta$ . For  $\theta_2$  and  $\theta_3$ , we can prove the statements in a similar way.

**Lemma 5.** If p is not the bottom element of any of semilattices of the distributive trice T, then not all of  $\theta_1$ ,  $\theta_2$  and  $\theta_3$  are equal.

**Proof.** Suppose that all the congruences are equal. Let  $x <_1 p$ . Then,  $x \theta_1 p$ . As congruences are the same,  $x \theta_2 p$  and  $x \theta_3 p$ . Hence  $x \leq_2 p$  and  $x \leq_3 p$ , and thus  $((x \nearrow_1 p) \searrow_2 p) \downarrow_3 p = p$ , and finally from our assumption we obtain p = x, a contradiction.

**Lemma 6.** There are no subdirectly irreducible distributive trices with more than three elements.

**Proof.** Suppose that T is a subdirectly irreducible distributive trice with four or more elements. Then, there is an element, say  $p \in T$ , which is not the bottom element in any of the semilattices. This element determines three congruences  $\theta_1$ ,  $\theta_2$  and  $\theta_3$ , defined by formulas (16) – (18). Those relations are all distinct from the identity relation by Lemma 4, and at least two of them are not equal by Lemma 5. Using Lemma 2 we easily prove that

$$\theta_1 \cap \theta_2 \cap \theta_3 = \Delta.$$

By the well known theorem on congruence lattice of subdirectly irreducible algebras (see e.g. [1], p. 57. Thm. 8.4), we have that T is not subdirectly irreducible.

**Lemma 7.** There are no subdirectly irreducible distributive trices with three elements.

**Proof.** There is only one (up to the isomorphism and the order of operations) distributive trice with three elements  $(T; \nearrow_1, \searrow_2, \downarrow_3)$ , diagrams of its semilattices given in Figure 1. It is not subdirectly irreducible. Indeed, congruences of this trice, besides  $\Delta$  and  $\nabla$ , are  $\{\{a, b\}, \{c\}\}$ , and  $\{\{a\}, \{b, c\}\}$ , that is, congruence lattice is the four element boolean algebra. Thus, this trice is not subdirectly irreducible.



Figure 1

By lemmas 1 - 7, we have:

**Theorem 1.** The only subdirectly irreducible distributive trices are, up to the isomorphism and the order of operations, the two element one, given in Figure 2, and the trivial one.



**Theorem 2.** Every non-trivial distributive trice is isomorphic to a subdirect product of two element trices.

**Proof.** This is a consequence of previous theorem and the Birkhoff theorem on subdirect products.

An obvious corrolary is that every distributive trice is a subtrice of the direct product of two element trices.

**Example 1.** In the sequel, we give representation of the three element distributive trice in Figure 1, as a subdirect product of two element trices.

**Proof.** Let  $T_1 = \{a, b\}$  and  $T_2 = \{c, d\}$ , with  $a \leq_1 b, a \leq_2 b, b \leq_3 a, c \leq_1 d, d \leq_2 c$  and  $d \leq_3 c$ . The direct product has four elements  $\{ac, bc, ad, bd\}$ . The mentioned three element trice is isomorphic with the subtrice  $\{ac, bc, bd\}$ .

### 3. Free distributive trices

In the sequel we consider free distributive trices.

Obviously, free distributive trice with one generator is the one element trivial trice. Now, consider n generators,  $x_1, \ldots, x_n$ . Every element of a free distributive trice can be written in the form  $F_1 \downarrow_3 F_2 \downarrow_3 \ldots \downarrow_3 F_m$ , where every  $F_i(i \in \{1, 2, \ldots m\})$  is of the form:  $g_1 \searrow_2 g_2 \searrow_2 \ldots \searrow_2 g_k$ , and every  $g_j(j \in \{1, 2, \ldots k\})$  is of the form:  $x_{i_1} \nearrow_1 x_{i_2} \nearrow_1 \ldots \nearrow_1 x_{i_l}$ , where all  $x_s$  appearing in the mentioned expression, are generators. We can easily prove, by using distributive laws, that every element of a free distributive trice have a representation of that form. And obviously, some elements have several different representations.

By the previous considerations, the following theorem is evident:

**Theorem 3.** Every free distributive trice with a finite set of generators is finite.

**Proof.** Let *n* be the number of generators. Let *G* be the set of all elements of the form:  $x_{i_1} \nearrow_1 x_{i_2} \nearrow_1 \dots \nearrow_1 x_{i_l}$ , where all  $x_s$  appearing in the mentioned expressions are generators. Then, the cardinality of *G* is not greater than  $2^n - 1$ . Let *F* be the set of all elements of the form:  $g_1 \searrow_2 g_2 \searrow_2 \dots \searrow_2 g_k$ , where  $g_i \in G$ , for all  $i \in \{1, \dots, k\}$ . Then, the cardinality of *F* is not greater than  $2^{2^n-1} - 1$ . As every element of a free distributive trice can be written in the form  $F_1 \downarrow_3 F_2 \downarrow_3 \dots \downarrow_3 F_m$ , with  $F_i \in F$ , the order of free distributive trice with *n* generators is not greater than  $2^{2^{2^n}-1} - 1$ . There is some possibility of overlapping. But, this completes the proof.

**Example 2.** Free distributive trice with two generators has 18 elements.

We effectively construct a free distributive trice with two generators x and y. The notations in the sequel is taken from the proof of the previous theorem. Now, the set G is  $\{x, y, x \nearrow_1 y\}$ . From  $x \searrow_2 y \searrow_2 (x \nearrow_1 y) = (x \searrow_2 y \searrow_2 x) \nearrow_1 (x \searrow_2 y \searrow_2 y) = x \searrow_2 y$ , it follows that the set F is  $\{x, y, x \nearrow_1 y, x \searrow_2 y, x \searrow_2 (x \nearrow_1 y), y \searrow_2 (x \nearrow_1 y)\}$ . In a similar way, we can deduce that the free distributive trice with two generators has 18 elements.

All different elements of that trice are represented by the following terms:

$$\begin{split} &(1) = x, \qquad (2) = y, \qquad (3) = x \nearrow_1 y, \\ &(4) = x \swarrow_2 y = x \searrow_2 y \searrow_2 (x \nearrow_1 y), \qquad (5) = x \searrow_2 (x \nearrow_1 y), \\ &(6) = y \searrow_2 (x \nearrow_1 y), \qquad (7) = x \downarrow_3 y, \qquad (8) = x \downarrow_3 (x \swarrow_1 y), \\ &(9) = x \downarrow_3 (x \searrow_2 y), \qquad (10) = y \downarrow_3 (x \swarrow_1 y), \qquad (11) = y \downarrow_3 (x \searrow_2 y), \\ &(9) = x \downarrow_3 (y \searrow_2 (x \nearrow_1 y)), \qquad (13) = y \downarrow_3 (x \searrow_2 (x \swarrow_1 y)), \\ &(14) = (x \searrow_2 y) \downarrow_3 (x \swarrow_1 y) = (x \searrow_2 (x \swarrow_1 y)) \downarrow_3 (y \searrow_2 (x \swarrow_1 y)), \\ &(15) = (x \swarrow_1 y) \downarrow_3 (x \searrow_2 (x \swarrow_1 y)), \qquad (16) = (x \swarrow_1 y) \downarrow_3 (y \searrow_2 (x \swarrow_1 y)), \\ &(17) = (x \searrow_2 y) \downarrow_3 (x \searrow_2 (x \swarrow_1 y)), \qquad (18) = (x \swarrow_2 y) \downarrow_3 (y \searrow_2 (x \swarrow_1 y)). \end{split}$$

Diagrams of the free distributive trice with two generators are presented by Figures 3.1–3.3. The orders in each of the semilattices of the trice are represented by arrows. (3), (4) and (7) are the top elements in the orders  $\leq_1$ ,  $\leq_2$  and  $\leq_3$ , respectively.





### References

- [1] S. Burris, and H.P. Sankappanavar, A Course in Universal Algebra, Springer-Verlag, New York 1981 (new, electronic version, 1999: is available at the address: www.thoralf.uwaterloo.ca).
- [2] K. Horiuchi, Trice and Two delegates operation, Sci. Math. 2 (1999), 373–384.
- [3] J.A. Kalman, Subdirect decomposition of distributive quasi-lattices, Fund. Math. 71 (1971), 161–163.

Received 17 May 1999 Revised 12 March 2001