

## ON EXPONENTIATION OF $n$ -ARY HYPERALGEBRAS

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### Abstract

For  $n$ -ary hyperalgebras we study a binary operation of exponentiation which to a given pair of  $n$ -ary hyperalgebras assigns their power, i.e., an  $n$ -ary hyperalgebra carried by the corresponding set of homomorphisms. We give sufficient conditions for the existence of such a power and for a decent behaviour of the exponentiation. As a consequence of our investigations we discover a cartesian closed subcategory of the category of  $n$ -ary hyperalgebras and homomorphisms between them.

**Keywords:**  $n$ -ary hyperalgebra, medial hyperalgebra, diagonal hyperalgebra, power of hyperalgebras, cartesian closed category.

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As for generality, hyperalgebras lie between relational systems and algebras. More precisely, algebras are just the relational systems that are both partial algebras and hyperalgebras. Exponentiation of relational systems, partial algebras and algebras has been investigated in [9], [11] and [10], respectively. In this note we focus on investigating exponentiation of hyperalgebras. The results obtained generalize some of those from [10].

Hyperalgebras proved to be useful for many applications to various branches of mathematics, especially to computer science (automata theory). This led to a rapid development of the theory of hyperalgebras in the last decade - see, e.g., [2], [3], [6], [7], [12], [13]. This paper is also aimed to contribute to this development. We introduce and study a binary

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operation of exponentiation for  $n$ -ary hyperalgebras. As a consequence of the results received we discover a cartesian closed subcategory of the category of  $n$ -ary hyperalgebras with homomorphisms as morphisms. Let us recall that cartesian closed categories have many applications because they possess well-behaved powers for all pairs of objects. For example, in computer science cartesian closed categories are used as models of typed  $\lambda$ -calculi.

Let  $n$  be a positive integer. By an  $n$ -ary *hyperalgebra* we understand a pair  $G = (X, p)$ , where  $X$  is a set, the so called *carrier* of  $G$ , which will be denoted by  $|G|$ , and  $p : X^n \rightarrow \exp(X) \setminus \{\emptyset\}$  is a map, the so called  $n$ -ary *hyperoperation* on  $X$ . Of course,  $n$ -ary hyperalgebras  $(X, p)$  can also be considered for  $n = 0$ , in which case  $p$  is nothing else than a nonempty subset of  $X$  (so that  $X$  itself has to be nonempty too). To avoid some nonwanted singularities, we do not consider the trivial case  $n = 0$  for  $n$ -ary hyperalgebras in this note. An  $n$ -ary hyperalgebra  $(X, p)$  with the property  $\text{card } p(x_1, \dots, x_n) = 1$  for any  $x_1, \dots, x_n \in X$  is called an  $n$ -ary *algebra*. Binary hyperalgebras are usually called *hypergroupoids* (analogously to binary algebras which are usually called groupoids). If  $(X, p)$  is an  $n$ -ary hyperalgebra and  $A_1, \dots, A_n$  nonempty subsets of  $X$ , then we put  $p(A_1, \dots, A_n) = \bigcup \{p(x_1, \dots, x_n); x_1 \in A_1, \dots, x_n \in A_n\}$ . Given a pair of  $n$ -ary hyperalgebras  $G = (X, p)$  and  $H = (Y, q)$ , by a *homomorphism* of  $G$  into  $H$  we mean any map  $f : X \rightarrow Y$  such that  $f(p(x_1, \dots, x_n)) \subseteq q(f(x_1), \dots, f(x_n))$  whenever  $x_1, \dots, x_n \in X$ . We denote by  $\text{Hom}(G, H)$  the set of all homomorphisms of  $G$  into  $H$ . If  $f$  is a bijection of  $X$  onto  $Y$  and  $f(p(x_1, \dots, x_n)) = q(f(x_1), \dots, f(x_n))$  for any  $x_1, \dots, x_n \in X$ , then  $f$  is called an *isomorphism* of  $G$  onto  $H$ . In other words, an isomorphism of  $G$  onto  $H$  is a bijection  $f : X \rightarrow Y$  such that  $f$  is a homomorphism of  $G$  into  $H$  and  $f^{-1}$  is a homomorphism of  $H$  into  $G$ . If there is an isomorphism of  $G$  onto  $H$ , then we write  $G \cong H$  and say that  $G$  and  $H$  are *isomorphic*. An  $n$ -ary hyperalgebra  $(X, p)$  is called a *subhyperalgebra* of an  $n$ -ary hyperalgebra  $(Y, q)$  if  $X \subseteq Y$  and  $p(x_1, \dots, x_n) = q(x_1, \dots, x_n)$  for any  $x_1, \dots, x_n \in X$ . By the *direct product* of a family of  $n$ -ary hyperalgebras  $G_i = (X_i, p_i)$ ,  $i \in I$ , we understand the  $n$ -ary hyperalgebra  $\prod_{i \in I} G_i = (\prod_{i \in I} X_i, p)$  where  $p(f_1, \dots, f_n) = \prod_{i \in I} p_i(f_1(i), \dots, f_n(i))$  for any  $f_1, \dots, f_n \in \prod_{i \in I} X_i$ . If the set  $I$  is finite, say  $I = \{1, \dots, m\}$ , then we write  $G_1 \times \dots \times G_m$  instead of  $\prod_{i \in I} G_i$ . If  $G$  is an  $n$ -ary hyperalgebra and  $G_i = G$  for all  $i \in I$ , then we write  $G^I$  instead of  $\prod_{i \in I} G_i$ .

**Definition 1.** An  $n$ -ary hyperalgebra  $(X, p)$  is called *medial* if for any  $n \times n$ -matrix  $A = (x_{ij})$ ,  $i, j = 1, \dots, n$ , over  $X$  the inclusion

$$p(p(x_{11}, \dots, x_{1n}), \dots, p(x_{n1}, \dots, x_{nn})) \subseteq p(x_1, \dots, x_n)$$

is satisfied whenever  $x_j \in p(x_{1j}, \dots, x_{nj})$  for all  $j = 1, \dots, n$ .

**Remark 1.** For  $n$ -ary algebras the introduced mediality coincides with the mediality dealt with in [10]. Medial groupoids are studied in [4].

Some examples of medial  $n$ -ary algebras can be found in [10]. Here we give an example of a medial hypergroupoid which is not a groupoid in general (of course, a hypergroupoid  $(X, *)$  is medial if and only if for any elements  $x, y, z, t \in X$  we have  $(x*y)*(z*t) \subseteq u*v$  whenever  $u \in x*z$  and  $v \in y*t$ ).

**Example 1.** Let  $(X, \leq)$  be a partially ordered set with a least element 0 and let  $A$  be the set of all atoms of  $(X, \leq)$ . For any  $x, y \in X$  put  $x*y = \{z \in X; z < x, z < y \text{ and } z \in A \cup \{0\}\}$  whenever  $x \neq 0 \neq y$ , and  $x*y = \{0\}$  whenever  $x = 0$  or  $y = 0$ . Then  $(X, *)$  is a medial hypergroupoid because clearly  $(x*y)*(z*t) = u*v = \{0\}$  whenever  $x, y, z, t \in X$  and  $u \in x*z, v \in y*t$ .

**Theorem 1.** Let  $G = (X, p), H = (Y, q)$  be  $n$ -ary hyperalgebras. If  $G$  is medial, then there is a subhyperalgebra of the direct product  $G^Y$  whose carrier is  $\text{Hom}(H, G)$ .

**Proof.** Let  $G^Y = (X^Y, r)$ . Then for any  $f_1, \dots, f_n \in X^Y$  we have  $r(f_1, \dots, f_n) = \{f \in X^Y; \forall y \in Y : f(y) \in p(f_1(y), \dots, f_n(y))\}$ . Let  $f_1, \dots, f_n \in \text{Hom}(H, G)$  and let  $f \in r(f_1, \dots, f_n)$  be an arbitrary element. For any elements  $y_1, \dots, y_n \in Y$  we have

$$\begin{aligned} f(q(y_1, \dots, y_n)) &= \{f(y); y \in q(y_1, \dots, y_n)\} \\ &\subseteq p(f_1(q(y_1, \dots, y_n)), \dots, f_n(q(y_1, \dots, y_n))) \\ &\subseteq p(p(f_1(y_1), \dots, f_1(y_n)), \dots, p(f_n(y_1), \dots, f_n(y_n))) \\ &\subseteq p(f(y_1), \dots, f(y_n)). \end{aligned}$$

Hence  $f \in \text{Hom}(H, G)$ , which proves the statement. ■

**Definition 2.** Let  $G = (X, p), H = (Y, q)$  be  $n$ -ary hyperalgebras and let  $G$  be medial. The subhyperalgebra of the direct product  $G^Y$  from Theorem 1 is called the *power* of  $G$  and  $H$  and denoted by  $G^H$ .

The direct product of a family of medial  $n$ -ary hyperalgebras is also a medial  $n$ -ary hyperalgebra. Thus, if  $G$  is a medial  $n$ -ary hyperalgebra, then the power  $G^H$  is medial for any  $n$ -ary hyperalgebra  $H$ . Let  $G, H, K$  be  $n$ -ary hyperalgebras. If  $G$  and  $H$  are medial, then there clearly holds the second exponential law  $(G \times H)^K \cong G^K \times H^K$ , but the first one  $(G^H)^K \cong G^{H \times K}$  is not satisfied in general. In what follows we will deal with the problem of the validity of the first exponential law for exponentiation of  $n$ -ary hyperalgebras.

An  $n$ -ary hyperalgebra  $(X, p)$  is said to be *idempotent* if, whenever  $x \in X$  and  $x_i = x$  for all  $i = 1, \dots, n$ , we have  $x \in p(x_1, \dots, x_n)$ .

**Theorem 2.** *Let  $G, H, K$  be  $n$ -ary hyperalgebras and let  $G$  be medial and  $H, K$  be idempotent. Then there exists an injective homomorphism  $\varphi : G^{H \times K} \rightarrow (G^H)^K$  given by  $(\varphi(f)(z))(y) = f(y, z)$  whenever  $f \in \text{Hom}(H \times K, G)$ ,  $z \in |K|$  and  $y \in |H|$ .*

**Proof.** Let  $G = (X, p)$ ,  $H = (Y, q)$ ,  $K = (Z, r)$ ,  $H \times K = (Y \times Z, s)$ ,  $G^H = (\text{Hom}(H, G), t)$ ,  $G^{H \times K} = (\text{Hom}(H \times K, G), u)$  and  $(G^H)^K = (\text{Hom}(K, G^H), v)$ . Let  $f \in \text{Hom}(H \times K, G)$  be an arbitrary element. Clearly,  $\varphi(f)$  maps  $K$  into  $\text{Hom}(H, G)$  because for any  $z \in Z$  we have  $\varphi(f)(z) = f \circ (\text{id}_Y, c_z)$ , where  $\text{id}_Y : Y \rightarrow Y$  is the identity map and  $c_z \in \text{Hom}(H, K)$  is the constant map with the value  $z$ . Let  $z_1, \dots, z_n \in Z$  and  $w \in \varphi(f)(r(z_1, \dots, z_n))$ . Then  $w = \varphi(f)(z)$  for some  $z \in r(z_1, \dots, z_n)$ . For any  $y \in Y$  we have  $w(y) = (\varphi(f)(z))(y) = f(y, z) \in f(q(y, \dots, y) \times r(z_1, \dots, z_n)) = f(s((y, z_1), \dots, (y, z_n))) \subseteq p(f(y, z_1), \dots, f(y, z_n)) = p((\varphi(f)(z_1))(y), \dots, (\varphi(f)(z_n))(y))$ . Hence  $w \in t(\varphi(f)(z_1), \dots, \varphi(f)(z_n))$ . Therefore  $\varphi(f)(r(z_1, \dots, z_n)) \subseteq t(\varphi(f)(z_1), \dots, \varphi(f)(z_n))$  and thus  $\varphi(f) \in \text{Hom}(K, G^H)$ . We have shown that  $\varphi$  maps  $\text{Hom}(H \times K, G)$  into  $\text{Hom}(K, G^H)$ . To show that  $\varphi$  is a homomorphism, let  $f_1, \dots, f_n \in \text{Hom}(H \times K, G)$  and  $w \in \varphi(u(f_1, \dots, f_n))$  be arbitrary elements. Then  $w = \varphi(f)$  for some  $f \in u(f_1, \dots, f_n)$ . For any  $y \in Y$  and any  $z \in Z$  we have  $w(z)(y) = (\varphi(f)(z))(y) = f(y, z) \in p(f_1(y, z), \dots, f_n(y, z)) = p((\varphi(f_1)(z))(y), \dots, (\varphi(f_n)(z))(y))$ . Hence  $w(z) \in t(\varphi(f_1)(z), \dots, \varphi(f_n)(z))$  for any  $z \in Z$ . Consequently,  $w \in v(\varphi(f_1), \dots, \varphi(f_n))$ . It follows that  $\varphi(u(f_1, \dots, f_n)) \subseteq v(\varphi(f_1), \dots, \varphi(f_n))$ , thus  $\varphi \in \text{Hom}(G^{H \times K}, (G^H)^K)$ . As  $\varphi$  is clearly injective, the proof is complete. ■

**Definition 3.** An  $n$ -ary hyperalgebra  $(X, p)$  is called *diagonal* if for any  $n \times n$ -matrix  $A = (x_{ij})$ ,  $i, j = 1, \dots, n$ , over  $X$  the inclusion

$$p(p(x_{11}, \dots, x_{1n}), \dots, p(x_{n1}, \dots, x_{nn})) \subseteq p(x_{11}, \dots, x_{nn})$$

is valid.

**Remark 2.** For  $n$ -ary algebras the introduced diagonality coincides with the diagonality from [10]. Idempotent and diagonal  $n$ -ary algebras are investigated in [5].

Obviously, the hypergroupoid from Example 1 is diagonal. Some examples of diagonal  $n$ -ary algebras can also be found in [10].

**Lemma 1.** *Let  $G$ ,  $H$  be  $n$ -ary hyperalgebras, let  $G$  be medial, and let  $e : |H| \times \text{Hom}(H, G) \rightarrow |G|$  be the map given by  $e(y, f) = f(y)$  whenever  $y \in |H|$  and  $f \in \text{Hom}(H, G)$ . If  $G$  is diagonal, then  $e \in \text{Hom}(H \times G^H, G)$ .*

**Proof.** Let  $G = (X, p)$ ,  $H = (Y, q)$ ,  $G^H = (\text{Hom}(H, G), r)$  and  $H \times G^H = (Y \times \text{Hom}(H, G), s)$ . Let  $G$  be diagonal and let  $(y_1, f_1), \dots, (y_n, f_n) \in Y \times \text{Hom}(H, G)$  be arbitrary elements. Then we have  $e(s((y_1, f_1), \dots, (y_n, f_n))) = e(q(y_1, \dots, y_n) \times r(f_1, \dots, f_n)) = \{f(y); y \in q(y_1, \dots, y_n), f \in r(f_1, \dots, f_n)\} = \bigcup \{f(q(y_1, \dots, y_n)); f \in r(f_1, \dots, f_n)\} \subseteq p(f_1(q(y_1, \dots, y_n)), \dots, f_n(q(y_1, \dots, y_n))) \subseteq p(p(f_1(y_1), \dots, f_1(y_n)), \dots, p(f_n(y_1), \dots, f_n(y_n))) \subseteq p(f_1(y_1), \dots, f_n(y_n)) = p(e(y_1, f_1), \dots, e(y_n, f_n))$ . Therefore  $e \in \text{Hom}(H \times G^H, G)$ . ■

The first exponential law has an important category-theoretical meaning - it is a characteristic property of the so-called cartesian closed categories. Therefore we will now study  $n$ -ary hyperalgebras from the categorical point of view. For the categorical terminology used see, e.g., [1]. All categories are considered to be constructs, i.e., concrete categories of structured sets and structure-compatible maps. We denote by  $Hal_n$  the category of  $n$ -ary hyperalgebras as objects and homomorphisms as morphisms. Of course,  $Hal_n$  is transportable and direct products of  $n$ -ary hyperalgebras are concrete products in  $Hal_n$ . It is also evident that  $Hal_n$  is well-fibred, i.e., it is fibre-small and for each object with at most one element the corresponding fibre has exactly one element. Further, we denote by  $IHal_n$  and  $MDHal_n$  the full subcategories of  $Hal_n$  whose objects are precisely the idempotent  $n$ -ary hyperalgebras and the  $n$ -ary hyperalgebras that are both medial and diagonal, respectively. Of course, both  $IHal_n$  and  $MDHal_n$  are productive in  $Hal_n$ , and in  $IHal_n$  all constant maps are morphisms. Finally, we put  $IMDHal_n = IHal_n \cap MDHal_n$ .

Given a category  $\mathcal{C}$  and a  $\mathcal{C}$ -object  $G$ , we denote by  $|G|$  the underlying set of  $G$ , and given a pair  $G, H$  of  $\mathcal{C}$ -objects, we denote by  $\text{Mor}_{\mathcal{C}}(G, H)$  the

set of all  $\mathcal{C}$ -morphisms from  $G$  to  $H$ . In [8] the following generalization of the cartesian closedness is given:

**Definition 4.** Let  $\mathcal{C}$  be a category with finite concrete products and  $\mathcal{D}$ ,  $\mathcal{E}$  be full isomorphism closed subcategories of  $\mathcal{C}$ . Let  $\mathcal{E}$  be finitely productive in  $\mathcal{C}$ . We say that  $\mathcal{E}$  is *exponential* for  $\mathcal{D}$  in  $\mathcal{C}$  provided that for any two objects  $G \in \mathcal{D}$  and  $H \in \mathcal{E}$  there exists an object  $G^H \in \mathcal{D} \cap \mathcal{E}$  with  $|G^H| = \text{Mor}_{\mathcal{C}}(H, G)$  such that

- (i) for any  $\mathcal{E}$ -object  $K$  and any  $f \in \text{Mor}_{\mathcal{C}}(H \times K, G)$  the map  $f^* : K \rightarrow G^H$  given by  $f^*(z)(y) = f(y, z)$  whenever  $z \in |K|$  and  $y \in |H|$  fulfils  $f^* \in \text{Mor}_{\mathcal{C}}(K, G^H)$ ,
- (ii) the (so-called evaluation) map  $e : H \times G^H \rightarrow G$  given by  $e(y, f) = f(y)$  whenever  $y \in |H|$  and  $f \in \text{Mor}_{\mathcal{C}}(H, G)$  fulfils  $e \in \text{Mor}_{\mathcal{C}}(H \times G^H, G)$ .

Let us note that the conjunction of the conditions (i) and (ii) from Definition 4 means that the pair  $(G^H, e)$ , where  $e : H \times G^H \rightarrow G$  is the evaluation map, is a co-universal arrow for  $G$  with respect to the functor  $H \times - : \mathcal{E} \rightarrow \mathcal{C}$ .

If a category  $\mathcal{C}$  is exponential for itself in itself, then  $\mathcal{C}$  is cartesian closed, i.e., the functor  $H \times - : \mathcal{C} \rightarrow \mathcal{C}$  has a right adjoint for each object  $H \in \mathcal{C}$  (and vice versa whenever in  $\mathcal{C}$  all constant maps are morphisms). Especially, if  $\mathcal{E}$  is exponential for  $\mathcal{D}$  in  $\mathcal{C}$  and if also  $\mathcal{D}$  is finitely productive in  $\mathcal{C}$ , then  $\mathcal{D} \cap \mathcal{E}$  is cartesian closed.

The objects  $G^H$  from Definition 4 are called *function spaces*. In [8] it is shown that function spaces fulfil the first exponential law  $(G^H)^K \cong G^{H \times K}$  (where  $\cong$  denotes the isomorphism in  $\mathcal{C}$ ), and that they are unique up to the isomorphisms that are (carried by) identity maps - hence unique whenever  $\mathcal{C}$  is transportable.

**Theorem 3.**  *$IHal_n$  is an exponential category for  $MDHal_n$  in  $Hal_n$  for any positive integer  $n$ .*

**Proof.** From Theorem 2 and Lemma 1 it follows that the corresponding function spaces are given by powers. ■

**Corollary 1.**  *$IMDHal_n$  is a cartesian closed category for any positive integer  $n$ .* ■

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