THE SŁUPECKI CRITERION BY DUALITY

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Abstract

A method is presented for proving primality and functional completeness theorems, which makes use of the operation–relation duality. By the result of Sierpiński, we have to investigate relations generated by the two-element subsets of A^k only. We show how the method applies for proving Słupecki's classical theorem by generating diagonal relations from each pair of k-tuples.

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An algebra $\mathbf{A} = (A; F)$, with a finite set A, is called *primal* if all possible operations on A are term operations of $\mathbf{A}$. Establishing primality is often facilitated by theorems asserting that if F contains operations with some properties, then $\mathbf{A}$ is primal. A natural way to prove such theorems is to construct all operations on A as compositions of those in F. Another way is provided by the operation–relation duality exhibited by Bodnarčuk, Kalužnin, Kotov, Romov [3], and Geiger [6]. First, we outline their result in a few sentences. Let A be a set and B a subset of A^k. We say that an operation f *preserves* a relation $R \subseteq A^k$ if R is a subuniverse of the algebra $(A; f)^k$. A set of operations on a fixed carrier set is called a *clone* if it contains all projections and is closed under superposition. A nonempty set of relations is called a *closed class of relations* if it is closed under direct products, projections onto arbitrary sets of its variables and diagonalizations. Considering the Galois connection between operations and

relations, it turns out that the clones (which are exactly the Galois-closed sets of operations), and the closed classes of relations (which are exactly the Galois-closed sets of relations) mutually define each other by means of preservation of relations by operations.

If we apply this result only for the clone of all operations, we get that $(A; F)$ is primal iff F preserves exactly the relations on A constituting the least closed class of relations; this is also a consequence of another more general fact on quasiprimal algebras due to P.H. Krauss ([9], [10]). More and detailed information concerning this topic can be found in [7], [11], [14], and [15]. Related ideas were used, e. g., in [1].

First, we need some definitions. We consider a k-ary relation as a set of unary functions $r : \mathbf{k} \rightarrow A$, $\mathbf{k} = \{1, 2, \ldots, k\}$. We say that a k-ary relation D is *diagonal,* if there exists an equivalence relation ρ_D on $\mathbf{k}$ such that

$$D = \{r \colon \mathbf{k} \rightarrow A \mid r(u) = r(v) \text{ if } u\rho_D v, \ u, v \in \mathbf{k}\} .$$

All the diagonal relations on A form the minimal closed class of relations on A. Notice that a diagonal relation and the corresponding equivalence relation mutually define each other, so we may use the denotation D_ρ for the diagonal relation determined by an equivalence relation ρ on $\mathbf{k}$. Moreover, to each $r \in A^k$, we assign an equivalence relation ρ_r on the set $\mathbf{k}$ as follows:

$$u\rho_r v \qquad \text{iff} \qquad r(u) = r(v).$$

Clearly, for any diagonal relation D, we have $\rho_D = \bigcap_{r \in D} \rho_r$. Now let $R \subseteq A^k$. By $[R]$ we mean the underlying set of the subalgebra of $\mathbf{A}^k$ generated by R.

Proposition. (Bodnarčuk-Kalužnin-Kotov-Romov [2], Geiger [6], Krauss [9], [10]) *A finite algebra* $\mathbf{A} = (A; F)$ *is primal, iff every relation preserved by all operations in F is diagonal.*

The following Lemma 1 is a reformulation of the well known fact that the clone O_A of all operations defined on a finite set A can be generated by binary operations (Sierpiński [12]).

Lemma 1. *Given an algebra* $\mathbf{A} = (A; F)$, *the following two conditions are equivalent:*

  (i) *For each $R \subseteq A^k$, the relation $[R]$ is diagonal.*
  (ii) *For each $x, y \in A^k$, the relation $[x, y]$ is diagonal.*

We can formulate the statement of Lemma 1 without using the notion of a diagonal relation as follows.

Lemma 1′. *The following two conditions are equivalent:*
  (i) *The algebra* $\mathbf{A} = (A; F)$ *is primal.*

  (ii) *For each* $x, y, z \in A^k$, *we have* $z \in [x, y]$ *whenever*

$$((\forall u, v \in \mathbf{k})[x(u) = x(v) \land y(u) = y(v) \rightarrow z(u) = z(v))].$$

By Lemma 1, the problem of proving a primality theorem simplifies into the investigation of some suitably chosen matrices. We demonstrate the method on the Słupecki Criterion in details. We cannot avoid using the Yablonskiĭ Lemma ([17], see also, e.g., [8]):

Lemma 2 (Yablonskiĭ [8]). *Let* $f = f(x_1, \ldots, x_n)$ *be an at least binary operation on* A *depending on* x_1 *and* x_2 *such that the range of* f *contains at least three elements. Then there exist* $a, b, a_2, \ldots, a_n, b_2, \ldots b_n \in A$ *such that the elements* $c_1 = f(a, a_2, \ldots a_n)$, $c_2 = f(b, a_2, \ldots, a_n)$, *and* $c_3 = f(a, b_2, \ldots, b_n)$ *are pairwise different.*

We call an operation *essential*, if it is surjective and at least binary.

Theorem 1 (Słupecki [13]). *Let* A *be a finite set with* $|A| > 2$. *If* F *contains an essential operation* f *and all the unary operations, then the algebra* $\mathbf{A} = (A; F)$ *is primal.*

Proof. We shall show that (ii) of Lemma 1 holds in such a way that we proceed by induction on the number t of blocks of the equivalence relation ρ showing, that $\rho \supseteq \rho_x \cap \rho_y$ implies $D_\rho \subseteq [x, y]$. In all matrices we use below, one row means one element $r \in A^k$. We represent each block of the intersection of the equivalence relations corresponding to k-tuples displayed in a matrix by one of its entries.

Let $x, y \in A^k$. Let f be n-ary. Case $t = 1$ is trivial by using unary constant operations only. Now we treat the case $t = 2$. If ρ is a two-block equivalence relation with $\rho \supseteq \rho_x \cap \rho_y$ having one block in common with $\rho_x \cap \rho_y$, then there exist two-block equivalence relations ρ', ρ'' with $\rho' \supseteq \rho_x$ and $\rho'' \supseteq \rho_y$ such that C' and C'' are blocks of $\rho_{d'}$ and $\rho_{d''}$ respectively and the blocks of ρ are $C' \cap C''$ and $\overline{C' \cap C''}$. We choose $a, b, a_2, \ldots, a_n, b_2, \ldots b_n \in A$ in such a way that $c_1 = f(a, a_2, \ldots a_n)$, $c_2 = f(b, a_2, \ldots, a_n)$, and $c_3 = f(a, b_2, \ldots, b_n)$ are pairwise different (the existence of which is guaranteed by Yablonskiĭ Lemma). Let $f(b, b_2, \ldots, b_n) = c_4$. We shall display a matrix,

the rows of which are suitably chosen elements of A^k. The first row is an element of the diagonal relation $D_{\rho'}$, the others except for the final one under the line are elements of the diagonal relation $D_{\rho''}$. The values of f when f is applied to the columns are in the bottom line.

$$(3) \quad \begin{matrix} b & a & a & b \\ a_2 & a_2 & b_2 & b_2 \\ \vdots & \vdots & \vdots & \vdots \\ a_n & a_n & b_n & b_n \\ \hline \\ c_2 & c_1 & c_3 & c_4 \end{matrix}$$

Now, if $c_1 \neq c_4$, then a unary operation φ for which $\varphi(c_1) = d_0 \in A$ and $\varphi(c_2) = \varphi(c_3) = \varphi(c_4) = d_1 \in A$ with $d_0 \neq d_1$ is enough to produce an element z of D_ρ with $\rho_z = \rho$. With the help of the unary operations, we can generate each element of the diagonal relation D_ρ. However, if $c_1 = c_4$, then we have only to rearrange the matrix in the following way:

$$(4) \quad \begin{matrix} a & b & b & a \\ a_2 & a_2 & b_2 & b_2 \\ \vdots & \vdots & \vdots & \vdots \\ a_n & a_n & b_n & b_n \\ \hline \\ c_1 & c_2 & c_4 & c_3. \end{matrix}$$

Now, $c_2 \neq c_1, c_3, c_4$, therefore a unary operation $\varphi \in F$ for which $\varphi(c_1) = \varphi(c_3) = \varphi(c_4) = d_0 \in A$ and $\varphi(c_2) = d_1$ with $d_0 \neq d_1 \in A$ does the job. It is easy to check, that if ρ does not have common block with $\rho_x \cap \rho_y$, then $C = \bigcup_{i=1}^{j}(C_i' \cap C_i'')$ holds for any of the blocks of ρ and by induction on j (using matrices (3) and (4) without their second column) case $t = 2$ is complete.

Let $t > 2$. By Lemma 1′, it is enough to show that $\rho \supseteq \rho_x \cap \rho_y$ implies $D_\rho \subseteq [x, y]$ for $\rho = \rho_z$ with some $z \in A^k$. Then ρ has at most $|A|$ blocks; therefore $t \leq |A|$. We have already chosen $a, b, a_2, \ldots, a_n, b_2, \ldots b_n \in A$ in such a way that $c_1 = f(a, a_2, \ldots a_n)$, $c_2 = f(b, a_2, \ldots, a_n)$, and $c_3 = f(a, b_2, \ldots, b_n)$ are pairwise different (Yablonskiĭ Lemma). For $4 \leq k \leq t$,

let c_k be such elements of A that $c_1, \ldots, c_t$ are pairwise different and select elements $d_{i,k} \in A$ such that $f(d_{1,k}, \ldots, d_{n,k}) = c_k$. Let ρ' and ρ'' $(t-1)$-block equivalence relations on $\mathbf{k}$, both having $t-3$ same blocks in common with ρ and satisfying $\rho' \supseteq \rho_x \cap \rho_y$ and $\rho'' \supseteq \rho_x \cap \rho_y$. Obviously, $D_{\rho'}, D_{\rho''} \subseteq [x, y]$ because of the induction hypothesis. We shall display a matrix again, the rows of which are suitably chosen elements of A^k. The first row is an element of the diagonal relation $D_{\rho'}$, the others except for the final one under the line are elements of the diagonal relation $D_{\rho''}$. The values of f are in the bottom line.

$$
\begin{array}{cccccc}
b & a & a & d_{1,4} & \ldots & d_{1,t} \\
a_2 & a_2 & b_2 & d_{2,4} & \ldots & d_{2,t} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
a_n & a_n & b_n & d_{n,4} & \ldots & d_{n,t} \\
\hline \\
c_2 & c_1 & c_3 & c_4 & \ldots & c_t.
\end{array}
$$

We can see in the last row (where the values of f are situated) that with the help of the unary operations we can produce each element of the t-block diagonal relation D_ρ. $\blacksquare$

There is an improvement of the Słupecki Criterion by Yablonskiĭ (see, e.g. [8]): if we omit the injective unary operations from F, then $(A; F)$ is still primal. Even though every one of the previous steps needs some reconsideration, this case can also be completed by the method facilitated by the Proposition and Lemma 1.

An algebra $\mathbf{A} = (A; F)$ is called *functionally complete* if all possible operations on the base set A are polynomials of $\mathbf{A}$. Proving functional completeness for $(A; F)$ is the same as proving primality for the algebra $(A; F \cup F_0)$ where F_0 is the set of all constant operations on A.

The above type matrices can be analyzed easily not only in case of the Słupecki Criterion but also in case of other primality and functional completeness results. We proved e.g. the functional completeness of the ternary discriminator [14], the dual discriminator (for $|A| \geq 3$) [5], the n-ary $(n \geq 3)$ near-projections [3] as well as the primality theorem of Foster [4] this way.

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