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LINEAR OPERATORS PRESERVING MAXIMAL COLUMN RANKS OF NONBINARY BOOLEAN MATRICES*

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Abstract

The maximal column rank of an m by n matrix is the maximal number of the columns of A which are linearly independent. We compare the maximal column rank with rank of matrices over a nonbinary Boolean algebra. We also characterize the linear operators which preserve the maximal column ranks of matrices over nonbinary Boolean algebra.

Keywords: Boolean matrix, semiring, linear operator on matrices, congruence operator on matrices, maximal column rank of a matrix, Boolean rank of a matrix.

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1. Introduction

There is much literature on the study of linear operators that preserve rank of matrices over several semirings. Boolean matrices also have been the subject of research by many authors ([1] – [6]).

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Hwang, Kim and Song [3] defined a maximal column rank of a matrix over a semiring and compared it with column rank. And they obtained characterization of the linear operators that preserve maximal column rank of binary Boolean matrices. Kirkland and Pullman [4] exhibited characterizations of the linear operators that preserve several invariants of matrices over nonbinary Boolean algebra. But they did not deal with the maximal column rank.

In this paper we continue the study of maximal column rank of matrices over nonbinary Boolean algebras. We also obtain characterizations of the linear operators that preserve maximal column ranks of nonbinary Boolean matrices.

2. Comparison of rank and maximal column rank of Boolean matrices

Let $\mathbb{B}$ be a finite Boolean algebra. We may assume that $\mathbb{B}$ consists of the subsets of a k-element set S_k. Union is denoted by $+$, and intersection by juxtaposition ; 0 denote the null set and 1 the set S_k. Let $\mathbb{M}_{m,n}(\mathbb{B})$ denote the set of all $m \times n$ matrices with entries in $\mathbb{B}$. Addition and multiplication of matrices over $\mathbb{B}$ are defined as if it were a filed, as are the zero matrix, O, and the identity matrix, I.

Let $\sigma_1, \sigma_2, \cdots, \sigma_k$ denote the *singleton subsets* of S_k. For each $p \times q$ matrix A over $\mathbb{B}$, the *l-th constituent* of A, A_l, is the $p \times q$ binary matrix whose (i, j) entry is 1 if and only if $\alpha_{ij} \supseteq \sigma_l$. By these constituents, A can be written uniquely as $\sum_l \sigma_l A_l$, which is called the *canonical form* of A.

From the canonical forms, we have that for all $p \times q$ matrices A, all $q \times r$ matrices B and C, and all $\alpha \in \mathbb{B}$, (a) $(AB)_l = A_l B_l$, (b) $(B+C)_l = B_l + C_l$, and (c) $(\alpha A)_l = \alpha_l A_l$, for all $1 \leq l \leq k$.

The *Boolean rank*, $b(A)$, of a nonzero $A \in \mathbb{M}_{m,n}(\mathbb{B})$ is defined as the least r such that $A = BC$ for some $B \in \mathbb{M}_{m,r}(\mathbb{B})$ and $C \in \mathbb{M}_{r,n}(\mathbb{B})$. The rank of zero matrix is zero ; in the case that $\mathbb{B} = \mathbb{B}_1 = \{0, 1\}$, we refer to $b(A)$ as the *binary Boolean rank*, and denote it by $b_1(A)$.

For a binary Boolean matrix A, we have $b(A) = b_1(A)$ by definition.

A set G of $m \times 1$ matrices over $\mathbb{B}$ is *linearly dependent* if for some $g \in G$, g is a linear combination of elements in $G - \{g\}$. Otherwise G is *linearly independent.*

The *maximal column rank* [3], $mc(A)$, of a matrix $A \in \mathbb{M}_{m,n}(\mathbb{B})$ is the maximal number of the columns of A which are linearly independent over $\mathbb{B}$. In the case that $\mathbb{B} = \mathbb{B}_1$, we denote it by $mc_1(A)$ for $A \in \mathbb{M}_{m,n}(\mathbb{B}_1)$.

It follows that

$$0 \leq b(A) \leq mc(A) \leq n \tag{2.1}$$

for all $m \times n$ matrices A over $\mathbb{B}$.

The inequality in (2.1) may be strict over $\mathbb{B}$. For example, we consider the matrix

$$\begin{bmatrix} \sigma_1 & \sigma_2 \\ \sigma_1 & \sigma_2 \end{bmatrix} \tag{2.2}$$

over $\mathbb{B}$, where σ_1 and σ_2 are distinct singletons. Then $A = \begin{bmatrix} 1 \\ 1 \end{bmatrix} [\sigma_1 \sigma_2]$ has Boolean rank 1, but $mc(A) = 2$ since the two columns are linearly independent over $\mathbb{B}$.

Let $\beta(\mathbb{B}, m, n)$ be the largest integer r such that for all $m \times n$ matrices A over $\mathbb{B}$, $b(A) = mc(A)$ if $b(A) \leq r$. The previous example shows that $\beta(\mathbb{B}, 2, 2) < 1$. In general $0 \leq \beta(\mathbb{B}, m, n) \leq n$. In the case that $\mathbb{B} = \mathbb{B}_1$, we denote it by $\beta(\mathbb{B}_1, m, n)$.

We also obtain that

$$b\left(\begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix}\right) = b(A) \qquad \textit{and} \qquad mc\left(\begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix}\right) = mc(A) \tag{2.3}$$

for all $m \times n$ matrices A over $\mathbb{B}$.

Lemma 2.1. *For an arbitrary matrix $A \in \mathbb{M}_{m,n}(\mathbb{B}_1)$, we have $mc(A) = mc_1(A)$.*

Proof. Assume $mc(A) = r$. Then there are r columns $\mathbf{a_1}, \mathbf{a_2}, \cdots, \mathbf{a_r}$ of A which are linearly independent over $\mathbb{B}$.

Consider the p-th constituents $(\mathbf{a_1})_p, (\mathbf{a_2})_p, \cdots, (\mathbf{a_r})_p$ for $p = 1, 2, \cdots, k$. If $(\mathbf{a_i})_p = \sum_{j \neq i}^{r} \alpha_j (\mathbf{a_j})_p$ with $\alpha_j \in \{0, 1\} = \mathbb{B}_1$, then

$$\mathbf{a_i} = (\mathbf{a_i})_p = \sum_{j \neq i}^{r} \alpha_j (\mathbf{a_j})_p = \sum_{j \neq i}^{r} \alpha_j \mathbf{a_j}.$$

This contradicts the assumption. Thus $(\mathbf{a_1})_p, (\mathbf{a_2})_p, \cdots, (\mathbf{a_r})_p$ are linearly independent over $\mathbb{B}_1$. But they are the same as $\mathbf{a_1}, \mathbf{a_2}, \cdots, \mathbf{a_r}$ and hence $mc_1(A) \geq r$.

Conversely, if $mc_1(A) = r$, then there are r columns $\mathbf{a_1}, \mathbf{a_2}, \cdots, \mathbf{a_r}$ which are linearly independent over $\mathbb{B}_1$. If $\mathbf{a_i} = \sum_{j \neq i}^{r} \alpha_j \mathbf{a_j}$ with $\alpha_j \in \mathbb{B}$, then

$$\mathbf{a_i} = (\mathbf{a_i})_p = \sum_{j \neq i}^{r} (\alpha_j)_p (\mathbf{a_j})_p = \sum_{j \neq i}^{r} (\alpha_j)_p \mathbf{a_j}.$$

for any $p = 1, 2, \cdots, k$. This contradicts the assumption. Thus $\mathbf{a_1}, \mathbf{a_2}, \cdots, \mathbf{a_r}$ are linearly independent over $\mathbb{B}$ and hence $mc(A) \geq r$. ■

Lemma 2.2. *If $mc(A) > b(A)$ for some $p \times q$ matrix A over $\mathbb{B}$, then for all $m \geq p$ and $n \geq q$, $\beta(\mathbb{B}, m, n) < b(A)$.*

Proof. Since $mc(A) > b(A)$ for some $p \times q$ matrix A, we have $\beta(\mathbb{B}, p, q) < b(A)$ from the definition of β. Let $B = \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix}$ be an $m \times n$ matrix containing A as a submatrix. Then

$$b(B) = b(A) < mc(A) = mc(B)$$

by (2.3). So, $\beta(\mathbb{B}, m, n) < b(B)$ for all $m \geq p$ and $n \geq q$. ■

Lemma 2.3. *For any $A \in \mathbb{M}_{m,n}(\mathbb{B}_1)$, we have $b_1(A) = 1$ if and only if $mc_1(A) = 1$.*

Proof. Suppose $b_1(A) = 1$. Then A can be factored as

$$\begin{aligned} A &= \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_m \end{bmatrix} [b_1, b_2, \cdots, b_n] \\ &= \begin{bmatrix} a_1 b_1 & \cdots & a_1 b_i & \cdots & a_1 b_n \\ a_2 b_1 & \cdots & a_2 b_i & \cdots & a_2 b_n \\ \vdots & & \vdots & & \vdots \\ a_m b_1 & \cdots & a_m b_i & \cdots & a_m b_n \end{bmatrix}. \end{aligned}$$

If there exist nonzero b_p and b_q for $p \neq q$, then $b_p = b_q = 1$. Thus the pth and qth columns of A are the same and hence they are linearly dependent.

This implies that any two nonzero columns of A are the same. Therefore $mc_1(A) = 1$.

The converse is obvious from (2.1) ■

But as we have seen from the matrix in (2.2), it is not necessary that $b(A) = 1$ if and only if $mc(A) = 1$.

Example 2.4. Consider

$$A = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} \in \mathbb{M}_{3,4}(\mathbb{B}_1).$$

Then $b_1(A) = 3$ but $mc_1(A) = 4$.

For, all columns of A are linearly independent over $\mathbb{B}_1$, so we have $mc_1(A) = 4$. And $2 \leq b_1(A) \leq 3$ by Lemma 2.3.

If $b_1(A) = 2$, then A can be factored as

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} & b_{13} & b_{14} \\ b_{21} & b_{22} & b_{23} & b_{24} \end{bmatrix} = [(a_{i1}b_{1j} + a_{i2}b_{2j})].$$

If we take $a_{11} = 0$ in this factorization, then we have $b_{21} = b_{24} = 0$ and $a_{12} = b_{22} = b_{23} = 1$ from the comparison of the first row of A, and $a_{21} = b_{11} = b_{14} = 1$ and $a_{22} = b_{12} = b_{13} = 0$ from the comparison of the second row of A. But $a_{31} = a_{32} = 0$ from the comparison of the third row of A, which is impossible. Similarly, if we take $a_{11} = 1$, we also have $a_{31} = a_{32} = 0$, which is impossible.

Therefore $b_1(A) \neq 2$, which implies $b_1(A) = 3$.

Theorem 2.5. *For $m \times n$ matrices over the binary Boolean algebra $\mathbb{B}_1$, we have the values of β as follows;*

$$\beta(\mathbb{B}_1, m, n) = \begin{cases} 1 & \text{if} \quad min\{m, n\} = 1, \\ 3 & m \geq 3 \quad \text{and} \quad n = 3, \\ 2 & \text{otherwise}. \end{cases}$$

Proof. For the case $min\{m, n\} = 1$, Lemma 2.3 implies that $\beta(\mathbb{B}_1, m, n) = 1$. Let

$$A = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}.$$

Then we have $mc_1(A) = 4$ and $b_1(A) = 3$ from Example 2.4. Thus Lemma 2.2 implies that $\beta(\mathbb{B}_1, m, n) \leq 2$ for all $m \geq 3$ and $n \geq 4$.

Suppose $m \geq 2$, $n \geq 2$ and $A \in \mathbb{M}_{m,n}(\mathbb{B}_1)$. If $mc_1(A) = 2$, then $b_1(A) = 1$ or 2. But $b_1(A) \neq 1$ by Lemma 2.3. Thus $b_1(A) = 2$. Conversely, if $b_1(A) = 2$, then there exist $F \in \mathbb{M}_{m,2}(\mathbb{B}_1)$ and $G \in \mathbb{M}_{2,n}(\mathbb{B}_1)$ such that $A = FG$. Since $b_1(F) = 2$, $mc_1(F) = 2$ by (2.1). But the two columns f_1 and f_2 of F span all the columns of A over $\mathbb{B}_1$. Hence each column of A has one of the forms f_1, f_2, $f_1 + f_2$ or 0. Then any three columns of them are linearly dependent, so $mc_1(A) \leq 2$. Hence $mc_1(A) = 2$. Therefore we get $\beta(\mathbb{B}_1, m, n) \geq 2$ for all $m \geq 2$ and $n \geq 2$.

Finally, consider the case $m \geq 3$ and $n = 3$. Let $A \in \mathbb{M}_{m,3}(\mathbb{B}_1)$. If $mc_1(A) = 3$, then $b_1(A) = 1$, 2 or 3. But $b_1(A)$ cannot be 1 by Lemma 2.3, and $b_1(A) \neq 2$ by the above argument. Thus $b_1(A) = 3$. Conversely, if $b_1(A) = 3$, then it is obvious that $mc_1(A) = 3$ for this case.

Therefore we have the values of β as required.

Kirkland and Pullman obtained the relation between the rank of a Boolean matrix and the binary ranks of its constituents in [4] as follows:

(2.4) The Boolean rank of a matrix in $\mathbb{M}_{m,n}(\mathbb{B})$ is the maximum of the binary Boolean ranks of its constituents.

But for the maximal column rank of the Boolean matrix, we do not have such relation as (2.4).

Example 2.6. Let $A = \begin{bmatrix} \sigma_1 & \sigma_1 & 1 \\ 0 & 1 & 1 \end{bmatrix} \in \mathbb{M}_{2,3}(\mathbb{B})$. Then any two columns cannot span the other column. Thus $mc(A) = 3$. But

$$mc_1(A_1) = mc_1\left(\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}\right) = 2$$

and

$$mc_1(A_p) = mc_1\left(\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}\right) = 2 \text{ for all } p = 2,\ 3,\ \cdots,\ k.$$

Therefore we have $mc(A) > max\{mc_1(A_p) | p = 1,\ 2,\ \cdots,\ k\}$.

In general, we obtain the following relation between the maximal column rank of a Boolean matrix and the maximal column rank of its constituents over binary Boolean algebra.

Proposition 2.7. *For $A \in \mathbb{M}_{m,n}(\mathbb{B})$, if $\sum \sigma_p A_p$ is the canonical form of A, then*

$$max\{mc_1(A_p)|1 \le p \le k\} \le mc(A).$$

Proof. Suppose $mc(A) = r$. If $mc_1(A_p) > r$ for some p, then there exist $r+1$ columns $(\mathbf{a_1})_p, (\mathbf{a_2})_p, \cdots, (\mathbf{a_{r+1}})_p$ which are linearly independent over $\mathbb{B}$. Since $mc(A) = r$, these columns $\mathbf{a_1}, \mathbf{a_2}, \cdots, \mathbf{a_{r+1}}$ are linearly dependent over $\mathbb{B}$. So $\mathbf{a_j} = \sum_{i \neq j}^{r+1} \alpha_i \mathbf{a_i}$, $\alpha_i \in \mathbb{B}$ for some j. Thus $(\mathbf{a_j})_p = (\sum_{i \neq j}^{r+1} \alpha_i \mathbf{a_i})_p = \sum_{i \neq j}^{r+1} (\alpha_i)_p (\mathbf{a_i})_p$, which contradicts the linearly independence. Therefore $mc_1(A_p) \le r = mc(A)$ for all $p = 1,\ 2,\ \cdots,\ k$. ■

Theorem 2.8. *For $m \times n$ matrices over nonbinary Boolean algebra $\mathbb{B}$, we have the values of β as follows;*

$$\beta(\mathbb{B}, m, n) = \begin{cases} 1 & if \quad n = 1, \\ 0 & if \quad n \ge 2. \end{cases}$$

Proof. For any $A \in \mathbb{M}_{m,1}(\mathbb{B})$, we have that $b(A) = 1$ if and only if $mc(A) = 1$. Thus $\beta(\mathbb{B}, m, 1) = 1$. Let $A = [\sigma_1, \sigma_2] \in \mathbb{M}_{1,2}(\mathbb{B})$, where σ_1 and σ_2 are distinct. Then $b(A) = 1$, but $mc(A) = 2$. Hence $\beta(\mathbb{B}, m, n) = 0$ for $n \ge 2$ by Lemma 2.2. ■

3. Maximal column rank preservers over $\mathbb{M}_{m,n}(\mathbb{B})$

In this section we obtain characterizations of the linear operators that preserve maximal column rank of matrices over nonbinary Boolean algebra.

A linear operator T on $\mathbb{M}_{m,n}(\mathbb{B})$ is said to *preserve maximal column rank* if $mc(T(A)) = mc(A)$ for all $A \in \mathbb{M}_{m,n}(\mathbb{B})$. It *preserves maximal column rank* r if $mc(T(A)) = r$ whenever $mc(A) = r$. For the terms Boolean rank preserver and Boolean rank r preserver, they are defined similarly [3].

If T is a linear operator on $\mathbb{M}_{m,n}(\mathbb{B})$, for each $1 \le p \le k$, define its *p-th constituent*, T_p, by $T_p(X) = (T(X))_p$ for every $X \in \mathbb{M}_{m,n}(\mathbb{B}_1)$. By the linearity of T, we have $T(A) = \sum \sigma_p T_p(A_p)$ for any matrix $A \in \mathbb{M}_{m,n}(\mathbb{B})$.

Since $\mathbb{M}_{m,n}(\mathbb{B})$ is a semiring, we can consider the invertible members of its multiplicative monoid. Wedderburn [6] showed that a Boolean matrix is invertible if and only if all its constituents are permutation matrices.

Lemma 3.1. *The maximal column rank of a Boolean matrix is preserved under pre-multiplication by an invertible matrix.*

Proof. Let $A \in \mathbb{M}_{m,n}(\mathbb{B})$ and U be an invertible matrix in $\mathbb{M}_{m,n}(\mathbb{B})$. If $mc(A) = h$, then there exist h linearly independent columns $\mathbf{a}_{\boldsymbol{i}(1)}, \mathbf{a}_{\boldsymbol{i}(2)}, \cdots, \mathbf{a}_{\boldsymbol{i}(\boldsymbol{h})}$ in A which are maximal. Then $U\mathbf{a}_{\boldsymbol{i}(1)}, U\mathbf{a}_{\boldsymbol{i}(2)}, \cdots, U\mathbf{a}_{\boldsymbol{i}(\boldsymbol{h})}$ are linearly independent columns of UA. Thus $mc(UA) \geq h$.

Conversely, if $mc(UA) = h$, then there exist h linearly independent columns $\mathbf{v}_{\boldsymbol{i}(1)}, \mathbf{v}_{\boldsymbol{i}(2)}, \cdots, \mathbf{v}_{\boldsymbol{i}(\boldsymbol{h})}$ which are maximal. Then $U^{-1}\mathbf{v}_{\boldsymbol{i}(1)}, U^{-1}\mathbf{v}_{\boldsymbol{i}(2)}, \cdots, U^{-1}\mathbf{v}_{\boldsymbol{i}(\boldsymbol{h})}$ are linearly independent columns of $U^{-1}(UA) = A$. Hence $mc(A) \geq h$. Therefore $mc(A) = mc(UA)$. ■

But the maximal column rank of a Boolean matrix is not preserved under post-multiplication by an invertible matrix.

Example 3.2. Let $\mathbb{B}$ be the Boolean algebra of subsets of $\{1, 2, 3\}$. Consider

$$U = \begin{bmatrix} \{1\} & \{2\} & \{3\} \\ \{2\} & \{3\} & \{1\} \\ \{3\} & \{1\} & \{2\} \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} \{1\} & \{2\} & \{3\} \\ \{2\} & \{3\} & \{1\} \end{bmatrix}.$$

Then U is an invertible matrix in $\mathbb{M}_{3,3}(\mathbb{B})$ whose inverse is U itself. And $mc(A) = 3$ since the three columns are linearly independent. But $mc(AU) = 2$ since

$$AU = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

Thus the post-multiplication by an invertible matrix does not preserve maximal column rank r for $r \geq 3$.

Lemma 3.3. *Assume that T is a linear operator on $\mathbb{M}_{m,n}(\mathbb{B})$. If T preserves maximal column rank r, then each constituent T_p preserves maximal column rank r on $\mathbb{M}_{m,n}(\mathbb{B}_1)$.*

Proof. Suppose that T preserves maximal column rank r. Let Y be any matrix over $\mathbb{B}_1$ such that $mc_1(Y) = r$. Then Lemma 2.1 implies that $mc(Y) = r$ and $mc(\sigma_p Y) = r$ for each $p = 1, 2, \cdots, k$. Since T preserves maximal coulmn rank r, $mc(T(\sigma_p Y)) = r$.

But

$$\begin{aligned} r = mc(T(\sigma_p Y)) &= mc(\sigma_p T(Y)) \\ &= mc(\sigma_p \textstyle\sum_i \sigma_i T_i(Y_i)) \\ &= mc(\sigma_p T_p(Y)). \end{aligned}$$

Therefore $mc(\sigma_p T_p(Y)) = r$ for each $p = 1, 2, \cdots, k$, and hence $mc(T_p(Y)) = r$. ■

Lemma 3.4. *Suppose T is a linear operator on $\mathbb{M}_{m,n}(\mathbb{B})$. Then T preserves Boolean rank r if and only if each constituent T_p is a binary Boolean rank preserving operator on $\mathbb{M}_{m,n}(\mathbb{B}_1)$.*

Proof. Let $b(A) = r$ for $A \in \mathbb{M}_{m,n}(\mathbb{B})$. Then there exists some p such that $b_1(A_p) = r$ and $b_1(A_q) \leq r$ for $1 \leq q \leq k$ by property (2.4). Thus $b_1(T_p(A_p)) = r$ and $b_1(T_q(A_q)) \leq r$ for $1 \leq q \leq k$. Since $b(T(A)) = max\{b_1(T_q(A_q))|1 \leq q \leq k\} = r$ by property (2.4), T preserves Boolean rank r.

For the converse, it is similar to the proof of Lemma 3.3. ■

Now we need the following definitions of linear operators on the $m \times n$ matrices over $\mathbb{B}$. For any fixed pair of invertible $m \times m$ and $n \times n$ Boolean matrices U and V, the operator $A \to UAV$ is called a *congruence operator*. Let σ^* denote the complement of σ for each σ in $\mathbb{B}$. For $1 \leq q \leq k$, we define the *q-th rotation operator*, $R^{(q)}$, by

$$R^{(q)}(A) = \sigma_q A_q^t + \sigma_q^* A,$$

where A_q^t is the transpose matrix of A_q. We see that $R^{(q)}$ has the effect of transposing A_q while leaving the remaining constituents unchanged. Each rotation operator is linear on $\mathbb{M}_{m,n}(\mathbb{B})$ and their product is the transposition operator, $R : A \to A^t$.

Example 3.5. Let

$$A = \begin{bmatrix} 0 & 0 & 0 \\ \sigma_1 & \sigma_1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

be a matrix on $\mathbb{M}_{3,3}(\mathbb{B})$. then $mc(A) = 3$ by Example 2.6 and property (2.3). But

$$\begin{aligned} R^{(1)}(A) &= \sigma_1 A_1^t + \sigma_1^* A \\ &= \sigma_1 \begin{pmatrix} 0 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}^t + \sigma_1^* \begin{pmatrix} 0 & 0 & 0 \\ \sigma_1 & \sigma_1 & 1 \\ 0 & 1 & 1 \end{pmatrix} \\ &= A^t \end{aligned}$$

which has maximal column rank 2. Thus the rotation operator does not preserve maximal column rank 3 on $\mathbb{M}_{m,m}(\mathbb{B})$ for $m \geq 3$ by property (2.3).

Lemma 3.6 ([4]). *If T is a linear operator on $\mathbb{M}_{m,m}(\mathbb{B})$, then the following are equivalent.*

(1) *T preserves Boolean ranks 1 and 2.*

(2) *T is in the group of operators generated by the congruence (if $m = n$, also the rotation) operators.*

∎

Theorem 3.7. *Suppose T is a linear operator on $\mathbb{M}_{m,n}(\mathbb{B})$ for $m \geq 3$ and $n > 1$. Then the following are equivalent.*

(1) *T preserves maximal column rank.*

(2) *T preserves maximal column ranks 1, 2 and 3.*

(3) *There exist an invertible matrix $U \in \mathbb{M}_{m,m}(\mathbb{B})$ and a permutation matrix $P \in \mathbb{M}_{n,n}(\mathbb{B})$ such that $T(A) = UAP$ for all $A \in \mathbb{M}_{m,n}(\mathbb{B})$.*

Proof. Obviously (1) implies (2). Assume that T preserves maximal column ranks 1, 2 and 3. Then each constituent T_p preserves binary maximal column ranks 1, 2 and 3 by Lemma 3.3. For $A \in \mathbb{M}_{m,n}(\mathbb{B}_1)$, Theorem 3.1 implies that $b_1(A) = mc_1(A)$ for $b_1(A) \leq 2$. Thus T_p preserves binary Boolean ranks 1 and 2, and hence T preserves Boolean ranks 1 and 2 by Lemma 3.4. So T is in the group of operators generated by congruence (if $m = n$, also the rotation) operators by Lemma 3.6. But the rotation operator does not preserve maximal column rank 3 by Example 3.5, and the post-multiplication by an invertible matrix does not preserve maximal column rank 3 by Example 3.2. But the operation of permuting the columns does not change the maximal number of linearly independent columns of the given matrix. Hence in order to preserve maximal column ranks 1, 2 and 3,

T has the form $T(A) = UAP$ for some invertible matrix $U \in \mathbb{M}_{m,m}(\mathbb{B})$ and some permutation matrix $P \in \mathbb{M}_{n,n}(\mathbb{B})$. That is, (2) implies (3). Finally, if we assume (3), then T preserves maximal column rank by Lemma 3.1 and the fact that the post-multiplication by a permutation matrix preserves the maximal column rank. Hence (3) implies (1). ∎

If $m \leq 2$, then the linear operators that preserve maximal column rank on $\mathbb{M}_{m,n}(\mathbb{B})$ are the same as the Boolean rank-preservers, which were characterized in [4].

Thus we have characterizations of the linear operators that preserve the maximal column rank of nonbinary Boolean matrices.

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References

[1] L.B. Beasley and N.J. Pullman, *Boolean rank-preserving operators and Boolean rank-1 spaces*, Linear Algebra Appl. **59** (1984), 55–77.

[2] L.B. Beasley and N.J. Pullman, *Semiring rank versus column rank*, Linear Algebra Appl. **101** (1988), 33–48.

[3] S.G. Hwang, S.J. Kim and S.Z. Song, *Linear operators that preserve maximal column rank of Boolean matrices*, Linear and Multilinear Algebra **36** (1994), 305–313.

[4] S. Kirkland and N. J. Pullman, *Linear operators preserving invariants of nonbinary matrices*, Linear and Multilinear Algebra **33** (1992), 295–300.

[5] S.Z. Song, *Linear operators that preserve Boolean column ranks*, Proc. Amer. Math. Soc. **119** (1993), 1085–1088.

[6] J.H.M. Wedderburn, *Boolean linear associative algebra*, Ann. of Math. **35** (1934), 185–194.

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