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### CONGRUENCES ON PSEUDOCOMPLEMENTED SEMILATTICES

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#### Abstract

It is known that congruence lattices of pseudocomplemented semilattices are pseudocomplemented [4]. Many interesting properties of congruences on pseudocomplemented semilattices were described by Sankappanavar in [4], [5], [6]. Except for other results he described congruence distributive pseudocomplemented semilattices [6] and he characterized pseudocomplemented semilattices whose congruence lattices are Stone, i.e. belong to the variety  $\mathcal{B}_1$  [5].

In this paper we give a partial solution to a more general question: Under what condition on a pseudocomplemented semilattice its congruence lattice is element of the variety  $\mathcal{B}_n$   $(n \ge 2)$ ?

In the last section we widen the Sankappanavar's result to obtain the description of pseudocomplemented semilattices with relative Stone congruence lattices. A partial solution of the description of pseudocomplemented semilattices with relative  $(L_n)$ -congruence lattices  $(n \ge 2)$  is also given.

**Keywords:** pseudocomplemented semilattice, congruence lattice, p-algebra, Stone algebra, (relative)  $(L_n)$ -lattice.

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#### 1. Preliminaries

A pseudocomplemented semilattice (PCS) is an algebra  $S = \langle S; \wedge, *, 0 \rangle$ , where  $\langle S; \wedge, 0 \rangle$  is a  $\wedge$ -semilattice with 0 and \* is the unary operation of pseudocomplementation defined by:

$$x \wedge a = 0$$
 iff  $x \leq a^*$ .

 $0^*$  is the largest element in S and is denoted by 1. An element  $a \in S$  is called *closed* if  $a = a^{**}$ . The set of all closed elements of S is denoted B(S). It is known that  $\langle B(S); +, \wedge, ^*, 0, 1 \rangle$  forms a Boolean algebra in which the operation of join is defined by  $a + b = (a^* \wedge b^*)^*$ . To denote the join of subset  $A \subseteq B(S)$  of closed elements we will use the symbol  $\sum A$ .

An element  $d \in S$  is called *dense* if  $d^* = 0$ . All dense elements form the set denoted D(S) which is a filter in S.

The set of all congruences on PCS S is denoted Con(S). It is known that Con(S) is an algebraic pseudocomplemented lattice [4] with  $\Delta$  and  $\nabla$ the least and the largest element, respectively.

For any pair  $a, b \in S$  the symbol  $\theta(a, b)$  denotes the *principal congruence* relation generated by a, b, i.e. the least congruence relation  $\theta$  on S for which  $(a, b) \in \theta$ .

The congruence relation  $\varphi$  defined by:

$$(x,y) \in \varphi$$
 iff  $x^* = y^*$ ,

is called the *Glivenko congruence relation*.

For arbitrary filter  $F \subseteq S$  we define binary relation  $\hat{F}$ :

 $(x,y) \in \hat{F}$  iff  $x \wedge f = y \wedge f$  for some  $f \in F$ .

Clearly  $\hat{F}$  is a semilattice congruence relation on S. For arbitrary element  $f \in S$  the interval  $[0, f] \subseteq S$  is a PCS such that the pseudocomplement  $a_{[0,f]}^*$  is equal to  $a^* \wedge f$ . It follows that  $\hat{F}$  is compatible also with the operation of pseudocomplementation and  $\hat{F} \in Con(S)$ . Similarly for arbitrary element  $a \in S$  we define binary relation  $\hat{a}$  by

$$(x,y) \in \hat{a}$$
 iff  $x \wedge a = y \wedge a$ .

Again  $\hat{a} \in Con(S)$ . One can easily verify that  $\hat{a} = \theta(a, 1)$  for arbitrary  $a \in S$ .

The following two facts were proved by Sankappanavar in [4] and [6].

**Lemma 1.1.** Let S be a PCS. If  $\psi \in Con(S)$  then  $\psi = ([1]\psi)^{\wedge} \lor (\psi \land \varphi)$ .

**Lemma 1.2.** Let S be a PCS. The following statesments are equivalent: (1) Con(S) is distributive,

- (2) S satisfies: (D)  $\forall x \forall y (x < y^{**} \Rightarrow x \le y \text{ or } y \le x),$
- (3) S satisfies:

(D<sub>w</sub>) 
$$\forall x \forall y (x^* = y^* \Rightarrow x \le y \text{ or } y \le x)$$
  
and  
(U')  $\forall x \forall y ((x = x^{**} \text{ and } x < y^{**}) \Rightarrow x < y),$   
(4)  $Con(S)$  is modular.

One can easily verify the next auxiliary lemma.

**Lemma 1.3.** Let S be a PCS satisfying (D). Let  $a, b \in S$  be such that a < b and  $a^* = b^*$ . Then

- (i)  $\theta(a,b) = [a,b] \times [a,b] \cup \Delta;$
- (ii)  $[1]\theta^*(a,b) = [b,1].$

A (distributive) *p*-algebra is an algebra  $L = \langle L; \vee, \wedge, *, 0, 1 \rangle$  where  $\langle L; \vee, \wedge, 0, 1 \rangle$  is a bounded (distributive) lattice and \* is the unary operation of pseudocomplementation. learly the congruence lattice of any congruence distributive PCS is a distributive *p*-algebra.

The class  $\mathcal{B}_{\omega}$  of all distributive *p*-algebras is equational. K.B. Lee proved in [3] that the lattice of all equational subclasses of  $\mathcal{B}_{\omega}$  is a chain

$$\mathcal{B}_{-1} \subset \mathcal{B}_0 \subset \mathcal{B}_1 \subset \ldots \subset \mathcal{B}_n \subset \ldots \subset \mathcal{B}_\omega$$

of type  $\omega + 1$ , where  $\mathcal{B}_{-1}, \mathcal{B}_0$  and  $\mathcal{B}_1$  denote the classes of all trivial, Boolean and Stone algebras, respectively. Moreover, he proved that for  $n \ge 1, L \in \mathcal{B}_n$ if and only if L satisfies the identity

$$(\mathbf{L}_n) \quad (x_1 \wedge x_2 \wedge \ldots \wedge x_n)^* \vee (x_1^* \wedge x_2 \wedge \ldots \wedge x_n)^* \vee \ldots \vee (x_1 \wedge x_2 \wedge \ldots \wedge x_n^*)^* = 1.$$

**Definition 1.4** ([2]; Definition 1). Let L be a distributive p-algebra and  $n \ge 1$ . L is said to be an  $(L_n)$ -*lattice* if  $L \in \mathcal{B}_n$ .

## 2. Pseudocomplemented semilattices with $(L_n)$ -congruence Lattices

In [5] H.P. Sankappanavar gave a description of those PCS S whose congruence lattice Con(S) is Stone, i.e. satisfies  $(L_1)$ . The aim of this paper is to continue in this direction and investigate the cases for which Con(S) satisfies  $(L_n)$  for  $n \ge 2$ .

**Theorem 2.1.** Let S be a PCS. If 
$$Con(S) \in \mathcal{B}_n$$
  $(n \ge 1)$ , then  
 $(C_n) \quad \forall x_i \ (x_i \neq x_i^{**} \ (i = 1, \dots, n+1) \text{ and } x_i \neq x_j \ (i \neq j)) \Rightarrow \bigwedge_{i=1}^{n+1} x_i = 0.$ 

**Proof.** For n = 1, the claim was proved by H.P. Sankappanavar in Theorem 3.2 of [5]. Assume that  $n \ge 2$ . Let  $x_1, x_2, \ldots, x_{n+1} \in S$  be such that  $x_i \ne x_i^{**}$   $(i = 1, 2, \ldots, n+1)$  and  $x_i \ne x_j$   $(i \ne j)$ . Suppose that  $w = \bigwedge_{i=1}^{n+1} x_i > 0$ .

Without loss of generality we can divide elements  $x_1, x_2, \ldots, x_{n+1}$  into k disjoint groups  $(1 \le k \le n+1)$ :

$$\{x_{11}, x_{12}, \ldots, x_{1m_1}\}, \{x_{21}, x_{22}, \ldots, x_{2m_2}\}, \ldots, \{x_{k1}, x_{k2}, \ldots, x_{km_k}\}$$

such that  $m_1 + m_2 + ... + m_k = n + 1$  and

$$x_{i1} < x_{i2} < \ldots < x_{im_i} < x_{i1}^{**} \quad (i = 1, \ldots, k).$$

Let us denote

$$\begin{aligned} \tau_i &= \theta(x_{1i}, x_{1i+1}), \quad i = 1, 2, \dots, m_1 - 1 \\ \tau_{m_1} &= \theta(x_{1m_1}, x_{11}^{**}), \\ \tau_{m_1+j} &= \theta(x_{2j}, x_{2j+1}), \quad j = 1, 2, \dots, m_2 - 1 \\ \tau_{m_1+m_2} &= \theta(x_{2m_2}, x_{21}^{**}), \\ \dots \\ \tau_{m_1+m_2+\dots+m_{k-1}+l} &= \theta(x_{kl}, x_{kl+1}), \quad l = 1, 2 \dots m_k - 1 \\ \tau_{m_1+m_2+\dots+m_k} &= \tau_{n+1} = \theta(x_{km_k}, x_{k1}^{**}). \end{aligned}$$

Let  $\theta_1 = \bigvee_{j=2}^{n+1} \tau_j$  and  $\theta_i = \bigvee_{j=1}^{i-1} \tau_j \vee \bigvee_{j=i+1}^{n+1} \tau_j$ ,  $i = 2, 3, \dots, n$ . From Lemma 1.3 follows that  $\theta_i^* \supseteq \tau_i$ ,  $i = 1, 2, \dots, n$ . Therefore, we have

$$\begin{array}{ll} \theta_{1} \wedge \theta_{2} \wedge \ldots \wedge \theta_{n} \supseteq \tau_{n+1} & \text{and} & (\theta_{1} \wedge \theta_{2} \wedge \ldots \wedge \theta_{n})^{*} \subseteq \tau_{n+1}^{*}; \\ \theta_{1}^{*} \wedge \theta_{2} \wedge \ldots \wedge \theta_{n} \supseteq \tau_{1} & \text{and} & (\theta_{1}^{*} \wedge \theta_{2} \wedge \ldots \wedge \theta_{n})^{*} \subseteq \tau_{1}^{*}; \\ & & \ddots \\ \theta_{1} \wedge \ldots \wedge \theta_{i}^{*} \wedge \ldots \wedge \theta_{n} \supseteq \tau_{i} & \text{and} & (\theta_{1} \wedge \ldots \wedge \theta_{i}^{*} \wedge \ldots \wedge \theta_{n})^{*} \subseteq \tau_{i}^{*}; \\ & & \ddots \\ \theta_{1} \wedge \theta_{2} \wedge \ldots \wedge \theta_{n}^{*} \supseteq \tau_{n} & \text{and} & (\theta_{1} \wedge \theta_{2} \wedge \ldots \wedge \theta_{n}^{*})^{*} \subseteq \tau_{n}^{*}. \end{array}$$

From our assumption that  $Con(S) \in \mathcal{B}_n$ , we obtain

$$\tau_{n+1}^* \lor \tau_1^* \lor \tau_2^* \lor \ldots \lor \tau_n^* = \bigvee_{i=1}^{n+1} \tau_i^* = \nabla.$$

It implies that there exists a sequence  $a_0 = 1, a_1, a_2, \ldots, a_m = 0 \subseteq S$  such that  $a_i \equiv a_{i+1}(\alpha_{j(i)}), (i = 0, 1, \ldots, m - 1),$  where  $\alpha_{j(i)} \in \{\tau_k^* : k = 1, 2, \ldots, n + 1\}.$ 

From Lemma 1.3, we obtain

$$[1]\tau_{1}^{*} \subseteq [x_{12}, 1] \subseteq [\bigwedge_{i=1}^{n+1} x_{i}, 1] = [w, 1],$$

$$[1]\tau_{2}^{*} \subseteq [x_{13}, 1] \subseteq [w, 1],$$

$$\dots$$

$$[1]\tau_{m_{1}+m_{2}+\dots+m_{r}+j}^{*} \subseteq [x_{r+1j+1}, 1] \subseteq [w, 1] \ (j = 1, 2, \dots, m_{r+1} - 1),$$

$$\dots$$

$$[1]\tau_{n+1}^{*} \subseteq [x_{k1}^{**}, 1] \subseteq [w, 1].$$

Clearly  $a_1 \geq w$  and  $a_{m-1}^* \geq w$ , since  $1 = a_0 \equiv a_1(\alpha_{j(0)})$  and  $a_{m-1}^* \equiv 1(\alpha_{j(m-1)}), \alpha_{j(m-1)} \in \{\tau_k^*: k = 1, 2, \dots, n+1\}$ . If we meet elements  $a_1, a_2, \dots, a_{m-1}$  with the element  $a_{m-1}^*$ , we obtain a new sequence  $b_1 = a_1 \wedge a_{m-1}^*, b_2 = a_2 \wedge a_{m-1}^*, \dots, b_{m-2} = a_{m-2} \wedge a_{m-1}^*, b_{m-1} = 0$  such that  $b_i \equiv b_{i+1}(\alpha_{j(i)}), \ (i = 1, 2, \dots, m-2) \text{ and } \alpha_{j(i)} \in \{\tau_k^*: k = 1, 2, \dots, n+1\}$ . Again  $b_1 = a_1 \wedge a_{m-1}^* \geq w$  and  $b_{m-2}^* \geq w$ . Repeating the previous step m-2 times we obtain  $y \equiv 0(\alpha_{j(0)}), \alpha_{j(0)} \in \{\tau_k^*: k = 1, 2, \dots, n+1\}$  such that  $y \geq w$ . Since  $y^* \equiv 1(\alpha_{j(0)}), y^* \geq w$ . Therefore,  $w \leq y \wedge y^* = 0$  which is a contradiction with our assumption that  $w = \bigwedge_{i=1}^{n+1} x_i > 0$ .

**Corollary 2.2.** Let S be a PCS such that  $Con(S) \in \mathcal{B}_n$   $(n \ge 1)$ . Then  $|[a]\varphi| \le n+1$  for arbitrary  $a \in S$ .

**Definition 2.3.** Let S be a PCS. We say that S is an  $(S_n)$ -semilattice  $(n \ge 1)$  iff S satisfies  $(C_n)$  and S satisfies (D). In other words, S is an  $(S_n)$ -semilattice if and only if S is a congruence distributive pseudocomplemented semilattice which satisfies the condition  $(C_n)$ .

In the next we will often deal with non-closed elements. We find it useful to introduce now a few notations.

$$N(S) = \{n \in S : n \text{ is non} - \text{closed }\}, \text{ i.e.}$$
$$N(S) = \{n \in S : n \neq n^{**}\};$$
$$N^{**}(S) = \{n^{**} : n \in N(S)\};$$
$$C(S) = \{c \in S : c \land n = 0; \forall n \in N(S)\};$$
$$C^{*}(S) = \{c^{*} : c \in C(S)\}.$$

One can easily verify that C(S) is an ideal in B(S) and  $0 \in C(S)$ . Moreover, if  $c \in C(S)$  and  $n \in N(S)$ , then  $c \wedge n^{**} = (c \wedge n)^{**} = 0$ . It follows that C(S)can be defined equivalently as  $C(S) = \{c \in S : c \wedge n^{**} = 0; \forall n \in N(S)\}$ . If there is no danger of confusion, we will write  $N, N^{**}, C$  and  $C^*$  instead of  $N(S), N^{**}(S), C(S)$  and  $C^*(S)$ , respectively.

**Definition 2.4.** Let S be a PCS and  $\psi \in Con(S)$ . Then

$$N_{\psi} = \{ n \in N : \ \theta(n, n^{**}) \land \psi \neq \Delta \},$$
$$N_{\psi}^{**} = \{ n^{**} : \ n \in N_{\psi} \}.$$

Clearly  $C_{\psi}$  is an ideal in B(S),  $N_{\psi} = N_{\psi \wedge \varphi}$  and  $N_{\varphi} = N$ .

3. Properties of congruences on  $(S_n)$ -semilattices

The following lemmas were inspired by [5]. The next lemma is obvious.

Lemma 3.1. Let S be a PCS. Then

$$\varphi = \bigvee \{ \theta(n, n^{**}) : n \in N(S) \}.$$

For arbitrary  $A \subseteq S$  the symbol  $A^u$  denotes the set of all upper bounds of A.

**Lemma 3.2.** Let S be a PCS. Then  $(N^{**})^u = C^*$ .

**Proof.** Let  $n \in N$  and  $c \in C$  be arbitrary. Then  $c \wedge n^{**} = 0$ . Therefore,  $n^{**} \leq c^*$  and  $C^* \subseteq (N^{**})^u$ . Take arbitrary  $y \in (N^{**})^u$ . Clearly  $y \in B(S)$ . It means that  $y = y^{**} \geq n^{**}$  for arbitrary  $n \in N$ . Thus  $y^* \leq n^*$  and  $y^* \wedge n = y^* \wedge n^{**} = 0$ . It follows that  $y^* \in C$  and since y is a closed element  $y \in C^*$ .

**Lemma 3.3.** Let S be a PCS satisfying (D) and  $X \subseteq N(S) = N$ . Then  $N_{((X^{**})^u)^{\wedge}} \subseteq N \setminus X$ .

**Proof.** Suppose that  $n \in X \cap N_{((X^{**})^u)^{\wedge}}$ . Then there exist  $n \leq n_1 < m_1 \leq n^{**}$  such that  $n_1 \wedge f = m_1 \wedge f$  and  $f \in (X^{**})^u$ . Since  $n \in X$ , it follows that  $f \geq n^{**}$ . Thus  $n_1 \wedge f = n_1 = m_1 = m_1 \wedge f$  contrary to our assumption  $n_1 < m_1$ . Therefore,  $N_{((X^{**})^u)^{\wedge}} \cap X = \emptyset$  and  $N_{((X^{**})^u)^{\wedge}} \subseteq N \setminus X$ .

**Lemma 3.4.** Let S be a PCS satisfying (D) and  $\beta \in Con(S)$  be such that  $\beta \subseteq \varphi$ . Then  $((N_{\beta}^{**})^u)^{\wedge} \subseteq \beta^*$ .

**Proof.** Let  $(x, y) \in \beta \land ((N_{\beta}^{**})^{u})^{\land}$ . Without loss of generality we can assume that  $x < y \leq x^{**}$ . Then  $x \land f = y \land f$  for some  $f \in (N_{\beta}^{**})^{u}$ . Since  $(x, y) \in \beta$ , we obtain that  $\theta(x, x^{**}) \land \beta \neq \Delta$  and  $x \in N_{\beta}$ . It implies that  $f \geq x^{**} \geq y > x$  and  $x \land f = x = y = y \land f$  contrary to our assumption x < y. So, we can conclude that  $((N_{\beta}^{**})^{u})^{\land} \subseteq \beta^{*}$ .

**Corollary 3.5.** Let S be a PCS satisfying (D). Then  $\varphi^* = ((N^{**})^u)^{\wedge} = (C^*)^{\wedge}$ .

**Lemma 3.6.** Let S be an  $(S_n)$ -semilattice  $(n \ge 1)$ . Let  $\psi \in Con(S)$  be such that  $|[1]\psi \cap N| \ge n$ . Then  $\psi^* = \Delta$ .

**Proof.** Two cases can occur:  $|[1]\psi \cap N| \ge n+1$  or  $|[1]\psi \cap N| = n$ . In the first case  $\psi = \nabla$  since S is an  $(S_n)$ -semilattice. Thus  $\psi^* = \Delta$ .

In the second case we first claim that  $\varphi \subseteq \psi$ . If  $N \subseteq [1]\psi$  then it is true. Assume that  $N \subsetneq [1]\psi$ . Let  $[1]\psi \cap N = \{n_i : i = 1, ..., n\}$ . Let  $s \in N \setminus [1]\psi$ . Since  $\bigwedge_{i=1}^n n_i \equiv 1(\psi)$  and S is an  $(S_n)$ -semilattice, we obtain that  $s \wedge \bigwedge_{i=1}^n n_i = 0 \equiv s(\psi)$ . Therefore,  $s \equiv s^{**}(\psi)$  for arbitrary  $s \in N$  and  $\varphi \subseteq \psi$ .

To complete the proof it suffices to show that also  $\varphi^* \subseteq \psi$ . Let  $f \in (N^{**})^u$ . Then  $f \geq n_i^{**}$  for any  $n_i \in [1]\psi \cap N$ . It implies that  $(N^{**})^u \subseteq [1]\psi$ . Thus  $\varphi^* = ((N^{**})^u)^{\wedge} \subseteq ([1]\psi)^{\wedge} \subseteq \psi$ . Hence  $\varphi \lor \varphi^* \subseteq \psi$ . Therefore, we obtain  $\psi^* \subseteq (\varphi \lor \varphi^*)^* = \varphi^* \land \varphi^{**} = \Delta$  proving the lemma. **Definition 3.7** Let S be a PCS satisfying (D) and  $A \subseteq C$ . Then we define

$$d_C(A) = \{ c \in C : c \land a = 0, a \in A \}.$$

**Lemma 3.8.** Let S be a PCS satisfying (D) and  $I \subseteq C$  be an ideal in B(S). Then  $(N^{**} \cup I \cup d_C(I))^u = \{1\}.$ 

**Proof.** Let  $f \in (N^{**} \cup I \cup d_C(I))^u$ . Then

$$f \ge n^{**} \ (n^{**} \in N^{**}) \quad \text{and} \quad f^* \land n^{**} = 0;$$
  
$$f \ge a \ (a \in I) \qquad \text{and} \quad f^* \land a = 0;$$
  
$$f \ge c \ (c \in d_C(I)) \qquad \text{and} \quad f^* \land c = 0.$$

From  $f^* \wedge n^{**} = 0$  follows  $f^* \in C$ . Since  $f^* \wedge a = 0$  for all  $a \in I$ , it follows  $f^* \in d_C(I)$ . Since  $f^* \wedge c = 0$  for all  $c \in d_C(I)$ , we obtain that also  $f^* \wedge f^* = f^* = 0$ . Hence, f is a dense element.  $f \in (N^{**})^u$  implies that f is closed. So we can conclude f = 1 proving the lemma.

By taking  $I = \{0\}$ , we immediately obtain

**Corollary 3.9.** Let S be a PCS satisfying (D). Then  $\{N^{**} \cup C\}^u = \{1\}$ .

**Lemma 3.10.** Let S be a PCS satisfying (D) and  $F \subseteq S$  be a Boolean filter, i.e.  $F \subseteq B(S)$ . Then  $F \subseteq ((N^{**} \setminus N_{\hat{F}}^{**}) \cup d_C(C_{\hat{F}}))^u$ .

**Proof.** Let  $f \in F$  be such that  $f \notin (N^{**} \setminus N_{\hat{F}}^{**})^u$ . Thus  $f \not\geq n^{**}$  for some  $n^{**} \in N^{**} \setminus N_{\hat{F}}^{**}$ . Then  $f \wedge n^{**} < n^{**}$  and, since Con(S) is distributive, two possibilities may occure.

First suppose that  $f \wedge n^{**} \leq n < n^{**}$ . Since  $f \equiv 1(\hat{F}), f \wedge n^{**} \equiv n^{**}(\hat{F})$ , we obtain that  $n \equiv n^{**}(\hat{F})$ . Hence,  $\theta(n, n^{**}) \wedge \hat{F} \neq \Delta$ . Therefore,  $n \in N_{\hat{F}}, n^{**} \in N_{\hat{F}}^{**}$  contrary to assumption  $n^{**} \in N^{**} \setminus N_{\hat{F}}^{**}$ . Now suppose that  $n \leq f \wedge n^{**} < n^{**}$ . Since  $f \equiv 1(\hat{F}), f \wedge n^{**} \equiv n^{**}(\hat{F})$ , we again obtain that  $\theta(n, n^{**}) \wedge \hat{F} \neq \Delta$ . Therefore,  $n \in N_{\hat{F}}, n^{**} \in N_{\hat{F}}^{**}$  contrary to assumption  $n^{**} \in N_{\hat{F}}^{**}$  contrary to assumption  $n^{**} \in N_{\hat{F}}^{**}$ . Thus  $F \subseteq (N^{**} \setminus N_{\hat{F}}^{**})^u$ .

Let  $f \in F$  and  $y \in d_C(C_{\hat{F}})$ . Since  $f^* \wedge f = 0 \wedge f$ , we have  $f^* \equiv 0(\hat{F})$ and also  $f^* \wedge y \equiv 0(\hat{F})$ . Thus,  $f^* \wedge y \in C_{\hat{F}}$ . From this, we get  $(f^* \wedge y) \wedge y =$  $f^* \wedge y = 0$ . Hence,  $y \leq f^{**} = f$  proving that  $F \subseteq d_C(C_{\hat{F}})^u$ . So we can conclude that  $F \subseteq ((N^{**} \setminus N_{\hat{F}}^{**}) \cup d_C(C_{\hat{F}}))^u$ .

**Lemma 3.11.** Let S be a PCS satisfying (D) and let  $F \subseteq S$  be a Boolean filter. Then  $((N_{\hat{F}}^{**} \cup C_{\hat{F}})^u)^{\wedge} \subseteq (\hat{F})^*$ .

**Proof.** Let  $(x, y) \in \hat{F} \land ((N_{\hat{F}}^{**} \cup C_{\hat{F}})^u)^{\land}, x < y$ . It means that  $x \land f = y \land f$  for some  $f \in F$  and  $x \land h = y \land h$  for some  $h \in (N_{\hat{F}}^{**} \cup C_{\hat{F}})^u$ . Therefore,  $x^{**} \land f = y^{**} \land f$  and  $x^{**} \land h^{**} = y^{**} \land h^{**}$ . Since  $x^{**}, y^{**}, f, h^{**} \in B(S)$ , it follows that  $x^{**} \land (f + h^{**}) = y^{**} \land (f + h^{**})$ . Since  $f \in F \subseteq ((N^{**} \setminus N_{\hat{F}}^{**}) \cup d_C(C_{\hat{F}}))^u$  and  $h^{**} \in (N_{\hat{F}}^{**} \cup C_{\hat{F}})^u$ , from the two previous lemmas we obtain that  $f + h^{**} \in \{(N^{**} \setminus N_{\hat{F}}^{**}) \cup d_C(C_{\hat{F}}) \cup N_{\hat{F}}^{**} \cup C_{\hat{F}}\}^u = (N^{**} \cup C_{\hat{F}} \cup d_C(C_{\hat{F}}))^u = \{1\}$ . Thus, we see that  $x^{**} = y^{**}$ . Since  $(x, y) \in \hat{F}, x < y$  and  $x^* = y^*$  we obtain that  $\theta(x, x^{**}) \land \hat{F} \neq \Delta$  and  $x \in N_{\hat{F}}$ . Therefore,  $h \ge x^{**} \ge y > x$  and  $x \land h = x = y = y \land h$  which is a contradiction with our assumption x < y. Thus the lemma is proved.

**Theorem 3.12.** Let S be an  $(S_n)$ -semilattice  $(n \ge 1)$  such that B(S) is a complete Boolean algebra. Then Con(S) is an  $(L_n)$ -lattice.

**Proof.** For n = 1 the claim follows from [5] (see Theorem 3.27). Assume that  $n \ge 2$ . Let  $\theta_1, \theta_2, \ldots, \theta_n$  be arbitrary elements of Con(S). For the sake of simplicity let us denote

$$\alpha_0 = \theta_1 \wedge \theta_2 \wedge \ldots \wedge \theta_n,$$
  

$$\alpha_1 = \theta_1^* \wedge \theta_2 \wedge \ldots \wedge \theta_n,$$
  

$$\ldots$$
  

$$\alpha_i = \theta_1 \wedge \ldots \wedge \theta_i^* \wedge \ldots \wedge \theta_n,$$
  

$$\ldots$$
  

$$\alpha_n = \theta_1 \wedge \theta_2 \wedge \ldots \wedge \theta_n^*.$$

We want to prove that  $\alpha_0^* \vee \alpha_1^* \vee \ldots \vee \alpha_n^* = \nabla$ . From Lemma 1.1, follows that  $\alpha_i^* = (\alpha_i \wedge \varphi)^* \wedge (([1]\alpha_i)^{\wedge})^*$   $(i = 0, 1, \ldots, n)$ . Three possibilities may occur:

- (1)  $[1]\alpha_i \cap N \neq \emptyset$  for  $i = 0, 1, \ldots, n$ ;
- (2)  $[1]\alpha_i \subseteq B(S)$  for i = 0, 1, ..., n;
- (3) There exist  $I, J \subseteq \{0, 1, ..., n\}$  such that  $I \neq \emptyset \neq J$ ,  $I \cap J = \emptyset$ ,  $I \cup J = \{0, 1, ..., n\}$  and  $[1]\alpha_i \subseteq B(S)$  for  $i \in I$  and  $[1]\alpha_j \cap N \neq \emptyset$  for  $j \in J$ .

Ad (1): Suppose that  $n_i \in [1]\alpha_i \cap N$  (i = 0, 1, ..., n). It means that  $\theta(n_i, 1) \subseteq \alpha_i$ . Since  $\alpha_i \wedge \alpha_j = \Delta$  for  $i, j \in \{0, 1, ..., n\}, i \neq j$ , we obtain

that  $\theta(n_i, 1) \subseteq \alpha_i^*$  for arbitrary  $j \neq i$ . It follows that

$$\alpha_0^* \lor \alpha_1^* \lor \ldots \lor \alpha_n^* \supseteq \bigvee_{i=0}^n \theta(n_i, 1).$$

Let  $\bigvee_{i=0}^{n} \theta(n_i, 1) = \theta$ . Then  $n_i \equiv 1(\theta)$  (i = 0, 1, ..., n) and therefore  $\bigwedge_{i=0}^{n} n_i \equiv 1(\theta)$ . Since  $\alpha_i$  (i = 0, 1, ..., n) are pairwise disjoint, congruences  $n_i$  (i = 0, 1, ..., n) are pairwise different nonclosed elements. From the assumption that S is an  $(S_n)$ -semilattice we obtain  $\bigwedge_{i=0}^{n} n_i = 0 \equiv 1(\theta)$  hence  $\alpha_0^* \vee \alpha_1^* \vee ... \vee \alpha_n^* = \nabla$ .

Ad (2): Suppose that  $[1]\alpha_i \subseteq B(S)$  for  $i = 0, 1, \ldots, n$ . From Lemma 3.4 and Lemma 3.11 follows that  $\alpha_i^* \supseteq ((N_{\alpha_i \wedge \varphi}^{**})^u)^{\wedge} \wedge ((N_{([1]\alpha_i)^{\wedge}}^{**} \cup C_{([1]\alpha_i)^{\wedge}})^u)^{\wedge}$ ,  $i = 0, 1, \ldots, n$ . Let  $\sum N_{\alpha_i \wedge \varphi}^{**} = a_i$ ,  $\sum N_{([1]\alpha_i)^{\wedge}}^{**} = b_i$ ,  $\sum C_{([1]\alpha_i)^{\wedge}} = c_i$ ,  $(i = 0, 1, \ldots, n)$ . Since  $([1]\alpha_i)^{\wedge} \subseteq \alpha_i$  and  $N_{\alpha_i \wedge \varphi}^{**} = N_{\alpha_i}^{**}$ , we have  $a_i = \sum N_{\alpha_i}^{**} \ge \sum N_{([1]\alpha_i)^{\wedge}}^{**} = b_i$   $(i = 0, 1, \ldots, n)$ . Hence,  $\alpha_i^* \supseteq \hat{a}_i \wedge (b_i + c_i)^{\wedge} = \theta(a_i, 1) \wedge \theta(b_i + c_i, 1) \supseteq \theta(a_i + b_i + c_i, 1) = \theta(a_i + c_i, 1) = \theta(\Lambda_{i=0}^n (a_i + c_i), 1)$ . We claim that  $a_i \wedge c_j = 0$  for arbitrary  $i, j \in \{0, 1, \ldots, n\}$ .

From the assumption that B(S) is a complete Boolean algebra, it follows that B(S) satisfies the join infinite distributive identity and its dual meet infinite distributive identity. Let  $\sum N^{**} = m$ . Since  $c \wedge n^{**} = 0$  for arbitrary  $c \in C, n \in N$ , we obtain that  $m = \sum N^{**} \leq c^*$  and therefore  $c \leq m^*$  for arbitrary  $c \in C$ . Thus  $\sum C \leq m^*$ . It follows that  $a_i \wedge c_j = \sum N^{**}_{\alpha_i} \wedge$  $\sum C_{([1]\alpha_j)^{\wedge}} \leq \sum N^{**} \wedge \sum C \leq m \wedge m^* = 0$ . Thus we obtain that  $\bigwedge_{i=0}^n (a_i + c_i) = \bigwedge_{i=0}^n a_i + \bigwedge_{j=0}^n c_j$ .

We claim that  $\bigwedge_{j=0}^{n} c_j = 0$ . Take arbitrary  $i, j \in \{0, 1, \ldots, n\}$  such that  $i \neq j$ . Then  $c_i \wedge c_j = \sum C_{([1]\alpha_i)^{\wedge}} \wedge \sum C_{([1]\alpha_j)^{\wedge}} = \sum \{d \wedge e : d \in C_{([1]\alpha_i)^{\wedge}}$  and  $e \in C_{([1]\alpha_j)^{\wedge}} \}$ . Since  $([1]\alpha_i)^{\wedge} \wedge ([1]\alpha_j)^{\wedge} = \Delta$ , we have  $d \wedge e = 0$  for arbitrary  $d \in C_{([1]\alpha_i)^{\wedge}}$  and  $e \in C_{([1]\alpha_j)^{\wedge}}$ . Hence,  $c_i \wedge c_j = 0$ .

Next we will prove that  $\bigwedge_{i=0}^{n} a_i = 0$ . Using the fact that B(S) satisfies both the join and meet infinite distributive identities we obtain that  $\bigwedge_{i=0}^{n} a_i = \bigwedge_{i=0}^{n} \sum N_{\alpha_i}^{**} = \sum \{\bigwedge_{i=0}^{n} n_i^{**} : n_i^{**} \in N_{\alpha_i}^{**}\}$ . Take arbitrary (n + 1)-tuple  $(n_i^{**} : n_i^{**} \in N_{\alpha_i}^{**}, i = 0, 1, \dots, n)$ . Clearly some elements  $n_i^{**}$   $(i = 0, 1, \dots, n)$  may coincide. Suppose that  $n_i^{**} = m^{**}$  for  $i \in I \subseteq \{0, 1, \dots, n\}$ . It means that there exist elements  $r_i, s_i$  such that  $r_i < s_i, r_i^{**} = s_i^{**} = m^{**}$  and  $(r_i, s_i) \in \alpha_i$  for  $i \in I$ . Since  $\alpha_i$  are pairwise disjoin congruences, it follows that  $\theta(r_i, s_i)$   $(i \in I)$  are also pairwise disjoint congruences. Thus  $|I| \leq |[m^{**}]\varphi \cap N| \leq n$ .

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From the previous consideration follows that  $\bigwedge_{i=0}^{n} n_i^{**} = \bigwedge_{j=1}^{k} m_j^{**}$  where  $m_j^{**} \neq m_l^{**}$  for  $j \neq l$ ;  $n_i^{**} = m_j^{**}$  for  $i \in I_j \subset \{0, 1, \ldots, n\}$ ,  $j = 1, 2, \ldots, k$ ;  $I_j \cap I_l = \emptyset$  for  $j \neq l$ ;  $\bigcup_{j=1}^{k} I_j = \{0, 1, \ldots, n\}$  and  $|[m_j^{**}]\varphi \cap N| \geq |I_j|, j = 1, 2, \ldots, k$ ; Thus, we can write  $\bigwedge_{j=1}^{k} m_j^{**} = \bigwedge_{j=1}^{k} (\bigwedge \{s^{**} : s \in [m_j^{**}]\varphi \cap N\}) = \bigwedge_{j=1}^{k} (\bigwedge \{s : s \in [m_j^{**}]\varphi \cap N\})^{**} = (\bigwedge_{j=1}^{k} (\bigwedge \{s : s \in [m_j^{**}]\varphi \cap N\}))^{**}$ . From  $|[m_j^{**}]\varphi \cap N| \geq |I_j|$   $(j = 1, 2, \ldots, k)$  and  $\bigcup_{j=1}^{k} I_j = \{0, 1, \ldots, n\}$ , it follows that  $\bigwedge_{j=1}^{k} (\bigwedge \{s : s \in [m_j^{**}]\varphi \cap N\})$  is meet of at least (n + 1) different nonclosed elements. Hence,  $\bigwedge_{j=1}^{k} (\bigwedge \{s : s \in [m_j^{**}]\varphi \cap N\}) = 0$ . Thus we obtain  $\bigwedge_{i=0}^{n} a_i = \sum \{\bigwedge_{i=0}^{n} n_i^{**} : n_i^{**} \in N_{\alpha_i}^{**}\} = 0$  which implies  $\bigvee_{i=0}^{n} \alpha_i^{*} \supseteq \theta(0, 1) = \nabla$  and  $Con(S) \in \mathcal{B}_n$ .

Ad (3): Suppose that  $[1]\alpha_i \subseteq B(S)$  for  $i \in I$  and  $[1]\alpha_j \cap N \neq \emptyset$  for  $j \in J$ where  $I \neq \emptyset \neq J$ ,  $I \cap J \neq \emptyset$  and  $I \cup J = \{0, 1, \ldots, n\}$ . Using the previous part of the proof we obtain that  $\bigvee_{i \in I} \alpha_i^* \supseteq \theta(\bigwedge_{i \in I} a_i, 1)$ , where  $a_i = \sum N_{\alpha_i \wedge \varphi}^{**}, i \in I$ . Let  $m_j \in [1]\alpha_j \cap N$  for  $j \in J$ . Then  $\theta(m_j, 1) \wedge \alpha_i = \Delta$  and  $\alpha_i^* \supseteq \theta(m_j, 1)$ for arbitrary  $i \in I$  and  $j \in J$ . It follows that  $\bigvee_{i \in I} \alpha_i^* \supseteq \theta(\bigwedge_{i \in I} a_i, 1) \lor \bigvee_{j \in J} \theta(m_j, 1) = \theta(\bigwedge_{i \in I} a_i, 1) \lor \theta(\bigwedge_{j \in J} m_j, 1) = \theta(\bigwedge_{i \in I} a_i \land \bigwedge_{j \in J} m_j, 1)$ . Next we will prove that  $\bigwedge_{i \in I} a_i \land \bigwedge_{j \in J} m_j^{**} = 0$ . Since  $\bigwedge_{i \in I} a_i = \sum_{i \in I} \bigwedge_{i \in I} m_i^{**} :$  $n_i^{**} \in N_{\alpha_i}^{**}\}$ , we can write  $\bigwedge_{i \in I} a_i \land \bigwedge_{j \in J} m_j^{**} = \sum_{i \in I} \bigwedge_{i \in I} n_i^{**} \in N_{\alpha_i}^{**}$  ( $j \in J$ ). Repeating the same consideration as in the part (2) of this proof we obtain that  $\bigwedge_{i \in I} n_i^{**} \land \bigwedge_{j \in J} m_j^{**} = 0$  for arbitrary |I|-tuple  $(n_i^{**} : n_i^{**} \in N_{\alpha_i}^{**} i \in I)$ . Therefore,  $\bigwedge_{i \in I} a_i \land \bigwedge_{j \in J} m_j \le \bigwedge_{i \in I} a_i \land \bigwedge_{j \in J} m_j^{**} = 0$  and  $\bigvee_{i=0}^n \alpha_i^* \supseteq$  $\bigvee_{i \in I} \alpha_i^* \supseteq \theta(0, 1) = \nabla$ , hence  $Con(S) \in \mathcal{B}_n$ .

**Corollary 3.13.** Let S be a PCS such that B(S) is a complete Boolean algebra. For arbitrary  $n \ge 1$  the following statesments are equivalent:

- (i) Con(S) is an  $(L_n)$ -lattice,
- (ii) S is an  $(S_n)$ -semilattice.

# 4. Pseudocomplemented semilattices with relative $(L_n)$ -congruence lattices

**Definition 4.1** ([2], Definition 2). Let L be a distributive lattice. L is said to be a *relative*  $(L_n)$ -*lattice* if every interval [a, b] in L is an  $(L_n)$ -lattice.

**Lemma 4.2** ([2], Theorem 2). Let L be a distributive lattice with 1. The following conditions are equivalent:

- (i) L is a relative  $(L_n)$ -lattice,
- (ii) for every  $a \in L$ , [a, 1] is an  $(L_n)$ -lattice.

**Lemma 4.3.** Let S be a PCS. Then S is an  $(S_n)$ -semilattice  $(n \ge 1)$  iff the quotient semilattice  $S/\theta$  is an  $(S_n)$ -semilattice for arbitrary  $\theta \in Con(S)$ .

**Proof.** Let S be a PCS. Suppose that S is an  $(S_n)$ -semilattice for some  $n \ge 1$ . We claim that for arbitrary  $\theta \in Con(S)$  the following is true: if  $[a]\theta \in N(S/\theta)$  then  $[a]\theta \subseteq N(S)$ .

Suppose that  $[a]\theta \neq ([a]\theta)^{**} = [a^{**}]\theta$  and there exists  $x \in [a]\theta$  such that  $x = x^{**}$ . Then  $[a]\theta = [x]\theta = [x^{**}]\theta = ([x]\theta)^{**} = ([a]\theta)^{**}$  which is a contradiction to our assumption.

Let  $[x_1]\theta, [x_2]\theta, \ldots, [x_{n+1}]\theta \in S/\theta$  be such that  $[x_i]\theta \neq [x_i^{**}]\theta$   $i = 1, \ldots, n+1$  and  $[x_i]\theta \neq [x_j]\theta$ ,  $i \neq j$ . From the previous part of proof follows that  $x_i$   $(i = 1, \ldots, n+1)$  are pairwise distinct non-closed elements from S. Since S is an  $(S_n)$ -semilattice we obtain  $\bigwedge_{i=1}^{n+1} [x_i]\theta = \left[\bigwedge_{i=1}^{n+1} x_i\right]\theta = [0]\theta$ . Thus  $S/\theta$  satisfies the condition  $(C_n)$ .

Since  $Con(S/\theta) \cong [\theta, \nabla] \subseteq Con(S)$  the congruence distributivity of S implies that the condition (D) is satisfied also in the quotient semilattice  $S/\theta$ . The sufficient condition is obvious.

From the previous result and from Theorem 3.28 of [5], we immediately obtain

**Corollary 4.4.** Let S be a PCS. The following statesments are equivalent:

- (i) Con(S) is a relative Stone lattice,
- (ii) S satisfies  $(C_1)$  and for arbitrary congruence  $\theta \in Con(S)$  the quotient PCS  $S/\theta$  satisfies:

$$(S_{2(\theta)})$$
 (a) if  $A \subseteq N^{**}(S/\theta)$ , then  $\sum A$  exists;  
(b) if  $K \subseteq C(S/\theta)$ , then  $\sum K$  exists.

**Corollary 4.5.** Let S be a PCS such that the Boolean algebra  $B(S|\theta)$  is complete for arbitrary congruence  $\theta \in Con(S)$ . For arbitrary  $n \ge 1$  the following statesments are equivalent:

(i) Con(S) is an  $(L_n)$ -lattice,

- (ii) Con(S) is a relative  $(L_n)$ -lattice,
- (iii) S is an  $(S_n)$ -semilattice.

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