THE ORDER OF NORMAL FORM HYPERSUBSTITUTIONS OF TYPE (2)

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Abstract

In [2] it was proved that all hypersubstitutions of type \( \tau = (2) \) which are not idempotent and are different from the hypersubstitution which maps the binary operation symbol \( f \) to the binary term \( f(y, x) \) have infinite order. In this paper we consider the order of hypersubstitutions within given varieties of semigroups. For the theory of hypersubstitution see [3].

Keywords: hypersubstitutions, terms, idempotent elements, elements of infinite order.

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1 Preliminaries

In [1] hypersubstitutions were defined to make the concept of a hyperidentity more precise. In this paper we consider the type \( \tau = (2) \) and the binary operation symbol \( f \). Type (2) hypersubstitutions seem to be simple enough to be accessible, yet rich enough to provide an interesting structure.
An identity $s \approx t$ of type $\tau = (2)$ is called a hyperidentity of a variety $V$ of this type if for every substitution of terms built up by at most two variables (binary terms) for $f$ in $s \approx t$, the resulting identity holds in $V$. This shows that we are interested in mappings

$$\sigma : \{f\} \to W(X_2),$$

where $W(X_2)$ is the set of all terms constructed by $f$ and the variables from the two-element alphabet $X_2 = \{x, y\}$. Any such mapping is called a hypersubstitution of type $\tau = (2)$. By $\sigma_t$ we denote the hypersubstitution $\sigma : \{f\} \to \{t\}$.

A hypersubstitutions $\sigma$ can be uniquely extended to a mapping $\hat{\sigma}$ on $W(X)$ (the set of all terms built up by $f$ and variables from the countably infinite alphabet $X = \{x, y, z, \ldots\}$) inductively defined by

(i) if $t = x$ for some variable $x$, then $\hat{\sigma}[t] = x$,

(ii) if $t = f(t_1, t_2)$ for some terms $t_1, t_2$, then $\hat{\sigma}[t] = \sigma(f)(\hat{\sigma}[t_1], \hat{\sigma}[t_2])$.

By $Hyp$ we denote the set of all hypersubstitutions of type $\tau = (2)$. For any two hypersubstitutions $\sigma_1, \sigma_2$ we define a product

$$\sigma_1 \circ_h \sigma_2 := \hat{\sigma}_1 \circ \sigma_2$$

and obtain together with $\sigma_{id} = \sigma_{xy}$, i.e., $\sigma_{id}(f) = xy$, a monoid $Hyp = (Hyp; \circ_h, \sigma_{id})$. We will refer to this monoid as to $Hyp$. In [2] Denecke and Wismath described all idempotent elements of $Hyp$.

We use the following denotation: Let $W_x$ denote the set of all words using only the letter $x$, and dually for $W_y$. We set

$$E_x = \{\sigma_{xu} | u \in W_x\}, \ E_y = \{\sigma_{vy} | v \in W_y\}, \ E = E_x \cup E_y,$$

where $xu$ abbreviates $f(x, u)$.

Clearly, for any element $xu$ with $u \in W_x$ we have

$$\sigma_{xu} \circ_h \sigma_{xu} = \sigma_{xu},$$

and for any element $vy$ with $v \in W_y$ we have

$$\sigma_{vy} \circ_h \sigma_{vy} = \sigma_{vy}.$$

This shows that all elements of $E$ are idempotent. The hypersubstitutions $\sigma_x, \sigma_y$ mapping the binary operation symbol $f$ to $x$ and to $y$, respectively, and the identity hypersubstitution are also idempotent.
The hypersubstitution $\sigma_{yx}$ satisfies the equation
\[ \sigma_{yx} \circ_h \sigma_{yx} = \sigma_{xy}. \]

Further we have:

**Proposition 1.1** (see [2]). If $\sigma_s \circ_h \sigma_t = \sigma_{id}$, then either $\sigma_s = \sigma_t = \sigma_{id}$ or $\sigma_s = \sigma_t = \sigma_{yx}$.

In the following theorem we will use the concept of the length of a term as number of occurrences of variables in the term.

In [2] was proved

**Theorem 1.2.**

(i) If $\sigma \in \text{Hyp}$ is an idempotent, then $\sigma \in E \cup \{\sigma_x, \sigma_y, \sigma_{xy}\}$.

(ii) If $\sigma \in \text{Hyp} \setminus (E \cup \{\sigma_x, \sigma_y, \sigma_{xy}, \sigma_{yx}\})$, then $\sigma^n \neq \sigma^{n+1}$ for all $n \in \mathbb{N}$ with $n \geq 1$ (i.e. $\sigma$ has infinite order).

(iii) If $\sigma \in \text{Hyp} \setminus (E \cup \{\sigma_x, \sigma_y, \sigma_{xy}, \sigma_{yx}\})$, then the length of the word $(\sigma \circ_h \sigma)(f)$ is greater than the length of $\sigma(f)$.

If we set $G := \text{Hyp} \setminus (E \cup \{\sigma_x, \sigma_y, \sigma_{xy}, \sigma_{yx}\})$, then $G$ does not form a sub-semigroup of Hyp. In fact, we consider the hypersubstitution $\sigma_{wx}$ where $w$ is a term different from $x$ and from $y$. Then $\sigma_{wx} \in G$. Let $u \in W_x$ and let $\overline{wu} \in W_x$ be the term formed from $xu$ by substitution of all occurrences of the letters $x$ by $y$, then $\sigma_{\overline{wu}} \in G$. But then we see
\[ \sigma_{\overline{wu}} \circ_h \sigma_{wx} = \sigma_{xu} \]
and the product of these elements from $G$ is outside of $G$.

If we want to check whether an equation $s \approx t$ is satisfied as a hyperidentity in a given variety $V$ of semigroups, it is not necessary to test all hypersubstitutions from $\text{Hyp}$. Depending on the identities satisfied in $V$ we may restrict ourselves to a smaller subset of $\text{Hyp}$. By definition of a binary operation on this subset, we will define a new algebra which, in general is not a monoid and will determine the order of elements of those algebras.

## 2 Normal Form hypersubstitutions

In [4] J. Plonka defined a binary relation on the set of all hypersubstitutions of an arbitrary type with respect to a variety of this type.
Definition 2.1. Let $V$ be a variety of semigroups, and let $\sigma_1, \sigma_2 \in \text{Hyp}$. Then

$$\sigma_1 \sim_V \sigma_2 :\iff \sigma_1(f) \approx \sigma_2(f) \in \text{Id}_V.$$ 

Clearly, the relation $\sim_V$ is an equivalence relation on $\text{Hyp}$ and has the following properties:

Proposition 2.2 ([3]). Let $V$ be a variety of semigroups and let $\sigma_1, \sigma_2 \in \text{Hyp}$.

(i) If $\sigma_1 \sim_V \sigma_2$, then for any term $t$ of type $\tau = (2)$ the equation $\hat{\sigma}_1[t] \approx \hat{\sigma}_2[t]$ is an identity of $V$.

(ii) If $s \approx t \in \text{Id}_V, \hat{\sigma}_1[s] \approx \hat{\sigma}_1[t] \in \text{Id}_V$ and $\sigma_1 \sim_V \sigma_2 \in \text{Id}_V$, then $\hat{\sigma}_2[s] \approx \hat{\sigma}_2[t] \in \text{Id}_V$.

In general, the relation $\sim_V$ is not a congruence relation on $\text{Hyp}$. A variety is called solid if every identity in $V$ is satisfied as a hyperidentity. For a solid variety $V$ the relation $\sim_V$ is a congruence relation on $\text{Hyp}$ and the factor monoid $\text{Hyp}/\sim_V$ exists.

In the arbitrary case we form also $\text{Hyp}/\sim_V$ and consider a choice function $\varphi : \text{Hyp}/\sim_V \to \text{Hyp}$, with $\varphi([\sigma_{id}]_{\sim_V}) = \sigma_{id}$, which selects from each equivalence class exactly one element. Then we obtain the set $\text{Hyp}_{\varphi}(V) := \varphi(\text{Hyp}/\sim_V)$ of all normal form hypersubstitutions with respect to $V$ and $\varphi$.

On the set $\text{Hyp}_{\varphi}(V)$ we define a binary operation

$$\circ_N : \text{Hyp}_{\varphi}(V) \times \text{Hyp}_{\varphi}(V) \to \text{Hyp}_{\varphi}(V)$$

by $\sigma_1 \circ_N \sigma_2 = \varphi(\sigma_1 \circ_h \sigma_2)$. This mapping is well-defined, but in general not associative. Therefore, $(\text{Hyp}_{\varphi}(V); \circ_N, \sigma_{id})$ is not a monoid. We call this structure groupoid of normal form hypersubstitutions. We ask, how to characterize the idempotent elements of $\text{Hyp}_{\varphi}(V)$ since for practical work normal form hypersubstitutions are more important than usual hypersubstitutions.

Proposition 2.3. Let $V$ be a variety of semigroups and let $\varphi : \text{Hyp}/\sim_V \to \text{Hyp}$ be a choice function. Then
The order of normal form hypersubstitutions of type (2) 187

(i) $\sigma \in Hyp_{N}\phi(V)$ is an idempotent element iff $\sigma \circ h \sigma \sim_{V} \sigma$.

(ii) $\sigma_{yx} \circ_{N} \sigma_{yx} = \sigma_{xy}$ if $\sigma_{yx} \in Hyp_{N}\phi(V)$.

Proof. (i) If $\sigma$ is an idempotent of $Hyp_{N}\phi(V)$, then $\sigma \circ_{N} \sigma = \sigma \sim_{V} \sigma \circ h \sigma$. If conversely $\sigma \sim_{V} \sigma \circ h \sigma$, then $\sigma \circ_{N} \sigma \sim_{V} \sigma$. But then $\sigma \circ_{N} \sigma = \sigma$ because of $\sigma \in Hyp_{N}\phi(V)$.

(ii) $\sigma_{yx} \circ_{N} \sigma_{yx} \sim_{V} \sigma_{yx} \circ_{h} \sigma_{yx} = \sigma_{xy} \in Hyp_{N}\phi(V)$. Therefore, $\sigma_{yx} \circ_{N} \sigma_{yx} = \sigma_{xy}$.

As a consequence we have: if $\sigma$ is an idempotent of $Hyp$ and $\sigma \in Hyp_{N}\phi(V)$, then it is also an idempotent in $Hyp_{N}\phi(V)$ for any variety $V$ of semigroups and any choice function $\phi$. But in general $Hyp_{N}\phi(V)$ has idempotents which are not idempotents in $Hyp$.

3 Idempotents in $Hyp_{N}\phi(V)$

Now we want to consider the following variety of semigroups: $V = Mod\{xyz \approx x(yz), xyuv \approx xuyv, x^{3} \approx x\}$, i.e., the variety of all medial semigroups satisfying $x^{3} \approx x$.

Let $f$ be our binary operation symbol. As usual instead of $f(x, y)$ we will also write $xy$. The elements of $W(X_{2})/IdV$ where $X_{2} = \{x, y\}$ is a two-element alphabet, have the following form: $[x^{n}y^{m}]_{IdV}, [y^{n}x^{m}]_{IdV}, [xy^{m}x^{n}]_{IdV}, [yx^{m}y^{n}]_{IdV}$ where $0 \leq m, n \leq 2$. So we get the set

$W(X_{2})/IdV =$

$= \{[x]_{IdV}, [x^{2}]_{IdV}, [xy]_{IdV}, [xy^{2}]_{IdV}, [x^{2}y]_{IdV}, [x^{2}y^{2}]_{IdV}, [xyx]_{IdV}, [xy^{2}x]_{IdV},$

$[xy^{2}x]_{IdV}, [xy^{2}x^{2}]_{IdV}, [y]_{IdV}, [y^{2}]_{IdV}, [yx]_{IdV}, [yx^{2}]_{IdV}, [yx^{2}x]_{IdV}, [yx^{2}x^{2}]_{IdV}, [y^{2}x]_{IdV}, [y^{2}x^{2}]_{IdV}, [y^{2}x^{2}x]_{IdV}, [y^{2}x^{2}x^{2}]_{IdV}, [y^{2}x^{2}x^{2}]_{IdV}\}$

From each class we exchange a normal form term using a certain choice function $\phi$ and obtain the following set of normal form hypersubstitutions: $Hyp_{N}\phi(V) = \{\sigma_{x}, \sigma_{x^{2}}, \sigma_{xy}, \sigma_{xy^{2}}, \sigma_{x^{2}y}, \sigma_{x^{2}y^{2}}, \sigma_{xy^{2}}, \sigma_{xy^{2}x}, \sigma_{xy^{2}x^{2}}, \sigma_{y}, \sigma_{y^{2}}, \sigma_{y^{2}x}, \sigma_{y^{2}x^{2}}, \sigma_{y^{2}x^{2}y}, \sigma_{y^{2}x^{2}y^{2}}\}$.

The multiplication in the groupoid $(Hyp_{N}\phi(V); \circ_{N}, \sigma_{id})$ is given by the following table.
The table shows that there are many idempotents in $Hyp_{N_ϕ}(V)$ which are not idempotents in $Hyp$.

The following example shows that $(Hyp_N(V); o_N, σ_{id})$ is not a monoid:

$$(σ_{x^2} o_N σ_{xy^2}) o_N σ_{x^2} = σ_{x^2} o_N σ_{x^2} = σ_{x^2},$$

$$(σ_{x^2} o_N (σ_{xy^2} o_N σ_{x^2})) = σ_{x^2} o_N σ_x = σ_x.$$ 

All idempotent elements of $Hyp_N(V)$ are

$$\{σ_{xy}, σ_x, σ_{x^2}, σ_{xy^2}, σ_{x^2y}, σ_{xy^2y}, σ_{x^2y^2}, σ_{xy^2y^2}, σ_y, σ_{y^2}, σ_{y^2y}, σ_{y^2y^2}, σ_{y^2y^2y}\}.$$

Now we ask for which varieties at most the idempotents of $Hyp$ are idempotents of $Hyp_{N_ϕ}(V)$.

**Theorem 3.1.** For a variety $V$ of semigroups the following are equivalent:

(i) $\{ (xy)z ≈ x(yz), xy ≈ yx \} ⊆ V$,

(ii) $\{ σ | σ ∈ Hyp_{N_ϕ}(V) \text{ and } σ o_N σ = σ \} = \{ σ | σ ∈ Hyp \text{ and } σ o_h σ = σ \} \cap Hyp_{N_ϕ}(V)$ for each choice function $ϕ$.

**Proof.** "(i)⇒(ii)" Let $ϕ$ be an arbitrary choice function and let $σ ∈ Hyp_{N_ϕ}(V)$ be an idempotent element of $Hyp_{N_ϕ}(V)$. Then $σ = σ o_N σ \sim_V σ o_h σ$. Let $u$ and $v$ be the words corresponding to $σ$ and to $σ o_h σ$, respectively. By $ℓ(u)$ we denote the length of $u$. Assume that $σ ∉ E \cup \{ σ_{id}, σ_x, σ_y \}$.

By Theorem 1.2 (iii) the length of $v$ is greater than that of $u$ since $σ ∉ τ_{f(y,x)}$ by Theorem 2.3 (ii). But then $u ≈ v ∉ IdMod\{ (xy)z ≈ (xy)z, xy ≈ yx \}$ since from the associative and the commutative identity one can derive only identities $u ≈ v$ with $ℓ(u) = ℓ(v)$. But by assumption, $u ≈ v ∈ IdV ⊆ IdMod\{ (xy)z ≈ x(yz), xy ≈ yx \}$, a contradiction. This shows

$$\{ σ | σ ∈ Hyp_{N_ϕ}(V) \text{ and } σ o_N σ = σ \} ⊆ (E \cup \{ σ_x, σ_y, σ_{id} \}) \cap Hyp_{N_ϕ}(V).$$

If conversely $σ$ is an idempotent of $Hyp$, i.e. $σ o_h σ = σ$, then $σ o_N σ \sim_V σ o_h σ = σ$ and thus $σ o_N σ = σ$, since $σ ∈ Hyp_{N_ϕ}(V)$ and $σ$ is an idempotent of $Hyp_{N_ϕ}(V)$. Therefore we have equality.

"(ii)⇒(i)" Assume that $Mod\{ (xy)z ≈ x(yz), xy ≈ yx \} ⊆ V$. Then there exists an identity $x^k ≈ x^n ∈ IdV$ with $1 ≤ k < n ∈ N$. Now we construct an idempotent element of $Hyp_{N_ϕ}(V)$ which is not in $E ∪ \{ σ_x, σ_y, σ_{id} \}$. We set $m := n − k$ and $w := x^2u$ for some word $u ∈ W_x$ with $ℓ(u) = 3km − 2$. 


Clearly, $\sigma_w \not\in E \cup \{\sigma_x, \sigma_y, \sigma_{id}\}$. It is easy to see that the length of $w$ is $3km$ and the length of the word $v$ corresponding to $\sigma_w \circ_h \sigma_w$ is $(3km)^2$. In fact, from $x^k \approx x^n \in IdV$ it follows $x^a \approx x^{a + bm} \in IdV$ for all natural numbers $a \geq k$ and $b \geq 1$ and in particular we have $x^{3km} \approx x^{3km + (9km^2 - 3km)} = x^{(3km)^2}$. Thus

$$(\sigma_w \circ_h \sigma_w)(f) \approx x^{(3km)^2} \approx x^{3km} \approx f(f(x, x), u) = \sigma_w(f).$$

Therefore, $\sigma_w \circ_h \sigma_w \sim_V \sigma_w$ and $\sigma_w \circ_{N} \sigma_w \sim_V \sigma_w \circ_h \sigma_w \sim_V \sigma_w$. Let $\varphi$ be a choice function with $\sigma_w \in Hyp_{N^e}(V)$. Then from $\sigma_w \circ_N \sigma_w \sim_V \sigma_w$ it follows $\sigma_w \circ_N \sigma_w = \sigma_w$, a contradiction. 

4 Elements of infinite order

We remember that the order of an element of a groupoid is the cardinality of the subgroupoid generated by this element if this cardinality is finite and the order is infinite otherwise. By $O(\sigma)$ we denote the order of the hypersubstitution $\sigma \in Hyp_{N^e}(V)$. By Theorem 1.2 (ii), the hypersubstitution $\sigma_{f(x, f(y, x))}$ has infinite order in $Hyp$, but in $Hyp_{N^e}(V) = \{\sigma_x, \sigma_x^2, \sigma_{xy}, \sigma_{x^2y}, \sigma_{xy^2}, \sigma_{x^2y^2}, \sigma_{xy^3}, \sigma_{xy^2z}, \sigma_{y^2z}, \sigma_y, \sigma_y^2, \sigma_{yx}, \sigma_{yx^2}, \sigma_{yx^2y}, \sigma_{yx^2y^2}, \sigma_{y^2x}, \sigma_{y^2x^2}, \sigma_{y^2x^2y}, \sigma_{yx}, \sigma_{yx^2}, \sigma_{yx^2y}, xyuv \approx xyuv, x^3 \approx x \}$ we have

$$\sigma_{xy} \circ_N \sigma_{xy} = \sigma_{xy^2} \sigma_{x^2}$$

and

$$\sigma_{xy} \circ_N \sigma_{xy^2} \sigma_{x^2} = \sigma_{xy^2} = \sigma_{x^2} \circ_N \sigma_{xy},$$

thus

$$\sigma_{xy}^3 = \sigma_{xy}^2$$

and $\sigma_{xy}$ has finite order. Now we characterize elements of infinite order with respect to varieties of semigroups which contain the variety of commutative semigroups.

By $\langle \sigma \rangle_{O_N}$ we denote the subgroupoid of $Hyp_{N^e}(V)$ generated by the hypersubstitution $\sigma$.

**Theorem 4.1.** Let $V$ be a variety of semigroups. Then the following are equivalent:
(i) \( Mod\{(xy)z \approx x(yz), xy \approx yx\} \subseteq V \)

(ii) \( \{\sigma | \sigma \in Hyp_{N\phi}(V) \text{ and the order of } \sigma \text{ is infinite}\} = Hyp_{N\phi}(V) \setminus (E \cup \{x, y, \sigma_x, \sigma_y, \sigma_{xy}\} \cup A_1 \cup A_2) \), where

\[ A_1 = \{\sigma | \sigma \in Hyp_{N\phi}(V) \cap (\{v | v \in W_x \} \setminus (E_x \cup \{x\})) \text{ and } \langle \sigma \rangle_{\sigma_N} \cap \{xu | u \in W(X_2)\} = \emptyset\} \]

and

\[ A_2 = \{\sigma | \sigma \in Hyp_{N\phi}(V) \cap (\{u | u \in W_y\} \setminus (E_y \cup \{y\})) \text{ and } \langle \sigma \rangle_{\sigma_N} \cap \{uy | u \in W(X_2)\} = \emptyset\} \] for each choice function \( \varphi \).

Proof. "(i)⇒(ii)" Let \( \varphi \) be a choice function. Let \( \sigma \) be an element of \( Hyp_{N\phi}(V) \) with \( O(\sigma) = \infty \). By Theorem 3.1 and Proposition 2.3, \( \sigma \notin E \cup \{x, y, \sigma_x, \sigma_y, \sigma_{xy}\} \).

If we assume that \( \sigma \) belongs to \( A_1 \), then there exists a word \( u \in W(X_2) \) such that \( \sigma_{xu} \in \langle \sigma \rangle_{\sigma_N} \). Clearly, there exists a natural number \( n \geq 1 \) such that \( \ell(\sigma_{xy}) = n \). Moreover, we have

\[ \sigma \circ_{\sigma_N} \sigma_{xu} \sim_V \sigma \circ_h \sigma_{xu} = \sigma, \]

since the word corresponding to \( \sigma \) is in \( W_x \). Because of \( \sigma \in Hyp_{N\phi}(V) \) we get

\[ \sigma \circ_{\sigma_N} \sigma_{xu} = \sigma \]

and \( \ell(\sigma) + \ell(\sigma_{xu}) = n + 1 \). But this means, \( O(\sigma) \leq n \). Thus \( \sigma \notin A_1 \). In a similar way we show \( \sigma \notin A_2 \). This shows \( \{\sigma | \sigma \in Hyp_{N\phi}(V) \text{ and the order of } \sigma \text{ is infinite}\} \subseteq Hyp_{N\phi}(V) \setminus (E \cup \{x, y, \sigma_x, \sigma_y, \sigma_{xy}\} \cup A_1 \cup A_2) \).

Suppose that \( \sigma \in Hyp_{N\phi}(V) \setminus (E \cup \{x, y, \sigma_x, \sigma_y, \sigma_{xy}\} \cup A_1 \cup A_2) \). Let \( u \) be the word corresponding to \( \sigma \).

If \( u \in W_x \), then \( \langle \sigma \rangle_{Hyp_{N\phi}(V)} \subseteq \{v | v \in W_x\} \). Otherwise there exists an identity \( a \approx b \in IdV \) such that \( a \in W_x \) and \( b \) uses the letter \( y \). Clearly, \( a \approx b \notin IdMod\{(xy)z \approx x(yz), xy \approx yx\} \) which contradicts \( a \approx b \in IdV \subseteq IdMod\{(xy)z \approx x(yz), xy \approx yx\} \). Moreover, \( \langle \sigma \rangle_{\sigma_N} \cap \{xu | u \in W(X_2)\} = \emptyset \) and \( \sigma_x \notin \langle \sigma \rangle_{\sigma_N} \). Therefore, for \( \sigma_1, \sigma_2 \in \langle \sigma \rangle_{Hyp_{N\phi}(V)} \) the length of the word corresponding to \( \sigma_1 \circ_h \sigma_2 \) is greater than the length of \( u \). Hence for each \( \sigma' \in \langle \sigma \rangle_{\sigma_N} \) with \( \ell(\sigma') \geq 2 \) the length of the word corresponding to \( \sigma' \) is greater than the length of \( u \). Otherwise there would exist an identity \( c \approx d \in IdV \) such that the length of \( d \) is greater than that of \( c \). Clearly, \( c \approx d \notin IdMod\{(xy)z \approx x(yz), xy \approx yx\} \), what contradicts \( c \approx d \in IdV \subseteq IdMod\{(xy)z \approx x(yz), xy \approx yx\} \). Therefore, for all \( \sigma_a, \sigma_b \in \langle \sigma \rangle_{\sigma_N} \) there holds \( \sigma_a \circ_{\sigma_N} \sigma_b \neq \sigma \), i.e. \( O(\sigma) = \infty \). If \( u \in W_y \), then we get \( O(\sigma) = \infty \) in the dual way.
If $u$ uses both letters $x$ and $y$, then $\langle \sigma \rangle_{O_N} \subseteq \{\sigma_v | v \in W(X_2) \smallsetminus (W_x \cup W_y)\}$. Otherwise there is an identity $a \approx b \in IdV$ such that $a \in W_x \cup W_y$ and $b$ uses both letters $x$ and $y$. Clearly, $a \approx b \notin IdMod\{(xy)z \approx x(yz), xy \approx yx\}$ which contradicts $a \approx b \in IdV \subseteq IdMod\{(xy)z \approx x(yz), xy \approx yx\}$.

The same argumentation as above (using also $\sigma \notin \{\sigma_{xy}, \sigma_{yx}\}$) shows that for each $\sigma' \in \langle \sigma \rangle_{O_N}$ with $\ell(\sigma') \geq 2$ the length of the word corresponding to $\sigma'$ is greater than the length of $u$. This means there don’t exist hypersubstitutins $\sigma_a, \sigma_b \in \langle \sigma \rangle_{O_N}$ such that $\sigma_a \circ_N \sigma_b = \sigma$ and hence $O(\sigma) = \infty$. This shows $\{\sigma | \sigma \in Hyp_{N_v}(V) \text{ and the order of } \sigma \text{ is infinite}\} \supseteq Hyp_{N_v}(V) \setminus (E \cup \{\sigma_x, \sigma_y, \sigma_{id}, \sigma_{yx}\} \cup A_1 \cup A_2)$.

"(ii) $\Rightarrow$ (i)": Assume that $Mod\{(xy)z \approx x(yz), xy \approx yx\} \not\subseteq V$. Then there exists an identity $x^k \approx x^n \in IdV$ with $1 \leq k < n \in \mathbb{N}$. We set $m := n - k$ and $w := f(f(\ldots f(x, y), \ldots), y)$, where $w$ has the length $km + 1$. It is easy to check that $(\sigma_w \circ_h \sigma_w)(f) = v \approx xy^{(km)^2}$. In fact, from $x^k \approx x^n \in IdV$ and $m := n - k$, it follows $x^{km} \approx x^c \in IdV$ with $c = km + (k^2m - k)m = k^2m^2$. Therefore, $(\sigma_w \circ_h \sigma_w)(f) = v \approx xy^{k^2m^2} \approx xy^{km} \approx \sigma_w(f)$, i.e. $\sigma_w \circ_h \sigma_w \sim_V \sigma_w$ and thus $\sigma_w \circ_N \sigma_w \sim_V \sigma_w \circ_h \sigma_w \sim_V \sigma_w$.

Let $\varphi$ be a choice function such that $\sigma_w \in Hyp_{N_v}(V)$. Obviously, $\sigma_w \in Hyp_{N_v}(V) \setminus (E \cup \{\sigma_x, \sigma_y, \sigma_{id}, \sigma_{f(y,x)}\} \cup A_1 \cup A_2)$ and thus $O(\sigma) = \infty$. But $\sigma_w \in Hyp_{N_v}(V)$ forces $\sigma_w \circ_N \sigma_w = \sigma_w$ and $O(\sigma) = 2$, what contradicts $O(\sigma) = \infty$. Therefore $Mod\{(xy)z \approx x(yz), xy \approx yx\} \subseteq V$. \hfill $\blacksquare$

References


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