THE ORDER OF NORMAL FORM HYPERSUBSTITUTIONS OF TYPE (2)

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Abstract

In [2] it was proved that all hypersubstitutions of type \( \tau = (2) \) which are not idempotent and are different from the hypersubstitution which maps the binary operation symbol \( f \) to the binary term \( f(y, x) \) have infinite order. In this paper we consider the order of hypersubstitutions within given varieties of semigroups. For the theory of hypersubstitution see [3].

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1 Preliminaries

In [1] hypersubstitutions were defined to make the concept of a hyperidentity more precise. In this paper we consider the type \( \tau = (2) \) and the binary operation symbol \( f \). Type (2) hypersubstitutions seem to be simple enough to be accessible, yet rich enough to provide an interesting structure.
An identity $s \approx t$ of type $\tau = (2)$ is called a hyperidentity of a variety $V$ of this type if for every substitution of terms built up by at most two variables (binary terms) for $f$ in $s \approx t$, the resulting identity holds in $V$. This shows that we are interested in mappings

$$\sigma : \{f\} \to W(X_2),$$

where $W(X_2)$ is the set of all terms constructed by $f$ and the variables from the two-element alphabet $X_2 = \{x, y\}$. Any such mapping is called a hypersubstitution of type $\tau = (2)$. By $\sigma_t$ we denote the hypersubstitution $\sigma : \{f\} \to \{t\}$.

A hypersubstitutions $\sigma$ can be uniquely extended to a mapping $\hat{\sigma}$ on $W(X)$ (the set of all terms constructed by $f$ and variables from the countably infinite alphabet $X = \{x, y, z, \cdots\}$) inductively defined by

1. if $t = x$ for some variable $x$, then $\hat{\sigma}[t] = x$,
2. if $t = f(t_1, t_2)$ for some terms $t_1, t_2$, then $\hat{\sigma}[t] = \sigma(f)(\hat{\sigma}[t_1], \hat{\sigma}[t_2])$.

By $Hyp$ we denote the set of all hypersubstitutions of type $\tau = (2)$. For any two hypersubstitutions $\sigma_1, \sigma_2$ we define a product

$$\sigma_1 \circ_h \sigma_2 := \hat{\sigma}_1 \circ \sigma_2$$

and obtain together with $\sigma_{id} = \sigma_{xy}$, i.e., $\sigma_{id}(f) = xy$, a monoid $Hyp = (Hyp; \circ_h, \sigma_{id})$. We will refer to this monoid as to $Hyp$. In [2] Denecke and Wismath described all idempotent elements of $Hyp$.

We use the following denotation: Let $W_x$ denote the set of all words using only the letter $x$, and dually for $W_y$. We set

$$E_x = \{\sigma_{xu} \mid u \in W_x\}, \quad E_y = \{\sigma_{vy} \mid v \in W_y\}, \quad E = E_x \cup E_y,$$

where $xu$ abbreviates $f(x, u)$.

Clearly, for any element $xu$ with $u \in W_x$ we have

$$\sigma_{xu} \circ_h \sigma_{xu} = \sigma_{xu},$$

and for any element $vy$ with $v \in W_y$ we have

$$\sigma_{vy} \circ_h \sigma_{vy} = \sigma_{vy}.$$

This shows that all elements of $E$ are idempotent. The hypersubstitutions $\sigma_x, \sigma_y$ mapping the binary operation symbol $f$ to $x$ and to $y$, respectively, and the identity hypersubstitution are also idempotent.
The hypersubstitution $\sigma_{yx}$ satisfies the equation

$$\sigma_{yx} \circ_h \sigma_{yx} = \sigma_{xy}.$$ 

Further we have:

**Proposition 1.1** (see [2]). If $\sigma_s \circ_h \sigma_t = \sigma_{id}$, then either $\sigma_s = \sigma_t = \sigma_{id}$ or $\sigma_s = \sigma_t = \sigma_{yx}$. □

In the following theorem we will use the concept of the length of a term as number of occurrences of variables in the term.

In [2] was proved

**Theorem 1.2.**

(i) If $\sigma \in \text{Hyp}$ is an idempotent, then $\sigma \in E \cup \{\sigma_x, \sigma_y, \sigma_{xy}\}$.

(ii) If $\sigma \in \text{Hyp} \setminus (E \cup \{\sigma_x, \sigma_y, \sigma_{xy}, \sigma_{yx}\})$, then $\sigma^n \neq \sigma^{n+1}$ for all $n \in \mathbb{N}$ with $n \geq 1$ (i.e. $\sigma$ has infinite order).

(iii) If $\sigma \in \text{Hyp} \setminus (E \cup \{\sigma_x, \sigma_y, \sigma_{xy}, \sigma_{yx}\})$, then the length of the word $(\sigma \circ_h \sigma)(f)$ is greater than the length of $\sigma(f)$. □

If we set $G := \text{Hyp} \setminus (E \cup \{\sigma_x, \sigma_y, \sigma_{xy}, \sigma_{yx}\})$, then $G$ does not form a subsemigroup of $\text{Hyp}$. In fact, we consider the hypersubstitution $\sigma_{wx}$ where $w$ is a term different from $x$ and from $y$. Then $\sigma_{wx} \in G$. Let $u \in W_x$ and let $\sigma_{wu} \in W_x$ be the term formed from $xu$ by substitution of all occurrences of the letters $x$ by $y$, then $\sigma_{wu} \in G$. But then we see

$$\sigma_{wu} \circ_h \sigma_{wx} = \sigma_{xu}$$

and the product of these elements from $G$ is outside of $G$.

If we want to check whether an equation $s \approx t$ is satisfied as a hyperidentity in a given variety $V$ of semigroups, it is not necessary to test all hypersubstitutions from $\text{Hyp}$. Depending on the identities satisfied in $V$ we may restrict ourselves to a smaller subset of $\text{Hyp}$. By definition of a binary operation on this subset, we will define a new algebra which, in general is not a monoid and will determine the order of elements of those algebras.

2 Normal Form hypersubstitutions

In [4] J. Plonka defined a binary relation on the set of all hypersubstitutions of an arbitrary type with respect to a variety of this type.
Definition 2.1. Let $V$ be a variety of semigroups, and let $\sigma_1, \sigma_2 \in Hyp$. Then

\[
\sigma_1 \sim_V \sigma_2 :\iff \sigma_1(f) \approx \sigma_2(f) \in IdV.
\]

Clearly, the relation $\sim_V$ is an equivalence relation on $Hyp$ and has the following properties:

Proposition 2.2 ([3]). Let $V$ be a variety of semigroups and let $\sigma_1, \sigma_2 \in Hyp$.

(i) If $\sigma_1 \sim_V \sigma_2$, then for any term $t$ of type $\tau = (2)$ the equation $\hat{\sigma}_1[t] \approx \hat{\sigma}_2[t]$ is an identity of $V$.

(ii) If $s \approx t \in IdV, \hat{\sigma}_1[s] \approx \hat{\sigma}_1[t] \in IdV$ and $\sigma_1 \sim_V \sigma_2 \in IdV$, then $\hat{\sigma}_2[s] \approx \hat{\sigma}_2[t] \in IdV$.

In general, the relation $\sim_V$ is not a congruence relation on $Hyp$. A variety is called solid if every identity in $V$ is satisfied as a hyperidentity. For a solid variety $V$ the relation $\sim_V$ is a congruence relation on $Hyp$ and the factor monoid $Hyp/\sim_V$ exists.

In the arbitrary case we form also $Hyp/\sim_V$ and consider a choice function

\[
\varphi : Hyp/\sim_V \to Hyp, \text{ with } \varphi([\sigma_{id}]_{\sim_V}) = \sigma_{id},
\]

which selects from each equivalence class exactly one element. Then we obtain the set $Hyp_{N_\varphi}(V) := \varphi(Hyp/\sim_V)$ of all normal form hypersubstitutions with respect to $V$ and $\varphi$.

On the set $Hyp_{N_\varphi}(V)$ we define a binary operation

\[
\circ_N : Hyp_{N_\varphi}(V) \times Hyp_{N_\varphi}(V) \to Hyp_{N_\varphi}(V)
\]

by $\sigma_1 \circ_N \sigma_2 = \varphi(\sigma_1 \circ_h \sigma_2)$. This mapping is well-defined, but in general not associative. Therefore, $(Hyp_{N_\varphi}(V); \circ_N, \sigma_{id})$ is not a monoid. We call this structure groupoid of normal form hypersubstitutions. We ask, how to characterize the idempotent elements of $Hyp_{N_\varphi}(V)$ since for practical work normal form hypersubstitutions are more important than usual hypersubstitutions.

Proposition 2.3. Let $V$ be a variety of semigroups and let

\[
\varphi : Hyp/\sim_V \to Hyp
\]

be a choice function. Then
(i) $\sigma \in Hyp_{N\varphi}(V)$ is an idempotent element iff $\sigma \circ_h \sigma \sim_V \sigma$.

(ii) $\sigma_{yx} \circ_N \sigma_{yx} = \sigma_{xy}$ if $\sigma_{yx} \in Hyp_{N\varphi}(V)$.

**Proof.** (i) If $\sigma$ is an idempotent of $Hyp_{N\varphi}(V)$, then $\sigma \circ_N \sigma = \sigma \sim_V \sigma \circ_h \sigma$. If conversely $\sigma \sim_V \sigma \circ_h \sigma$, then $\sigma \circ_N \sigma \sim_V \sigma$. But then $\sigma \circ_N \sigma = \sigma$ because of $\sigma \in Hyp_{N\varphi}(V)$.

(ii) $\sigma_{yx} \circ_N \sigma_{yx} \sim_V \sigma_{yx} \circ_h \sigma_{yx} = \sigma_{xy} \in Hyp_{N\varphi}(V)$. Therefore, $\sigma_{yx} \circ_N \sigma_{yx} = \sigma_{xy}$.

As a consequence we have: if $\sigma$ is an idempotent of $Hyp$ and $\sigma \in Hyp_{N\varphi}(V)$, then it is also an idempotent in $Hyp_{N\varphi}(V)$ for any variety $V$ of semigroups and any choice function $\varphi$. But in general $Hyp_{N\varphi}(V)$ has idempotents which are not idempotents in $Hyp$.

## 3 Idempotents in $Hyp_{N\varphi}(V)$

Now we want to consider the following variety of semigroups: $V = Mod\{xy\}z \approx x(yz), xyuv \approx xuyv, x^3 \approx x\}$, i.e., the variety of all medial semigroups satisfying $x^3 \approx x$.

Let $f$ be our binary operation symbol. As usual instead of $f(x, y)$ we will also write $xy$. The elements of $W(X_2)/IdV$ where $X_2 = \{x, y\}$ is a two-element alphabet, have the following form: $[x^n y^m]_{IdV}, [y^n x^m]_{IdV}, [x y^m x^n]_{IdV}, [y x^m y^n]_{IdV}$ where $0 \leq m, n \leq 2$. So we get the set

$$W(X_2)/IdV =$$

$$= \{[x]_{IdV}, [x^2]_{IdV}, [xy]_{IdV}, [xy^2]_{IdV}, [x^2 y]_{IdV}, [x y^2]_{IdV}, [x^2 y^2]_{IdV}, [xy^2 x]_{IdV},$$

$$[xy x^2]_{IdV}, [xy^2 x^2]_{IdV}, [y]_{IdV}, [y^2]_{IdV}, [x y]_{IdV}, [y x]_{IdV}, [y^2 x]_{IdV}, [y x^2]_{IdV}, [x y^2]_{IdV}, [x^2 y^2]_{IdV}\}.$$

From each class we exchange a normal form term using a certain choice function $\varphi$ and obtain the following set of normal form hypersubstitutions:

$$Hyp_{N\varphi}(V) = \{\sigma_x, \sigma_x^2, \sigma_{xy}, \sigma_{xy^2}, \sigma_{xy^2 y}, \sigma_{x^2 y}, \sigma_{x^2 y^2}, \sigma_{x^2 y^2 x}, \sigma_{y}, \sigma_{y^2}, \sigma_{y^2 x}, \sigma_{y^2 x^2}, \sigma_{y^2 y}, \sigma_{y^2 y^2}, \sigma_{y^2 y^2 x}, \sigma_{y^2 y^2 x^2}, \sigma_{y^2 y^2 y}, \sigma_{y^2 y^2 y^2}\}.$$

The multiplication in the groupoid $(Hyp_{N\varphi}(V); \circ_N, \sigma_{id})$ is given by the following table.
The table shows that there are many idempotents in $Hyp_{N_b}(V)$ which are not idempotents in $Hyp$.

The following example shows that $(Hyp_N(V); \circ_N, \sigma_{id})$ is not a monoid:

$$(\sigma_{x^2} \circ_N \sigma_{xy^2}) \circ_N \sigma_{x^2} = \sigma_{x^2} \circ_N \sigma_{x^2} = \sigma_{x^2},$$

$$\sigma_{x^2} \circ_N (\sigma_{xy^2} \circ_N \sigma_{x^2}) = \sigma_{x^2} \circ_N \sigma_{x} = \sigma_{x}.$$ 

All idempotent elements of $Hyp_N(V)$ are

$$\{\sigma_{xy}, \sigma_{x}, \sigma_{x^2}, \sigma_{xy^2}, \sigma_{x^2y}, \sigma_{x^2y^2}, \sigma_{xy^2x}, \sigma_{x^2y^2}, \sigma_{xy^2}, \sigma_{y^2}, \sigma_{yx^2}, \sigma_{y^2}, \sigma_{yx^2}, \sigma_{yx^2y^2}\}.$$ 

Now we ask for which varieties at most the idempotents of $Hyp$ are idempotents of $Hyp_{N_b}(V)$.

**Theorem 3.1.** For a variety $V$ of semigroups the following are equivalent:

(i) $\text{Mod}\{(xy)z \approx x(yz), xy \approx yx\} \subseteq V$,

(ii) $\{\sigma|\sigma \in Hyp_{N_b}(V) \text{ and } \sigma \circ_N \sigma = \sigma\} = \{\sigma|\sigma \in Hyp \text{ and } \sigma \circ_h \sigma = \sigma\} \cap Hyp_N(V)$ for each choice function $\varphi$.

**Proof.** "(i)⇒(ii)" Let $\varphi$ be an arbitrary choice function and let $\sigma \in Hyp_{N_b}(V)$ be an idempotent element of $Hyp_N(V)$. Then $\sigma = \sigma \circ_N \sigma \sim_V \sigma \circ_h \sigma$. Let $u$ and $v$ be the words corresponding to $\sigma$ and to $\sigma \circ_h \sigma$, respectively. By $\ell(u)$ we denote the length of $u$. Assume that $\sigma \notin E \cup \{\sigma_{id}, \sigma_{x}, \sigma_{y}\}$. By Theorem 1.2 (iii) the length of $v$ is greater than that of $u$ since $\sigma \neq f(y,x)$ by Theorem 2.3 (ii). But then $u \approx v \notin IdMod\{(xy)z \approx (x(y)z), xy \approx yx\}$ since from the associative and the commutative identity one can derive only identities $u \approx v$ with $\ell(u) = \ell(v)$. But by assumption, $u \approx v \in IdV \subseteq IdMod\{(xy)z \approx x(yz), xy \approx yx\}$, a contradiction. This shows

$$\{\sigma|\sigma \in Hyp_{N_b}(V) \text{ and } \sigma \circ_N \sigma = \sigma\} \subseteq (E \cup \{\sigma_{x}, \sigma_{y}, \sigma_{id}\}) \cap Hyp_N(V).$$

If conversely $\sigma$ is an idempotent of $Hyp$, i.e. $\sigma \circ_h \sigma = \sigma$, then $\sigma \circ_N \sigma \sim_V \sigma \circ_h \sigma = \sigma$ and thus $\sigma \circ_N \sigma = \sigma$, since $\sigma \in Hyp_{N_b}(V)$ and $\sigma$ is an idempotent of $Hyp_{N_b}(V)$. Therefore we have equality.

"(ii)⇒(i)" Assume that $Mod\{(xy)z \approx x(yz), xy \approx yx\} \not\subseteq V$. Then there exists an identity $x^k \approx x^n \in IdV$ with $1 \leq k < n \in N$. Now we construct an idempotent element of $Hyp_{N_b}(V)$ which is not in $E \cup \{\sigma_{x}, \sigma_{y}, \sigma_{id}\}$. We set $m := n - k$ and $w := x^2u$ for some word $u \in W_x$ with $\ell(u) = 3km - 2$. 


Clearly, $\sigma_w \not\in E \cup \{\sigma_x, \sigma_y, \sigma_{id}\}$. It is easy to see that the length of $w$ is $3km$ and the length of the word $v$ corresponding to $\sigma_w \circ_h \sigma_w$ is $(3km)^2$. In fact, from $x^k \approx x^n \in Id_V$ it follows $x^a \approx x^{a+bm} \in Id_V$ for all natural numbers $a \geq k$ and $b \geq 1$ and in particular we have $x^{3km} \approx x^{3km+(9k^2m-3km)} = x^{(3km)^2}$. Thus

\[
(\sigma_w \circ_h \sigma_w)(f) \approx x^{(3km)^2} \approx x^{3km} \approx f(f(x, x), u) = \sigma_w(f).
\]

Therefore, $\sigma_w \circ_h \sigma_w \sim_V \sigma_w$ and $\sigma_w \circ_N \sigma_w \sim_V \sigma_w \circ_h \sigma_w \sim_V \sigma_w$. Let $\varphi$ be a choice function with $\sigma_w \in Hyp_{N\varphi}(V)$. Then from $\sigma_w \circ_N \sigma_w \sim_V \sigma_w$ it follows $\sigma_w \circ_N \sigma_w = \sigma_w$, a contradiction.

4 Elements of infinite order

We remember that the order of an element of a groupoid is the cardinality of the subgroupoid generated by this element if this cardinality is finite and the order is infinite otherwise. By $O(\sigma)$ we denote the order of the hypersubstitution $\sigma \in Hyp_{N\varphi}(V)$. By Theorem 1.2 (ii), the hypersubstitution $\sigma_{f(x,f(y,x))}$ has infinite order in $Hyp_p$, but in $Hyp_{N\varphi}(V) = \{\sigma_x, \sigma_{x^2}, \sigma_{xy}, \sigma_{xyz}, \sigma_{x^2y}, \sigma_{x^2yz}, \sigma_{xy^2}, \sigma_{x^2y^2}, \sigma_{xy^2z}, \sigma_{xyz^2}, \sigma_{xy^2z^2}, \sigma_{xyz^2z}, \sigma_{x^{2y}z}, \sigma_{x^{2y^2}z^2}, \sigma_{x^{2y}z^2}, \sigma_{x^{2y^2}z^2}, \sigma_{x^{2y^2}z^2}\}$, where $V = \text{Mod}\{(xy)z \approx x(yz), xuvy \approx xuwv, x^3 \approx x\}$ we have

\[
\sigma_{xyz} \circ_N \sigma_{xyz} = \sigma_{xyz^2}x^2
\]

and

\[
\sigma_{xyx} \circ_N \sigma_{xy^2x^2} = \sigma_{xy^2x^2} = \sigma_{xy^2x^2} \circ_N \sigma_{xyx},
\]

thus

\[
\sigma_{xyx}^3 = \sigma_{xyx}^2
\]

and $\sigma_{xyx}$ has finite order. Now we characterize elements of infinite order with respect to varieties of semigroups which contain the variety of commutative semigroups.

By $\langle \sigma \rangle_{O_N}$ we denote the subgroupoid of $Hyp_{N\varphi}(V)$ generated by the hypersubstitution $\sigma$.

**Theorem 4.1.** Let $V$ be a variety of semigroups. Then the following are equivalent:
The order of normal form hypersubstitutions of type (2) is in $V$.

(i) $\text{Mod}\{(xy)z \approx x(yz), xy \approx yx\} \subseteq V$

(ii) $\{\sigma | \sigma \in \text{Hyp}_{N\varphi}(V) \text{ and the order of } \sigma \text{ is infinite} \} = \text{Hyp}_{N\varphi}(V) \cap (E \cup \{\sigma_x, \sigma_y, \sigma_{id}, \sigma_{yx}\} \cup A_1 \cup A_2)$, where $A_1 = \{\sigma | \sigma \in \text{Hyp}_{N\varphi}(V) \cap \{\sigma_v | v \in W_x \} \cap (E_x \cup \{\sigma_x\}) \text{ and } \langle \sigma \rangle_{\circ N} \cap \{\sigma_xu | u \in W(X_2)\} \neq \emptyset \}$ and $A_2 = \{\sigma | \sigma \in \text{Hyp}_{N\varphi}(V) \cap \{\sigma_v | v \in W_y\} \cap (E_y \cup \{\sigma_y\}) \text{ and } \langle \sigma \rangle_{\circ N} \cap \{\sigma_{uy} | u \in W(X_2)\} \neq \emptyset \}$ for each choice function $\varphi$.

**Proof.** "(i)$\Rightarrow$(ii)": Let $\varphi$ be a choice function. Let $\sigma$ be an element of $\text{Hyp}_{N\varphi}(V)$ with $O(\sigma) = \infty$. By Theorem 3.1 and Proposition 2.3, $\sigma \notin E \cup \{\sigma_x, \sigma_y, \sigma_{id}, \sigma_{yx}\}$.

If we assume that $\sigma$ belongs to $A_1$, then there exists a word $u \in W(X_2)$ such that $\sigma_{xu} \in \langle \sigma \rangle_{\circ N}$. Clearly, there exists a natural number $n \geq 1$ such that $\ell(\sigma_{xy}) = n$. Moreover, we have

$$\sigma \circ_{N} \sigma_{xu} \sim_{V} \sigma \circ_{h} \sigma_{xu} = \sigma,$$

since the word corresponding to $\sigma$ is in $W_x$. Because of $\sigma \in \text{Hyp}_{N\varphi}(V)$ we get

$$\sigma \circ_{N} \sigma_{xu} = \sigma$$

and $\ell(\sigma) + \ell(\sigma_{xu}) = n + 1$. But this means, $O(\sigma) \leq n$. Thus $\sigma \notin A_1$. In a similar way we show $\sigma \notin A_2$. This shows $\{\sigma | \sigma \in \text{Hyp}_{N\varphi}(V) \text{ and the order of } \sigma \text{ is infinite} \} \subseteq \text{Hyp}_{N\varphi}(V) \cap (E \cup \{\sigma_x, \sigma_y, \sigma_{id}, \sigma_{yx}\} \cup A_1 \cup A_2)$.

Suppose that $\sigma \in \text{Hyp}_{N\varphi}(V) \cap (E \cup \{\sigma_x, \sigma_y, \sigma_{id}, \sigma_{yx}\} \cup A_1 \cup A_2)$. Let $u$ be the word corresponding to $\sigma$.

If $u \in W_x$, then $\langle \sigma \rangle_{\text{Hyp}_{N\varphi}(V)} \subseteq \{\sigma_v | v \in W_x\}$. Otherwise there exists an identity $a \approx b \in IdV$ such that $a \in W_x$ and $b$ uses the letter $y$. Clearly, $a \approx b \notin Id\text{Mod}\{(xy)z \approx x(yz), xy \approx yx\}$ which contradicts $a \approx b \in IdV \subseteq Id\text{Mod}\{(xy)z \approx x(yz), xy \approx yx\}$. Moreover, $\langle \sigma \rangle_{\circ N} \cap \{\sigma_{xu} | u \in W(X_2)\} = \emptyset$ and $\sigma_x \notin \langle \sigma \rangle_{\circ N}$. Therefore, for $\sigma_1, \sigma_2 \in \langle \sigma \rangle_{\text{Hyp}_{N\varphi}(V)}$ the length of the word corresponding to $\sigma_1 \circ_{h} \sigma_2$ is greater than the length of $u$. Hence for each $\sigma' \in \langle \sigma \rangle_{\circ h}$ with $\ell(\sigma') \geq 2$ the length of the word corresponding to $\sigma'$ is greater than the length of $u$. Otherwise there would exist an identity $c \approx d \in IdV$ such that the length of $d$ is greater than that of $c$. Clearly, $c \approx d \notin Id\text{Mod}\{(xy)z \approx x(yz), xy \approx yx\}$, what contradicts $c \approx d \in IdV \subseteq Id\text{Mod}\{(xy)z \approx x(yz), xy \approx yx\}$. Therefore, for all $\sigma_a, \sigma_b \in \langle \sigma \rangle_{\circ N}$ there holds $\sigma_a \circ_{N} \sigma_b \neq \sigma$, i.e. $O(\sigma) = \infty$. If $u \in W_y$, then we get $O(\sigma) = \infty$ in the dual way.
If \( u \) uses both letters \( x \) and \( y \), then \( \langle \sigma \rangle \circ N \subseteq \{ \sigma_v | v \in W(X_2) \setminus (W_x \cup W_y) \} \). Otherwise there is an identity \( a \approx b \in \text{IdV} \) such that \( a \in W_x \cup W_y \) and \( b \) uses both letters \( x \) and \( y \). Clearly, \( a \approx b \notin \text{IdMod}\{ (xy)z \approx x(yz), xy \approx yx \} \) which contradicts \( a \approx b \in \text{IdV} \subseteq \text{IdMod}\{ (xy)z \approx x(yz), xy \approx yx \} \).

The same argumentation as above (using also \( \sigma \notin \{ \sigma_{xy}, \sigma_{yx} \} \)) shows that for each \( \sigma' \in \langle \sigma \rangle_o \), with \( \ell(\sigma') \geq 2 \) the length of the word corresponding to \( \sigma' \) is greater than the length of \( u \). This means there don’t exist hypersubstitutions \( \sigma_a, \sigma_b \in \langle \sigma \rangle_o \) such that \( \sigma_a \circ N \sigma_b = \sigma \) and hence \( O(\sigma) = \infty \). This shows \( \{ \sigma | \sigma \in \text{Hyp}_{N_o}(V) \} \) and the order of \( \sigma \) is infinite \( \supseteq \text{Hyp}_{N_o}(V) \setminus (E \cup \{ \sigma_x, \sigma_y, \sigma_{id}, \sigma_{yx} \}) \cup A_1 \cup A_2 \).

”(ii) \( \Rightarrow \) (i)” Assume that \( \text{Mod}\{ (xy)z \approx x(yz), xy \approx yx \} \notin V \). Then there exists an identity \( x^k \approx x^n \in \text{IdV} \) with \( 1 \leq k < n \in \mathbb{N} \). We set \( m := n - k \) and \( w := f(f(\ldots f(x, y), \ldots, y), y) \), where \( w \) has the length \( km + 1 \). It is easy to check that \( (\sigma_w \circ h \sigma_w)(f) = v \approx xy^{km} \). In fact, from \( x^k \approx x^n \in \text{IdV} \) and \( m := n - k \), it follows \( x^{km} \approx x^c \in \text{IdV} \) with \( c = km + (k^2 m - k)m = k^2 m^2 \). Therefore, \( (\sigma_w \circ h \sigma_w)(f) = v \approx xy^{k^2 m^2} \approx \sigma_w(f) \), i.e. \( \sigma_w \circ h \sigma_w \approx V \sigma_w \) and thus \( \sigma_w \circ N \sigma_w \approx V \sigma_w \circ h \sigma_w \approx V \sigma_w \). Let \( \varphi \) be a choice function such that \( \sigma_w \in \text{Hyp}_{N_o}(V) \). Obviously, \( \sigma_w \in \text{Hyp}_{N_o}(V) \setminus (E \cup \{ \sigma_x, \sigma_y, \sigma_{id}, \sigma_{f(y,x)} \}) \cup A_1 \cup A_2 \) and thus \( O(\sigma) = \infty \). But \( \sigma_w \in \text{Hyp}_{N_o}(V) \) forces \( \sigma_w \circ N \sigma_w = \sigma_w \) and \( O(\sigma) = 2 \), what contradicts \( O(\sigma) = \infty \). Therefore \( \text{Mod}\{ (xy)z \approx x(yz), xy \approx yx \} \subseteq V \).

References


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