THE ORDER OF NORMAL FORM HYPERSUBSTITUTIONS OF TYPE (2)

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Abstract

In [2] it was proved that all hypersubstitutions of type $\tau = (2)$ which are not idempotent and are different from the hypersubstitution which maps the binary operation symbol $f$ to the binary term $f(y, x)$ have infinite order. In this paper we consider the order of hypersubstitutions within given varieties of semigroups. For the theory of hypersubstitutions see [3].

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1 Preliminaries

In [1] hypersubstitutions were defined to make the concept of a hyperidentity more precise. In this paper we consider the type $\tau = (2)$ and the binary operation symbol $f$. Type (2) hypersubstitutions seem to be simple enough to be accessible, yet rich enough to provide an interesting structure.
An identity $s \approx t$ of type $\tau = (2)$ is called a hyperidentity of a variety $V$ of this type if for every substitution of terms built up by at most two variables (binary terms) for $f$ in $s \approx t$, the resulting identity holds in $V$. This shows that we are interested in mappings

$$\sigma : \{f\} \rightarrow W(X_2),$$

where $W(X_2)$ is the set of all terms constructed by $f$ and the variables from the two-element alphabet $X_2 = \{x, y\}$. Any such mapping is called a hypersubstitution of type $\tau = (2)$. By $\sigma_t$ we denote the hypersubstitution $\sigma : \{f\} \rightarrow \{t\}$.

A hypersubstitutions $\sigma$ can be uniquely extended to a mapping $\hat{\sigma}$ on $W(X)$ (the set of all terms built up by $f$ and variables from the countably infinite alphabet $X = \{x, y, z, \cdots\}$) inductively defined by

(i) if $t = x$ for some variable $x$, then $\hat{\sigma}[t] = x$,

(ii) if $t = f(t_1, t_2)$ for some terms $t_1, t_2$, then $\hat{\sigma}[t] = \sigma(f)(\hat{\sigma}[t_1], \hat{\sigma}[t_2])$.

By $\text{Hyp}$ we denote the set of all hypersubstitutions of type $\tau = (2)$. For any two hypersubstitutions $\sigma_1, \sigma_2$ we define a product

$$\sigma_1 \circ_H \sigma_2 := \hat{\sigma}_1 \circ \sigma_2$$

and obtain together with $\sigma_{id} = \sigma_{xy}$, i.e., $\sigma_{id}(f) = xy$, a monoid $\text{Hyp} = (\text{Hyp}, \circ_H, \sigma_{id})$. We will refer to this monoid as to $\text{Hyp}$. In [2] Denecke and Wismath described all idempotent elements of $\text{Hyp}$.

We use the following denotation: Let $W_x$ denote the set of all words using only the letter $x$, and dually for $W_y$. We set

$$E_x = \{\sigma_{xu} \mid u \in W_x\}, \quad E_y = \{\sigma_{vy} \mid v \in W_y\}, \quad E = E_x \cup E_y,$$

where $xu$ abbreviates $f(x, u)$.

Clearly, for any element $xu$ with $u \in W_x$ we have

$$\sigma_{xu} \circ_h \sigma_{xu} = \sigma_{xu},$$

and for any element $vy$ with $v \in W_y$ we have

$$\sigma_{vy} \circ_h \sigma_{vy} = \sigma_{vy}.$$

This shows that all elements of $E$ are idempotent. The hypersubstitutions $\sigma_x, \sigma_y$ mapping the binary operation symbol $f$ to $x$ and to $y$, respectively, and the identity hypersubstitution are also idempotent.
The hypersubstitution $\sigma_{yx}$ satisfies the equation

$$\sigma_{yx} \circ h \sigma_{yx} = \sigma_{xy}.$$ 

Further we have:

**Proposition 1.1** (see [2]). If $\sigma_s \circ h \sigma_t = \sigma_{id}$, then either $\sigma_s = \sigma_t = \sigma_{id}$ or $\sigma_s = \sigma_t = \sigma_{yx}$. □

In the following theorem we will use the concept of the length of a term as number of occurrences of variables in the term.

In [2] was proved

**Theorem 1.2.**

(i) If $\sigma \in \text{Hyp}$ is an idempotent, then $\sigma \in E \cup \{\sigma_x, \sigma_y, \sigma_{xy}\}$.

(ii) If $\sigma \in \text{Hyp} \setminus (E \cup \{\sigma_x, \sigma_y, \sigma_{xy}, \sigma_{yx}\})$, then $\sigma^n \neq \sigma^{n+1}$ for all $n \in \mathbb{N}$ with $n \geq 1$ (i.e. $\sigma$ has infinite order).

(iii) If $\sigma \in \text{Hyp} \setminus (E \cup \{\sigma_x, \sigma_y, \sigma_{xy}, \sigma_{yx}\})$, then the length of the word $(\sigma \circ h \sigma)(f)$ is greater than the length of $\sigma(f)$. □

If we set $G := \text{Hyp} \setminus (E \cup \{\sigma_x, \sigma_y, \sigma_{xy}, \sigma_{yx}\})$, then $G$ does not form a sub-semigroup of $\text{Hyp}$. In fact, we consider the hypersubstitution $\sigma_{wx}$ where $w$ is a term different from $x$ and from $y$. Then $\sigma_{wx} \in G$. Let $u \in W_x$ and let $\overline{wu} \in W_x$ be the term formed from $wu$ by substitution of all occurrences of the letters $x$ by $y$, then $\sigma_{\overline{wu}} \in G$. But then we see

$$\sigma_{\overline{wu}} \circ h \sigma_{wx} = \sigma_{xu}$$

and the product of these elements from $G$ is outside of $G$.

If we want to check whether an equation $s \approx t$ is satisfied as a hyperidentity in a given variety $V$ of semigroups, it is not necessary to test all hypersubstitutions from $\text{Hyp}$. Depending on the identities satisfied in $V$ we may restrict ourselves to a smaller subset of $\text{Hyp}$. By definition of a binary operation on this subset, we will define a new algebra which, in general is not a monoid and will determine the order of elements of those algebras.

2 Normal Form hypersubstitutions

In [4] J. Plonka defined a binary relation on the set of all hypersubstitutions of an arbitrary type with respect to a variety of this type.
Definition 2.1. Let $V$ be a variety of semigroups, and let $\sigma_1, \sigma_2 \in Hyp$. Then

$$\sigma_1 \sim_V \sigma_2 :\iff \sigma_1(f) \approx \sigma_2(f) \in IdV.$$ 

Clearly, the relation $\sim_V$ is an equivalence relation on $Hyp$ and has the following properties:

Proposition 2.2 ([3]). Let $V$ be a variety of semigroups and let $\sigma_1, \sigma_2 \in Hyp$.

(i) If $\sigma_1 \sim_V \sigma_2$, then for any term $t$ of type $\tau = (2)$ the equation $\hat{\sigma}_1[t] \approx \hat{\sigma}_2[t]$ is an identity of $V$.

(ii) If $s \approx t \in IdV, \hat{\sigma}_1[s] \approx \hat{\sigma}_1[t] \in IdV$ and $\sigma_1 \sim_V \sigma_2 \in IdV$, then $\hat{\sigma}_2[s] \approx \hat{\sigma}_2[t] \in IdV$. \hfill \Box

In general, the relation $\sim_V$ is not a congruence relation on $Hyp$. A variety is called solid if every identity in $V$ is satisfied as a hyperidentity. For a solid variety $V$ the relation $\sim_V$ is a congruence relation on $Hyp$ and the factor monoid $Hyp/\sim_V$ exists.

In the arbitrary case we form also $Hyp/\sim_V$ and consider a choice function

$$\varphi : Hyp/\sim_V \to Hyp,$$

which selects from each equivalence class exactly one element. Then we obtain the set $Hyp_{N,\varphi}(V) := \varphi(Hyp/\sim_V)$ of all normal form hypersubstitutions with respect to $V$ and $\varphi$.

On the set $Hyp_{N,\varphi}(V)$ we define a binary operation

$$\circ_N : Hyp_{N,\varphi}(V) \times Hyp_{N,\varphi}(V) \to Hyp_{N,\varphi}(V)$$

by $\sigma_1 \circ_N \sigma_2 = \varphi(\sigma_1 \circ_h \sigma_2)$. This mapping is well-defined, but in general not associative. Therefore, $(Hyp_{N,\varphi}(V); \circ_N, \sigma_{id})$ is not a monoid. We call this structure groupoid of normal form hypersubstitutions. We ask, how to characterize the idempotent elements of $Hyp_{N,\varphi}(V)$ since for practical work normal form hypersubstitutions are more important than usual hypersubstitutions.

Proposition 2.3. Let $V$ be a variety of semigroups and let

$$\varphi : Hyp/\sim_V \to Hyp$$

be a choice function. Then
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(i) \( \sigma \in Hyp_{N_{\varphi}}(V) \) is an idempotent element iff \( \sigma \circ \sigma \sim_{V} \sigma \).

(ii) \( \sigma_{yx} \circ_{N} \sigma_{yx} = \sigma_{xy} \) if \( \sigma_{yx} \in Hyp_{N_{\varphi}}(V) \).

**Proof.** (i) If \( \sigma \) is an idempotent of \( Hyp_{N_{\varphi}}(V) \), then \( \sigma \circ_{N} \sigma = \sigma \sim_{V} \sigma \circ_{h} \sigma \).

If conversely \( \sigma \sim_{V} \sigma \circ_{h} \sigma \), then \( \sigma \circ_{N} \sigma \sim_{V} \sigma \). But then \( \sigma \circ_{N} \sigma = \sigma \) because of \( \sigma \in Hyp_{N_{\varphi}}(V) \).

(ii) \( \sigma_{yx} \circ_{N} \sigma_{yx} \sim_{V} \sigma_{yx} \circ_{h} \sigma_{yx} = \sigma_{xy} \in Hyp_{N_{\varphi}}(V) \). Therefore, \( \sigma_{yx} \circ_{N} \sigma_{yx} = \sigma_{xy} \).

As a consequence we have: if \( \sigma \) is an idempotent of \( Hyp \) and \( \sigma \in Hyp_{N_{\varphi}}(V) \), then it is also an idempotent in \( Hyp_{N_{\varphi}}(V) \) for any variety \( V \) of semigroups and any choice function \( \varphi \). But in general \( Hyp_{N_{\varphi}}(V) \) has idempotents which are not idempotents in \( Hyp \).

3 Idempotents in \( Hyp_{N_{\varphi}}(V) \)

Now we want to consider the following variety of semigroups: \( V = Mod(\{xy\}; x(yz) = x(yz), x(yuv) = x(yuv), x^{3} \approx x) \), i.e., the variety of all medial semigroups satisfying \( x^{3} \approx x \).

Let \( f \) be our binary operation symbol. As usual instead of \( f(x, y) \) we will also write \( xy \). The elements of \( W(X_{2})/IdV \) where \( X_{2} = \{x, y\} \) is a two-element alphabet, have the following form: \( [x^{m}y^{n}]_{IdV}, [y^{m}x^{n}]_{IdV}, [xy^{m}x^{n}]_{IdV}, [yx^{m}y^{n}]_{IdV} \) where \( 0 \leq m, n \leq 2 \). So we get the set

\[
W(X_{2})/IdV =
\{ [x]_{IdV}, [x^{2}]_{IdV}, [xy]_{IdV}, [xy^{2}]_{IdV}, [x^{2}y]_{IdV}, [xyx]_{IdV}, [x^{2}y^{2}]_{IdV}, [xy^{2}x]_{IdV},
[xy^{2}x]_{IdV}, [xy^{2}x]_{IdV}, [yx^{2}]_{IdV}, [yx^{2}]_{IdV}, [y^{2}]_{IdV}, [yx]_{IdV}, [y^{2}]_{IdV}, [yx]_{IdV}, [y^{2}]_{IdV}, [yx]_{IdV},
[y^{2}]_{IdV}, [yx^{2}]_{IdV}, [yx^{2}]_{IdV}, [y^{2}]_{IdV}, [yx^{2}]_{IdV}, [y^{2}]_{IdV}, [yx^{2}]_{IdV}, [y^{2}]_{IdV}, [yx^{2}]_{IdV}\}
\]

From each class we exchange a normal form term using a certain choice function \( \varphi \) and obtain the following set of normal form hypersubstitutions:

\[
Hyp_{N_{\varphi}}(V) = \{ \sigma_{x}, \sigma_{x^{2}}, \sigma_{xy}, \sigma_{x^{2}y}, \sigma_{xy^{2}}, \sigma_{x^{2}y^{2}}, \sigma_{xy^{2}x}, \sigma_{x^{2}y^{2}}, \sigma_{y}, \sigma_{y^{2}}, \sigma_{yx}, \sigma_{y^{2}x}, \sigma_{yx^{2}}, \sigma_{yx^{2}y}, \sigma_{y^{2}x^{2}}, \sigma_{yx^{2}y^{2}} \}.
\]

The multiplication in the groupoid \( (Hyp_{N_{\varphi}}(V); \circ_{N}, \sigma_{id}) \) is given by the following table.
The table shows that there are many idempotents in \( \text{Hyp}_{N,\varphi}(V) \) which are not idempotents in \( \text{Hyp} \).

The following example shows that \( (\text{Hyp}_{N}(V); \circ_N, \sigma_{id}) \) is not a monoid:

\[
(\sigma_{x_2} \circ_N \sigma_{xy_2}) \circ_N \sigma_{x_2} = \sigma_{x_2} \circ_N \sigma_{x_2} = \sigma_{x_2},
\]
\[
\sigma_{x_2} \circ_N (\sigma_{xy_2} \circ_N \sigma_{x_2}) = \sigma_{x_2} \circ_N \sigma_{x} = \sigma_{x}.
\]

All idempotent elements of \( \text{Hyp}_{N}(V) \) are
\[
\{\sigma_{xy}, \sigma_{x}, \sigma_{x_2}, \sigma_{xy^2}, \sigma_{x_2y}, \sigma_{x_2y^2}, \sigma_{xy^2x}, \sigma_{xy^2y}, \sigma_{y}, \sigma_{y^2}, \sigma_{y^2y}, \sigma_{y^2y^2} \}.
\]

Now we ask for which varieties at most the idempotents of \( \text{Hyp} \) are idempotents of \( \text{Hyp}_{N,\varphi}(V) \).

**Theorem 3.1.** For a variety \( V \) of semigroups the following are equivalent:

(i) \( \text{Mod}\{ (xy)z \approx x(yz), xy \approx yx \} \subseteq V \),

(ii) \( \{ \sigma|\sigma \in \text{Hyp}_{N,\varphi}(V) \text{ and } \sigma \circ_N \sigma = \sigma \} \cap \text{Id}_V \neq \emptyset \text{ for each choice function } \varphi. \)

**Proof.** "(i)⇒(ii)" Let \( \varphi \) be an arbitrary choice function and let \( \sigma \in \text{Hyp}_{N,\varphi}(V) \) be an idempotent element of \( \text{Hyp}_{N,\varphi}(V) \). Then \( \sigma = \sigma \circ_N \sigma \sim_V \sigma \circ_h \sigma \). Let \( u \) and \( v \) be the words corresponding to \( \sigma \) and to \( \sigma \circ_h \sigma \), respectively. By \( \ell(u) \) we denote the length of \( u \). Assume that \( \sigma \notin E \cup \{ \sigma_{id}, \sigma_x, \sigma_y \} \). By Theorem 1.2 (iii) the length of \( v \) is greater than that of \( u \) since \( \sigma \neq \sigma_{f(y,x)} \) by Theorem 2.3 (ii). But then \( u \approx v \notin \text{IdMod}\{ (xy)z \approx (xy)z, xy \approx yx \} \) since from the associative and the commutative identity one can derive only identities \( u \approx v \) with \( \ell(u) = \ell(v) \). But by assumption, \( u \approx v \in \text{IdV} \subseteq \text{IdMod}\{ (xy)z \approx x(yz), xy \approx yx \} \), a contradiction. This shows
\[
\{ \sigma|\sigma \in \text{Hyp}_{N,\varphi}(V) \text{ and } \sigma \circ_N \sigma = \sigma \} \subseteq (E \cup \{ \sigma_x, \sigma_y, \sigma_{id} \}) \cap \text{Hyp}_{N,\varphi}(V).
\]

If conversely \( \sigma \) is an idempotent of \( \text{Hyp} \), i.e. \( \sigma \circ_h \sigma = \sigma \), then \( \sigma \circ_N \sigma \sim_V \sigma \circ_h \sigma = \sigma \) and thus \( \sigma \circ_N \sigma = \sigma \), since \( \sigma \in \text{Hyp}_{N,\varphi}(V) \) and \( \sigma \) is an idempotent of \( \text{Hyp}_{N,\varphi}(V) \). Therefore we have equality.

"(ii)⇒(i)" Assume that \( \text{Mod}\{ (xy)z \approx (xy)z, xy \approx yx \} \not\subseteq V \). Then there exists an identity \( x^k \approx x^n \in \text{IdV} \) with \( 1 \leq k < n \in \mathbb{N} \). Now we construct an idempotent element of \( \text{Hyp}_{N,\varphi}(V) \) which is not in \( E \cup \{ \sigma_x, \sigma_y, \sigma_{id} \} \). We set \( m := n - k \) and \( w := x^2u \) for some word \( u \in W_x \) with \( \ell(u) = 3km - 2 \).
Clearly, \( \sigma_w \notin E \cup \{ \sigma_x, \sigma_y, \sigma_{id} \} \). It is easy to see that the length of \( w \) is \( 3km \) and the length of the word \( v \) corresponding to \( \sigma_w \circ_h \sigma_w \) is \((3km)^2\). In fact, from \( x^k \approx x^n \in IdV \) it follows \( x^a \approx x^{a+bn} \in IdV \) for all natural numbers \( a \geq k \) and \( b \geq 1 \) and in particular we have \( x^{3km} \approx x^{3km+(9k^2m-3km)} = x^{(3km)^2} \). Thus
\[
(\sigma_w \circ_h \sigma_w)(f) \approx x^{(3km)^2} \approx x^{3km} \approx f(f(x,u)) = \sigma_w(f).
\]
Therefore, \( \sigma_w \circ_h \sigma_w \sim_V \sigma_w \) and \( \sigma_w \circ_N \sigma_w \sim_V \sigma_w \circ_h \sigma_w \sim_V \sigma_w \). Let \( \varphi \) be a choice function with \( \sigma_w \in Hyp_{N_\varphi}(V) \). Then from \( \sigma_w \circ_N \sigma_w \sim_V \sigma_w \) it follows \( \sigma_w \circ_N \sigma_w = \sigma_w \), a contradiction.

4 Elements of infinite order

We remember that the order of an element of a groupoid is the cardinality of the subgroupoid generated by this element if this cardinality is finite and the order is infinite otherwise. By \( O(\sigma) \) we denote the order of the hypersubstitution \( \sigma \in Hyp_{N_\varphi}(V) \). By Theorem 1.2 (ii), the hypersubstitution \( \sigma_{f(x,f(y,x))} \) has infinite order in \( Hyp \), but in \( Hyp_{N_\varphi}(V) = \{ \sigma_x, \sigma_{xy}, \sigma_{xyx}, \sigma_{xyxx}, \sigma_{xyxy}, \sigma_{xyxyx}, \sigma_{xyxyxx}, \sigma_{xyxyxx}, \sigma_{xyxyxy}, \sigma_{xyxyxyx}, \sigma_{xyxyxyxx} \} \), where \( V = Mod\{(xy)^2 \approx (xy)^2, xyuv \approx xyuv, x^3 \approx x \} \) we have
\[
\sigma_{xyx} \circ_N \sigma_{xyx} = \sigma_{xyx} \circ_N \sigma_{xyx},
\]
and
\[
\sigma_{xyx} \circ_N \sigma_{xyx} \approx \sigma_{xyx} \approx \sigma_{xyx} \circ_N \sigma_{xyx},
\]
thus
\[
\sigma_{xyx} \approx \sigma_{xyx} \approx \sigma_{xyx} \approx \sigma_{xyx}
\]
and \( \sigma_{xyx} \) has finite order. Now we characterize elements of infinite order with respect to varieties of semigroups which contain the variety of commutative semigroups.

By \( \langle \sigma \rangle_{\circ_N} \) we denote the subgroupoid of \( Hyp_{N_\varphi}(V) \) generated by the hypersubstitution \( \sigma \).

Theorem 4.1. Let \( V \) be a variety of semigroups. Then the following are equivalent:
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(i) \( \text{Mod}\{(xy)z \approx x(yz), xy \approx yx\} \subseteq V \)

(ii) \( \{\sigma|\sigma \in HypN_\varphi(V) \text{ and the order of } \sigma \text{ is infinite}\} = HypN_\varphi(V) \setminus (E \cup \{\sigma_x, \sigma_y, \sigma_{id}, \sigma_{yx}\} \cup A_1 \cup A_2) \), where \( A_1 = \{\sigma|\sigma \in HypN_\varphi(V) \cap (\{\sigma_v|v \in W_x\} \setminus (E_x \cup \{\sigma_x\}) \text{ and } \langle \sigma\rangle_{o_N} \cap \{\sigma_{ux}|u \in W(X_2)\} \neq \emptyset\} \) and \( A_2 = \{\sigma|\sigma \in HypN_\varphi(V) \cap (\{\sigma_v|v \in W_y\} \setminus (E_y \cup \{\sigma_y\}) \text{ and } \langle \sigma\rangle_{o_N} \cap \{\sigma_{uy}|u \in W(X_2)\} \neq \emptyset\} \) for each choice function \( \varphi \).

**Proof.** "(i)\(\Rightarrow\)(ii)"; Let \( \varphi \) be a choice function. Let \( \sigma \) be an element of \( HypN_\varphi(V) \) with \( O(\sigma) = \infty \). By Theorem 3.1 and Proposition 2.3, \( \sigma \notin E \cup \{\sigma_x, \sigma_y, \sigma_{xy}, \sigma_{yx}\} \).

If we assume that \( \sigma \) belongs to \( A_1 \), then there exists a word \( u \in W(X_2) \) such that \( \sigma_{ux} \in \langle \sigma\rangle_{o_N} \). Clearly, there exists a natural number \( n \geq 1 \) such that \( \ell(\sigma_{xy}) = n \). Moreover, we have

\[
\sigma \circ_N \sigma_{ux} \sim_V \sigma \circ_h \sigma_{ux} = \sigma,
\]

since the word corresponding to \( \sigma \) is in \( W_x \). Because of \( \sigma \in HypN_\varphi(V) \) we get

\[
\sigma \circ_N \sigma_{ux} = \sigma
\]

and \( \ell(\sigma) + \ell(\sigma_{ux}) = n + 1 \). But this means, \( O(\sigma) \leq n \). Thus \( \sigma \notin A_1 \). In a similar way we show \( \sigma \notin A_2 \). This shows \( \{\sigma|\sigma \in HypN_\varphi(V) \text{ and the order of } \sigma \text{ is infinite}\} \subseteq HypN_\varphi(V) \setminus (E \cup \{\sigma_x, \sigma_y, \sigma_{id}, \sigma_{yx}\} \cup A_1 \cup A_2) \).

Suppose that \( \sigma \in HypN_\varphi(V) \setminus (E \cup \{\sigma_x, \sigma_y, \sigma_{id}, \sigma_{yx}\} \cup A_1 \cup A_2) \). Let \( u \) be the word corresponding to \( \sigma \).

If \( u \in W_x \), then \( \langle \sigma\rangle_{HypN_\varphi(V)} \subseteq \{\sigma_v|v \in W_x\} \). Otherwise there exists an identity \( a \approx b \in IdV \) such that \( a \in W_x \) and \( b \) uses the letter \( y \). Clearly, \( a \approx b \notin IdMod\{(xy)z \approx x(yz), xy \approx yx\} \) which contradicts \( a \approx b \in IdV \subseteq IdMod\{(xy)z \approx x(yz), xy \approx yx\} \). Moreover, \( \langle \sigma\rangle_{o_N} \cap \{\sigma_{ux}|u \in W(X_2)\} = \emptyset \) and \( \sigma_x \notin \langle \sigma\rangle_{o_N} \). Therefore, for \( \sigma_1, \sigma_2 \in \langle \sigma\rangle_{HypN_\varphi(V)} \) the length of the word corresponding to \( \sigma_1 \circ_N \sigma_2 \) is greater than the length of \( u \). Hence for each \( \sigma' \in \langle \sigma\rangle_{o_N} \) with \( \ell(\sigma') \geq 2 \) the length of the word corresponding to \( \sigma' \) is greater than the length of \( u \). Otherwise there would exist an identity \( c \approx d \in IdV \) such that the length of \( d \) is greater than that of \( c \). Clearly, \( c \approx d \notin IdMod\{(xy)z \approx x(yz), xy \approx yx\} \), what contradicts \( c \approx d \in IdV \subseteq IdMod\{(xy)z \approx x(yz), xy \approx yx\} \). Therefore, for all \( \sigma_a, \sigma_b \in \langle \sigma\rangle_{o_N} \) there holds \( \sigma_a \circ_N \sigma_b \neq \sigma \), i.e. \( O(\sigma) = \infty \). If \( u \in W_y \), then we get \( O(\sigma) = \infty \) in the dual way.
If \( u \) uses both letters \( x \) and \( y \), then \( \langle \sigma \rangle_{\mathcal{O}_N} \subseteq \{ \sigma_v | v \in W(X_2) \setminus (W_x \cup W_y) \} \). Otherwise there is an identity \( a \approx b \in IdV \) such that \( a \in W_x \cup W_y \) and \( b \) uses both letters \( x \) and \( y \). Clearly, \( a \approx b \not\in IdMod(\langle xy \rangle z \approx x(yz), xy \approx yx \) which contradicts \( a \approx b \in IdV \subseteq IdMod(\langle xy \rangle z \approx x(yz), xy \approx yx \).

The same argumentation as above (using also \( \sigma \not\in \{ \sigma_{xy}, \sigma_{yx} \} \)) shows that for each \( \sigma' \in \langle \sigma \rangle_{\mathcal{O}_N} \) with \( \ell(\sigma') \geq 2 \) the length of the word corresponding to \( \sigma' \) is greater than the length of \( u \). This means there don’t exist hypersubstitutins \( \sigma_a, \sigma_b \in \langle \sigma \rangle_{\mathcal{O}_N} \) such that \( \sigma_a \circ \sigma_b = \sigma \) and hence \( O(\sigma) = \infty \). This shows \( \{ \sigma | \sigma \in Hyp_{\mathcal{N}_x}(V) \text{ and the order of } \sigma \text{ is infinite} \} \supseteq Hyp_{\mathcal{N}_x}(V) \setminus (E \cup \{ \sigma_x, \sigma_y, \sigma_{id}, \sigma_{xy} \} \cup A_1 \cup A_2) \).

"(ii) \Rightarrow (i)": Assume that \( Mod(\langle xy \rangle z \approx x(yz), xy \approx yx \) \( \not\subseteq V \). Then there exists an identity \( x^k \approx x^n \in IdV \) with \( 1 \leq k < n \in \mathbb{N} \). We set \( m := n - k \) and \( w := f(f(\ldots f(x,y),\ldots),y),y ) \), where \( w \) has the length \( km + 1 \). It is easy to check that \( (\sigma_w \circ_h \sigma_w)(f) = v \approx xy^{(km)^2} \). In fact, from \( x^k \approx x^n \in IdV \) and \( m := n - k \), it follows \( x^{km} \approx x^c \in IdV \) with \( c = km + (k^2m - k)m = k^2m^2 \). Therefore, \( (\sigma_w \circ_h \sigma_w)(f) = v \approx xy^{k^2m^2} \approx xy^{km} \approx \sigma_w(f) \), i.e. \( \sigma_w \circ_h \sigma_w \sim_V \sigma_w \) and thus \( \sigma_w \circ_N \sigma_w \sim_V \sigma_w \circ_h \sigma_w \sim_V \sigma_w \).

Let \( \varphi \) be a choice function such that \( \sigma_w \in Hyp_{\mathcal{N}_x}(V) \). Obviously, \( \sigma_w \in Hyp_{\mathcal{N}_x}(V) \setminus (E \cup \{ \sigma_x, \sigma_y, \sigma_{id}, \sigma_{f(y,x)} \} \cup A_1 \cup A_2) \) and thus \( O(\sigma) = \infty \). But \( \sigma_w \in Hyp_{\mathcal{N}_x}(V) \) forces \( \sigma_w \circ_N \sigma_w = \sigma_w \) and \( O(\sigma) = 2 \), what contradicts \( O(\sigma) = \infty \). Therefore \( Mod(\langle xy \rangle z \approx x(yz), xy \approx yx \) \( \subseteq V \).

References


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