THE ORDER OF NORMAL FORM
HYPERSUBSTITUTIONS OF TYPE (2)

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Abstract

In [2] it was proved that all hypersubstitutions of type $\tau = (2)$ which are not idempotent and are different from the hypersubstitution which maps the binary operation symbol $f$ to the binary term $f(y, x)$ have infinite order. In this paper we consider the order of hypersubstitutions within given varieties of semigroups. For the theory of hypersubstitution see [3].

Keywords: hypersubstitutions, terms, idempotent elements, elements of infinite order.

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1 Preliminaries

In [1] hypersubstitutions were defined to make the concept of a hyperidentity more precise. In this paper we consider the type $\tau = (2)$ and the binary operation symbol $f$. Type (2) hypersubstitutions seem to be simple enough to be accessible, yet rich enough to provide an interesting structure.
An identity \( s \approx t \) of type \( \tau = (2) \) is called a hyperidentity of a variety \( V \) of this type if for every substitution of terms built up by at most two variables (binary terms) for \( f \) in \( s \approx t \), the resulting identity holds in \( V \). This shows that we are interested in mappings

\[
\sigma : \{f\} \rightarrow W(X_2),
\]

where \( W(X_2) \) is the set of all terms constructed by \( f \) and the variables from the two-element alphabet \( X_2 = \{x, y\} \). Any such mapping is called a hypersubstitution of type \( \tau = (2) \). By \( \sigma_t \) we denote the hypersubstitution \( \sigma : \{f\} \rightarrow \{t\} \).

A hypersubstitutions \( \sigma \) can be uniquely extended to a mapping \( \hat{\sigma} \) on \( W(X) \) (the set of all terms built up by \( f \) and variables from the countably infinite alphabet \( X = \{x, y, z, \cdots\} \)) inductively defined by

(i) if \( t = x \) for some variable \( x \), then \( \hat{\sigma}[t] = x \),

(ii) if \( t = f(t_1, t_2) \) for some terms \( t_1, t_2 \), then \( \hat{\sigma}[t] = \sigma(f)(\hat{\sigma}[t_1], \hat{\sigma}[t_2]) \).

By \( \text{Hyp} \) we denote the set of all hypersubstitutions of type \( \tau = (2) \). For any two hypersubstitutions \( \sigma_1, \sigma_2 \) we define a product

\[
\sigma_1 \circ_h \sigma_2 := \hat{\sigma}_1 \circ \sigma_2
\]

and obtain together with \( \sigma_{id} = \sigma_{xy}, \) i.e., \( \sigma_{id}(f) = xy \), a monoid \( \text{Hyp} = (\text{Hyp}; \circ_h, \sigma_{id}) \). We will refer to this monoid as to \( \text{Hyp} \). In [2] Denecke and Wismath described all idempotent elements of \( \text{Hyp} \).

We use the following denotation: Let \( W_x \) denote the set of all words using only the letter \( x \), and dually for \( W_y \). We set

\[
E_x = \{\sigma_{xu} | u \in W_x\}, \quad E_y = \{\sigma_{vy} | v \in W_y\}, \quad E = E_x \cup E_y,
\]

where \( xu \) abbreviates \( f(x, u) \).

Clearly, for any element \( xu \) with \( u \in W_x \) we have

\[
\sigma_{xu} \circ_h \sigma_{xu} = \sigma_{xu},
\]

and for any element \( vy \) with \( v \in W_y \) we have

\[
\sigma_{vy} \circ_h \sigma_{vy} = \sigma_{vy}.
\]

This shows that all elements of \( E \) are idempotent. The hypersubstitutions \( \sigma_x, \sigma_y \) mapping the binary operation symbol \( f \) to \( x \) and to \( y \), respectively, and the identity hypersubstitution are also idempotent.
The hypersubstitution $\sigma_{yx}$ satisfies the equation

$$\sigma_{yx} \circ_h \sigma_{yx} = \sigma_{xy}.$$

Further we have:

**Proposition 1.1** (see [2]). If $\sigma \circ_h \sigma = \sigma_{id}$, then either $\sigma = \sigma_{id}$ or $\sigma = \sigma_{yx}$.

In the following theorem we will use the concept of the length of a term as number of occurrences of variables in the term.

In [2] was proved

**Theorem 1.2.**

(i) If $\sigma \in \text{Hyp}$ is an idempotent, then $\sigma \in E \cup \{\sigma_x, \sigma_y, \sigma_{xy}\}$.

(ii) If $\sigma \in \text{Hyp} \smallsetminus (E \cup \{\sigma_x, \sigma_y, \sigma_{xy}, \sigma_{yx}\})$, then $\sigma^n \neq \sigma^{n+1}$ for all $n \in \mathbb{N}$ with $n \geq 1$ (i.e. $\sigma$ has infinite order).

(iii) If $\sigma \in \text{Hyp} \smallsetminus (E \cup \{\sigma_x, \sigma_y, \sigma_{xy}, \sigma_{yx}\})$, then the length of the word $(\sigma \circ_h \sigma)(f)$ is greater than the length of $\sigma(f)$.

If we set $G := \text{Hyp} \smallsetminus (E \cup \{\sigma_x, \sigma_y, \sigma_{xy}, \sigma_{yx}\})$, then $G$ does not form a subsemigroup of $\text{Hyp}$. In fact, we consider the hypersubstitution $\sigma_{wx}$ where $w$ is a term different from $x$ and from $y$. Then $\sigma_{wx} \in G$. Let $u \in W_x$ and let $\overline{wu} \in W_x$ be the term formed from $xu$ by substitution of all occurrences of the letters $x$ by $y$, then $\sigma_{\overline{wu}} \in G$. But then we see

$$\sigma_{\overline{wu}} \circ_h \sigma_{wx} = \sigma_{xu}$$

and the product of these elements from $G$ is outside of $G$.

If we want to check whether an equation $s \approx t$ is satisfied as a hyperidentity in a given variety $V$ of semigroups, it is not necessary to test all hypersubstitutions from $\text{Hyp}$. Depending on the identities satisfied in $V$ we may restrict ourselves to a smaller subset of $\text{Hyp}$. By definition of a binary operation on this subset, we will define a new algebra which, in general is not a monoid and will determine the order of elements of those algebras.

### 2 Normal Form hypersubstitutions

In [4] J. Plonka defined a binary relation on the set of all hypersubstitutions of an arbitrary type with respect to a variety of this type.
Definition 2.1. Let $V$ be a variety of semigroups, and let $\sigma_1, \sigma_2 \in Hyp$. Then

$$\sigma_1 \sim_V \sigma_2 :\Leftrightarrow \sigma_1(f) \approx \sigma_2(f) \in IdV.$$ 

Clearly, the relation $\sim_V$ is an equivalence relation on $Hyp$ and has the following properties:

Proposition 2.2 ([3]). Let $V$ be a variety of semigroups and let $\sigma_1, \sigma_2 \in Hyp$.

(i) If $\sigma_1 \sim_V \sigma_2$, then for any term $t$ of type $\tau = (2)$ the equation $\hat{\sigma}_1[t] \approx \hat{\sigma}_2[t]$ is an identity of $V$.

(ii) If $s \approx t \in IdV, \hat{\sigma}_1[s] \approx \hat{\sigma}_1[t] \in IdV$ and $\sigma_1 \sim_V \sigma_2 \in IdV$, then $\hat{\sigma}_2[s] \approx \hat{\sigma}_2[t] \in IdV.$

In general, the relation $\sim_V$ is not a congruence relation on $Hyp$. A variety is called solid if every identity in $V$ is satisfied as a hyperidentity. For a solid variety $V$ the relation $\sim_V$ is a congruence relation on $Hyp$ and the factor monoid $Hyp/\sim_V$ exists.

In the arbitrary case we form also $Hyp/\sim_V$ and consider a choice function

$$\varphi : Hyp/\sim_V \to Hyp,$$

which selects from each equivalence class exactly one element. Then we obtain the set $Hyp_{N_\varphi}(V) := \varphi(Hyp/\sim_V)$ of all normal form hypersubstitutions with respect to $V$ and $\varphi$.

On the set $Hyp_{N_\varphi}(V)$ we define a binary operation

$$\circ_N : Hyp_{N_\varphi}(V) \times Hyp_{N_\varphi}(V) \to Hyp_{N_\varphi}(V)$$

by $\sigma_1 \circ_N \sigma_2 = \varphi(\sigma_1 \circ \sigma_2)$. This mapping is well-defined, but in general not associative. Therefore, $(Hyp_{N_\varphi}(V); \circ_N, \sigma_{id})$ is not a monoid. We call this structure groupoid of normal form hypersubstitutions. We ask, how to characterize the idempotent elements of $Hyp_{N_\varphi}(V)$ since for practical work normal form hypersubstitutions are more important than usual hypersubstitutions.

Proposition 2.3. Let $V$ be a variety of semigroups and let

$$\varphi : Hyp/\sim_V \to Hyp$$

be a choice function. Then
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(i) $\sigma \in Hyp_{N,\varphi}(V)$ is an idempotent element iff $\sigma \circ_h \sigma \sim_V \sigma$.

(ii) $\sigma_{yx} \circ_N \sigma_{yx} = \sigma_{xy}$ if $\sigma_{yx} \in Hyp_{N,\varphi}(V)$.

Proof. (i) If $\sigma$ is an idempotent of $Hyp_{N,\varphi}(V)$, then $\sigma \circ_N \sigma = \sigma \sim_V \sigma \circ_h \sigma$.

As a consequence we have: if $\sigma$ is an idempotent of $Hyp$ and $\sigma \in Hyp_{N,\varphi}(V)$, then it is also an idempotent in $Hyp_{N,\varphi}(V)$ for any variety $V$ of semigroups and any choice function $\varphi$. But in general $Hyp_{N,\varphi}(V)$ has idempotents which are not idempotents in $Hyp$.

3 Idempotents in $Hyp_{N,\varphi}(V)$

Now we want to consider the following variety of semigroups: $V = Mod\{(xy)z \approx x(yz), xyuv \approx xuyv, x^3 \approx x\}$, i.e., the variety of all medial semigroups satisfying $x^3 \approx x$.

Let $f$ be our binary operation symbol. As usual instead of $f(x, y)$ we will also write $xy$. The elements of $W(X_2)/IdV$ where $X_2 = \{x, y\}$ is a two-element alphabet, have the following form: $[x^n y^m]_{IdV}, [y^n x^m]_{IdV}, [xy^n x^m]_{IdV}, [yx^m y^n]_{IdV}$ where $0 \leq m, n \leq 2$. So we get the set

$$W(X_2)/IdV = \{[x]_{IdV}, [x^2]_{IdV}, [xy]_{IdV}, [xy^2]_{IdV}, [x^2 y]_{IdV}, [x y x]_{IdV}, [x^2 y^2]_{IdV}, [xy^2 x]_{IdV}, [xy^2 x y^2]_{IdV}, [x^2 y x]_{IdV}, [x y x y]_{IdV}, [y x y]_{IdV}, [y^2 x]_{IdV}, [y x y y]_{IdV}, [y y x]_{IdV}, [y^2 x^2]_{IdV}, [y^2 x^2 y]_{IdV}, [y x^2 y]_{IdV}, [x y^2 x]_{IdV}, [x y^2 x y]_{IdV}, [y x y^2]_{IdV}, [x y^2 x^2]_{IdV}, [x y^2 x^2 y]_{IdV}, [y y x^2]_{IdV}, [y y x^2 y]_{IdV}, [x y x^2 y]_{IdV}, [y y x^2 y^2]_{IdV}\}.$$ 

From each class we exchange a normal form term using a certain choice function $\varphi$ and obtain the following set of normal form hypersubstitutions:

$Hyp_{N,\varphi}(V) = \{\sigma_x, \sigma_{x^2}, \sigma_{xy}, \sigma_{xy^2}, \sigma_{x^2 y}, \sigma_{x^2 y^2}, \sigma_{xy^2 x}, \sigma_{xy^2 x^2}, \sigma_y, \sigma_{y^2}, \sigma_{yx}, \sigma_{yx^2}, \sigma_{yx^2 y}, \sigma_{yx^2 y^2}, \sigma_{xy x}, \sigma_{xy x y}, \sigma_{xy x y^2}, \sigma_{xy x y^2 y}, \sigma_{xy x y^2 y^2}\}.$

The multiplication in the groupoid $(Hyp_{N,\varphi}(V); \circ_N, \sigma_{id})$ is given by the following table.
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The table shows that there are many idempotents in $Hyp_{\phi}(V)$ which are not idempotents in $Hyp$.

The following example shows that $(Hyp_{\phi}(V); \circ_N, \sigma_{id})$ is not a monoid:

$$(\sigma x_2 \circ_N \sigma_{xy^2}) \circ_N \sigma x_2 = \sigma x_2 \circ_N \sigma x_2 = \sigma x_2,$$

$$\sigma x_2 \circ_N (\sigma_{xy^2} \circ_N \sigma x_2) = \sigma x_2 \circ_N \sigma x = \sigma x.$$

All idempotent elements of $Hyp_{\phi}(V)$ are

$$\{\sigma_{xy}, \sigma_x, \sigma_{x^2}, \sigma_{x^2 y}, \sigma_{x^2 y^2}, \sigma_{xy^2}, \sigma_{xy^2 x}, \sigma_{xy^2 x^2}, \sigma_{y}, \sigma_{y^2}, \sigma_{y^2 y}, \sigma_{y^2 y^2}, \sigma_{y^2 y^2} \}.$$

Now we ask for which varieties at most the idempotents of $Hyp$ are idempotents of $Hyp_{\phi}(V)$.

**Theorem 3.1.** For a variety $V$ of semigroups the following are equivalent:

1. $\{ (xy)z \approx x(yz), xy \approx yx \} \subseteq V$,
2. $\{ \sigma | \sigma \in Hyp_{\phi}(V) \text{ and } \sigma \circ_N \sigma = \sigma \} = \{ \sigma | \sigma \in Hyp \text{ and } \sigma \circ_h \sigma = \sigma \} \cap Hyp_{\phi}(V)$ for each choice function $\varphi$.

**Proof.** (i)⇒(ii)” Let $\varphi$ be an arbitrary choice function and let $\sigma \in Hyp_{\phi}(V)$ be an idempotent element of $Hyp_{\phi}(V)$. Then $\sigma = \sigma \circ_N \sigma \sim_V \sigma \circ_h \sigma$. Let $u$ and $v$ be the words corresponding to $\sigma$ and to $\sigma \circ_h \sigma$, respectively. By $\ell(u)$ we denote the length of $u$. Assume that $\sigma \notin E \cup \{ \sigma_{id}, \sigma_x, \sigma_y \}$. By Theorem 1.2 (iii) the length of $v$ is greater than that of $u$ since $\sigma \neq \sigma_{f(y,x)}$ by Theorem 2.3 (ii). But then $u \approx v \notin IdMod \{x(yz) \approx (xy)z, xy \approx yx\}$ since from the associative and the commutative identity one can derive only identities $u \approx v$ with $\ell(u) = \ell(v)$. But by assumption, $u \approx v \in IdV \subseteq IdMod \{x(yz) \approx x(yz), xy \approx yx\}$, a contradiction. This shows

$$\{ \sigma | \sigma \in Hyp_{\phi}(V) \text{ and } \sigma \circ_N \sigma = \sigma \} \subseteq (E \cup \{ \sigma_x, \sigma_y, \sigma_{id} \}) \cap Hyp_{\phi}(V).$$

If conversely $\sigma$ is an idempotent of $Hyp$, i.e. $\sigma \circ_h \sigma = \sigma$, then $\sigma \circ_N \sigma \sim_V \sigma \circ_h \sigma = \sigma$ and thus $\sigma \circ_N \sigma = \sigma$, since $\sigma \in Hyp_{\phi}(V)$ and $\sigma$ is an idempotent of $Hyp_{\phi}(V)$. Therefore we have equality.

”(ii)⇒(i)” Assume that $Mod \{ (xy)z \approx x(yz), xy \approx yx \} \nsubseteq V$. Then there exists an identity $x^k \approx x^n \in IdV$ with $1 \leq k < n \in \mathbb{N}$. Now we construct an idempotent element of $Hyp_{\phi}(V)$ which is not in $E \cup \{ \sigma_x, \sigma_y, \sigma_{id} \}$. We set $m := n - k$ and $w := x^2 u$ for some word $u \in W_x$ with $\ell(u) = 3km - 2$. 
Clearly, $\sigma_w \not\in E \cup \{\sigma_x, \sigma_y, \sigma_{\text{id}}\}$. It is easy to see that the length of $w$ is $3km$ and the length of the word $v$ corresponding to $\sigma_w \circ_h \sigma_w$ is $(3km)^2$. In fact, from $x^k \approx x^n \in IdV$ it follows $x^a \approx x^{a+b\ell n} \in IdV$ for all natural numbers $a \geq k$ and $b \geq 1$ and in particular we have $x^{3km} \approx x^{(3km) + (9k^2m - 3k)m} = x^{(3km)^2}$. Thus

$$(\sigma_w \circ_h \sigma_w)(f) \approx x^{(3km)^2} \approx x^{3km} \approx f(f(x, x), u) = \sigma_w(f).$$

Therefore, $\sigma_w \circ_h \sigma_w \sim_V \sigma_w$ and $\sigma_w \circ_N \sigma_w \sim_V \sigma_w \circ_h \sigma_w$. Let $\phi$ be a choice function with $\sigma_w \in Hyp_{N\ell}(V)$. Then from $\sigma_w \circ_N \sigma_w \sim_V \sigma_w$ it follows $\sigma_w \circ_N \sigma_w = \sigma_w$, a contradiction.

4 Elements of infinite order

We remember that the order of an element of a groupoid is the cardinality of the subgroupoid generated by this element if this cardinality is finite and the order is infinite otherwise. By $O(\sigma)$ we denote the order of the hypersubstitution $\sigma \in Hyp_{N\ell}(V)$. By Theorem 1.2 (ii), the hypersubstitution $\sigma_{f(x, f(y, x))}$ has infinite order in $Hyp_p$, but in $Hyp_{N\ell}(V) = \{\sigma_x, \sigma_{x^2}, \sigma_{xy}, \sigma_{x^2y}, \sigma_{x^2y^2}, \sigma_{xy^2}, \sigma_{xy^2x}, \sigma_{xy^2x^2}, \sigma_{y^3}, \sigma_{y^3x}, \sigma_{y^3x^2}, \sigma_{y^3x^2y}, \sigma_{y^3x^2y^2}, \sigma_{y^3x^2y^2x}, \sigma_{y^3x^2y^2x^2}, \sigma_{y^3x^2y^2x^2y}, \sigma_{y^3x^2y^2x^2y^2}\}$, where $V = \text{Mod}\{(xy)z \approx x(yz), x_{yuv} \approx x_{uyv}, x^3 \approx x\}$ we have

$$\sigma_{xy^2x^2} \circ_N \sigma_{xy^2x^2} = \sigma_{xy^2x^2}\sigma_{xy^2x^2}$$

and

$$\sigma_{xy^2x^2} \circ_N \sigma_{xy^2x^2} = \sigma_{xy^2x^2} \circ_N \sigma_{xy^2x^2},$$

thus

$$\sigma_{xy^2x^2}^3 = \sigma_{xy^2x^2}^2$$

and $\sigma_{xy^2x^2}$ has finite order. Now we characterize elements of infinite order with respect to varieties of semigroups which contain the variety of commutative semigroups.

By $\langle \sigma \rangle_{O_N}$ we denote the subgroupoid of $Hyp_{N\ell}(V)$ generated by the hypersubstitution $\sigma$.

Theorem 4.1. Let $V$ be a variety of semigroups. Then the following are equivalent:
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(i) \( \text{Mod}\{(xy)z \approx x(yz), xy \approx yx\} \subseteq V \)

(ii) \( \{\sigma | \sigma \in \text{Hyp}_N(V) \text{ and the order of } \sigma \text{ is infinite}\} \subseteq \text{Hyp}_N(V) \cap (E \cup \{\sigma_x, \sigma_y, \sigma_{id}, \sigma_{yx}\} \cup A_1 \cup A_2) \), where \( A_1 = \{\sigma | \sigma \in \text{Hyp}_N(V) \cap (\{\sigma_v | v \in W_x\} \setminus (E_x \cup \{\sigma_x\}) \text{ and } \langle \sigma \rangle \cap \{\sigma_v | v \in W_x\} = \emptyset\} \) and \( A_2 = \{\sigma | \sigma \in \text{Hyp}_N(V) \cap (\{\sigma_v | v \in W_x\} \setminus (E_y \cup \{\sigma_y\}) \text{ and } \langle \sigma \rangle \cap \{\sigma_w | u \in W(X_2)\} = \emptyset\} \) for each choice function \( \varphi \).

**Proof.** "(i)\( \Rightarrow \) (ii)" \( \Rightarrow \): Let \( \varphi \) be a choice function. Let \( \sigma \) be an element of \( \text{Hyp}_N(V) \) with \( O(\sigma) = \infty \). By Theorem 3.1 and Proposition 2.3, \( \sigma \notin E \cup \{\sigma_x, \sigma_y, \sigma_{id}, \sigma_{yx}\} \).

If we assume that \( \sigma \) belongs to \( A_1 \), then there exists a word \( u \in W(X_2) \) such that \( \sigma_{xu} \in \langle \sigma \rangle \). Clearly, there exists a natural number \( n \geq 1 \) such that \( \ell(\sigma_{xy}) = n \). Moreover, we have

\[
\sigma \circ \sigma_{xu} \sim V \sigma \circ_h \sigma_{xu} = \sigma,
\]

since the word corresponding to \( \sigma \) is in \( W_x \). Because of \( \sigma \in \text{Hyp}_N(V) \) we get

\[
\sigma \circ \sigma_{xu} = \sigma
\]

and \( \ell(\sigma) + \ell(\sigma_{xu}) = n + 1 \). But this means, \( O(\sigma) \leq n \). Thus \( \sigma \notin A_1 \). In a similar way we show \( \sigma \notin A_2 \). This shows \( \{\sigma | \sigma \in \text{Hyp}_N(V) \text{ and the order of } \sigma \text{ is infinite}\} \subseteq \text{Hyp}_N(V) \cap (E \cup \{\sigma_x, \sigma_y, \sigma_{id}, \sigma_{yx}\} \cup A_1 \cup A_2) \).

Suppose that \( \sigma \in \text{Hyp}_N(V) \setminus (E \cup \{\sigma_x, \sigma_y, \sigma_{id}, \sigma_{yx}\} \cup A_1 \cup A_2) \). Let \( u \) be the word corresponding to \( \sigma \).

If \( u \in W_x \), then \( \langle \sigma \rangle \setminus \{\sigma_v | v \in W_x\} \). Otherwise there exists an identity \( a \approx b \in IdV \) such that \( a \in W_x \) and \( b \) uses the letter \( y \). Clearly, \( a \approx b \notin IdMod\{(xy)z \approx x(yz), xy \approx yx\} \) which contradicts \( a \approx b \in IdV \). Moreover, \( \langle \sigma \rangle \cap \{\sigma_v | v \in W_x\} = \emptyset \) and \( \sigma \notin \langle \sigma \rangle \setminus \{\sigma_v | v \in W(x_2)\} = \emptyset \). Therefore, for \( \sigma_1, \sigma_2 \in \langle \sigma \rangle \setminus \{\sigma_v | v \in W(x_2)\} \) the length of the word corresponding to \( \sigma_1 \circ_{h} \sigma_2 \) is greater than the length of \( u \). Hence for each \( \sigma' \in \langle \sigma \rangle \setminus \{\sigma_v | v \in W(x_2)\} \) with \( \ell(\sigma') \geq 2 \) the length of the word corresponding to \( \sigma' \) is greater than the length of \( u \). Otherwise there would exist an identity \( c \approx d \in IdV \) such that the length of \( d \) is greater than that of \( c \). Clearly, \( c \approx d \notin IdMod\{(xy)z \approx x(yz), xy \approx yx\} \), what contradicts \( c \approx d \in IdV \setminus IdMod\{(xy)z \approx x(yz), xy \approx yx\} \). Therefore, for all \( \sigma_a, \sigma_b \in \langle \sigma \rangle \setminus \{\sigma_v | v \in W(x_2)\} \) holds \( \sigma_a \circ \sigma_b \neq \sigma \), i.e. \( O(\sigma) = \infty \). If \( u \in W_y \), then we get \( O(\sigma) = \infty \) in the dual way.
If \( u \) uses both letters \( x \) and \( y \), then \( \langle \sigma \rangle_{\circ N} \subseteq \{ \sigma_v \mid v \in W(X_2) \setminus (W_x \cup W_y) \} \). Otherwise there is an identity \( a \approx b \in IdV \) such that \( a \in W_x \cup W_y \) and \( b \) uses both letters \( x \) and \( y \). Clearly, \( a \approx b \notin IdMod\{(xy)z \approx x(yz), xy \approx yx\} \) which contradicts \( a \approx b \in IdV \subseteq IdMod\{(xy)z \approx x(yz), xy \approx yx\} \).

The same argumentation as above (using also \( \sigma \notin \{ \sigma_{xy}, \sigma_{yz} \} \)) shows that for each \( \sigma' \in \langle \sigma \rangle_{\circ N} \) with \( \ell(\sigma') \geq 2 \) the length of the word corresponding to \( \sigma' \) is greater than the length of \( u \). This means there don’t exist hypersubstitutins \( \sigma_a, \sigma_b \in \langle \sigma \rangle_{\circ N} \) such that \( \sigma_a \circ_N \sigma_b = \sigma \) and hence \( O(\sigma) = \infty \). This shows \( \{ \sigma \mid \sigma \in Hyp_{N_w}(V) \} \) and the order of \( \sigma \) is infinite \( \supseteq Hyp_{N_w}(V) \setminus (E \cup \{ \sigma_x, \sigma_y, \sigma_{id}, \sigma_{yz} \} \cup A_1 \cup A_2) \).

”(ii) \( \Rightarrow \) (i)” : Assume that \( Mod\{(xy)z \approx x(yz), xy \approx yx\} \nsubseteq V \). Then there exists an identity \( x^k \approx x^n \in IdV \) with \( 1 \leq k < n \in \mathbb{N} \). We set \( k := n - k \) and \( w := f(f(\ldots f(x,y),\ldots),y),y) \), where \( w \) has the length \( km + 1 \). It is easy to check that \( (\sigma_w \circ \sigma_w)(f) = v \approx xy^{(km)^2} \). In fact, from \( x^k \approx x^n \in IdV \) and \( m := n - k \), it follows \( x^{km} \approx x^c \in IdV \) with \( c = km + (k^2m - k)m = k^2m^2 \). Therefore, \( (\sigma_w \circ \sigma_w)(f) = v \approx xy^{k^2m^2} \approx xy^{km} \approx \sigma_w(f) \), i.e. \( \sigma_w \circ \sigma_w \sim \sigma_w \). Let \( \varphi \) be a choice function such that \( \sigma_w \in Hyp_{N_w}(V) \). Obviously, \( \sigma_w \in Hyp_{N_w}(V) \setminus (E \cup \{ \sigma_x, \sigma_y, \sigma_{id}, \sigma_{f(y,x)} \} \cup A_1 \cup A_2) \) and thus \( O(\sigma) = \infty \). But \( \sigma_w \in Hyp_{N_w}(V) \) forces \( \sigma_w \circ_N \sigma_w = \sigma_w \) and \( O(\sigma) = 2 \), what contradicts \( O(\sigma) = \infty \). Therefore \( Mod\{(xy)z \approx x(yz), xy \approx yx\} \subseteq V \). 

References


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