

## HYPERIDENTITIES IN ASSOCIATIVE GRAPH ALGEBRAS

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### Abstract

Graph algebras establish a connection between directed graphs without multiple edges and special universal algebras of type  $(2,0)$ . We say that a graph  $G$  satisfies an identity  $s \approx t$  if the corresponding graph algebra  $A(G)$  satisfies  $s \approx t$ . A graph  $G$  is called associative if the corresponding graph algebra  $A(G)$  satisfies the equation  $(xy)z \approx x(yz)$ . An identity  $s \approx t$  of terms  $s$  and  $t$  of any type  $\tau$  is called a hyperidentity of an algebra  $\underline{A}$  if whenever the operation symbols occurring in  $s$  and  $t$  are replaced by any term operations of  $\underline{A}$  of the appropriate arity, the resulting identities hold in  $\underline{A}$ .

In this paper we characterize associative graph algebras, identities in associative graph algebras and hyperidentities in associative graph algebras.

**Keywords:** identities, hyperidentities, associative graph algebras, terms.

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### 1. INTRODUCTION

An identity  $s \approx t$  of terms  $s, t$  of any type  $\tau$  is called *hyperidentity* of an algebra  $\underline{A}$  if whenever the operation symbols occurring in  $s$  and  $t$  are replaced by any term operations of  $\underline{A}$  of the appropriate arity, the resulting identity holds in  $\underline{A}$ . Hyperidentities can be defined more precisely using the concept of a hypersubstitution.

We fix a type  $\tau = (n_i)_{i \in I}$ ,  $n_i > 0$  for all  $i \in I$ , and operation symbols  $(f_i)_{i \in I}$ , where  $f_i$  is  $n_i$ -ary. Let  $W_\tau(X)$  be the set of all terms of type  $\tau$  over

some fixed alphabet  $X$ , and let  $Alg(\tau)$  be the class of all algebras of type  $\tau$ . Then a mapping

$$\sigma : \{f_i \mid i \in I\} \longrightarrow W_\tau(X)$$

which assigns to every  $n_i$ -ary operation symbol  $f_i$  an  $n_i$ -ary term will be called a *hypersubstitution of type  $\tau$*  (for short, a hypersubstitution). Let  $\hat{\sigma}[t]$  be the extension of the hypersubstitution  $\sigma$  to a mapping  $\hat{\sigma}$  defined on the set of all terms. The term  $\hat{\sigma}[t]$  can be defined inductively by:

for a term  $t \in W_\tau(X)$ ,

- (i)  $\hat{\sigma}[x] = x$  for any variable  $x$  in the alphabet  $X$ , and
- (ii)  $\hat{\sigma}[f_i(t_1, \dots, t_{n_i})] = \sigma(f_i)^{W_\tau(X)}(\hat{\sigma}[t_1], \dots, \hat{\sigma}[t_{n_i}])$ .

Here  $\sigma(f_i)^{W_\tau(X)}$  on the right hand side of (ii) is the operation induced by  $\sigma(f_i)$  on the term algebra  $W_\tau(X)$ .

Graph algebras have been invented in [8] to obtain examples of non-finitely based finite algebras. To recall this concept, let  $G = (V, E)$  be a (directed) graph with vertex set  $V$  and set of edges  $E \subseteq V \times V$ . Define the *graph algebra*  $A(G)$  corresponding to  $G$  to have underlying set  $V \cup \{\infty\}$ , where  $\infty$  is a symbol outside  $V$ , and two basic operations, a nullary operation pointing to  $\infty$  and a binary one denoted by juxtaposition, given for  $u, v \in V$  by

$$uv = \begin{cases} u & \text{if } (u, v) \in E \\ \infty & \text{otherwise.} \end{cases}$$

Graph identities were characterized in [3] by using the rooted graph of a term  $t$  where the vertices correspond to the variables occurring in  $t$ . Since on a graph algebra we have one nullary and one binary operation,  $\sigma(f)$  in this case is a binary term in  $W_\tau(X)$ , i.e. a term built up from variables of a two-element alphabet and a binary operation symbol  $f$  corresponding to the binary operation of the graph algebra.

In [6] R. Pöschel has shown that any term over the class of all graph algebras can be uniquely represented by a normal form term and that there is an algorithm to construct the normal form term to every given term  $t$ .

In [2] K. Denecke and T. Poomsa-ard characterized graph hyperidentities by using normal form graph hypersubstitutions.

A graph  $G = (V, E)$  is called *associative* if the corresponding graph algebra  $A(G)$  satisfies the equation  $x(yz) = (xy)z$ . In this paper we characterize associative graph algebras, identities in associative graph algebras and hyperidentities in associative graph algebras.

## 2. ASSOCIATIVE GRAPH ALGEBRAS

We begin with a more precise definition of terms of the type of graph algebras.

**Definition 2.1.** The set  $W_\tau(X)$  of all terms over the alphabet

$$X = \{x_0, x_1, x_2, \dots\}$$

is defined inductively as follows:

- (i) every variable  $x_i, i = 0, 1, 2, \dots$ , and  $\infty$  are terms;
- (ii) if  $s$  and  $t$  are terms, then  $f(s, t)$  is a term; instead of  $f(s, t)$  for short we will write  $(st)$ ;
- (iii)  $W_\tau(X)$  is the set of all terms which can be obtained from (i) and (ii) in finitely many steps.

Terms built up from the two-element set  $X_2 = \{x_1, x_2\}$  of variables are called binary. We denote the set of all binary terms by  $W_\tau(X_2)$ . The leftmost variable of a term  $t$  is denoted by  $L(t)$ . A term in which the symbol  $\infty$  occurs is called trivial.

**Definition 2.2.** To every non-trivial term  $t$  we assign a directed graph  $G(t) = (V(t), E(t))$  defined as follows: The vertex set  $V(t)$  is the set of variables occurring in  $t$ , and the set  $E(t)$  of edges is defined inductively by

$$E(t) = \phi \text{ if } t \text{ is a variable, } E(st) = E(s) \cup E(t) \cup \{(L(s), L(t))\}.$$

$L(t)$  is called the *root of the graph*  $G(t)$ , and the pair  $(G(t), L(t))$  is the *rooted graph corresponding to*  $t$ . Formally, to every trivial term  $t$  we assign the empty graph  $\phi$ .

**Definition 2.3.** We say, a graph  $G = (V, E)$  satisfies an identity  $s \approx t$  if the corresponding graph algebra  $A(G)$  satisfies  $s \approx t$  (i.e. we have  $s = t$  for every assignment  $V(s) \cup V(t) \rightarrow V \cup \{\infty\}$ ) and use the notation  $G \models s \approx t$ .

**Definition 2.4.** Let  $G = (V, E)$  and  $G' = (V', E')$  be graphs. A *homomorphism*  $f : G \rightarrow G'$  is a mapping  $f : V \rightarrow V'$  carrying edges to edges, i.e., for which  $(u, v) \in E$  we have  $(f(u), f(v)) \in E'$ .

In [3] was proved:

**Proposition 2.1.** *Let  $s$  and  $t$  be non-trivial terms from  $W_\tau(X)$  with variables  $V(s) = V(t) = \{x_0, x_1, \dots, x_n\}$  and  $L(s) = L(t)$ . Then a graph  $G = (V, E)$  satisfies  $s \approx t$  if and only if the graph algebra  $A(G)$  has the following*

*property:* A mapping  $h : V(s) \longrightarrow V$  is a homomorphism from  $G(s)$  into  $G$  iff it is a homomorphism from  $G(t)$  into  $G$ . ■

Proposition 2.1 gives a method to check whether a graph  $G = (V, E)$  has an associative graph algebra. Before to do this, we want to introduce some notation about graph.

For any graph  $G = (V, E)$ , let  $N(v) = \{w \in V \mid (v, w) \in E\}$  denote the set of all out-neighbours of a vertex  $v \in V$ . Graphs of the form  $(V, V \times V)$  are called *complete* graphs. For  $N \subseteq V$ , the *induced subgraph* is given by  $(N, E \cap (N \times N))$ . Then we have:

**Proposition 2.2.** *Let  $G = (V, E)$  be a graph. Then the following are equivalent:*

- (i)  $G$  has an associative graph algebra,
- (ii) for any edge  $(u, v) \in E$  and any vertex  $w \in V$ ,  $(u, w) \in E \iff (v, w) \in E$ ,
- (iii)  $E$  is transitive and for every  $v \in V$  the subgraph induced by  $N(v)$  is a complete graph.

**Proof.** (i)  $\Rightarrow$  (ii): Suppose  $G = (V, E)$  has an associative graph algebra. Let  $s$  and  $t$  be terms such that  $s := (xy)z, t := x(yz)$ . For any edge  $(u, v) \in E$  and any vertex  $w \in V$ , let  $h : V(s) \rightarrow V$  be the restriction of an evaluation function of the variables such that  $h(x) = u, h(y) = v$  and  $h(z) = w$ . We see that if  $(u, w) \in E$ , then  $h$  is a homomorphism from  $G(s)$  into  $G$ . By Proposition 2.1 we have  $h$  is a homomorphism from  $G(t)$  into  $G$ , that is  $(v, w) \in E$ . By the same way if  $(v, w) \in E$ , then  $(u, w) \in E$ .

(ii)  $\Rightarrow$  (iii): Clearly from (ii) we have  $E$  is transitive. Let  $v \in V$  and  $u, w \in N(v)$  we have  $(v, u), (v, w) \in E$ . By (ii) we get  $(u, w), (w, u), (u, u), (w, w) \in E$ . Hence subgraph of  $G$  induced by  $N(v)$  is a complete graph.

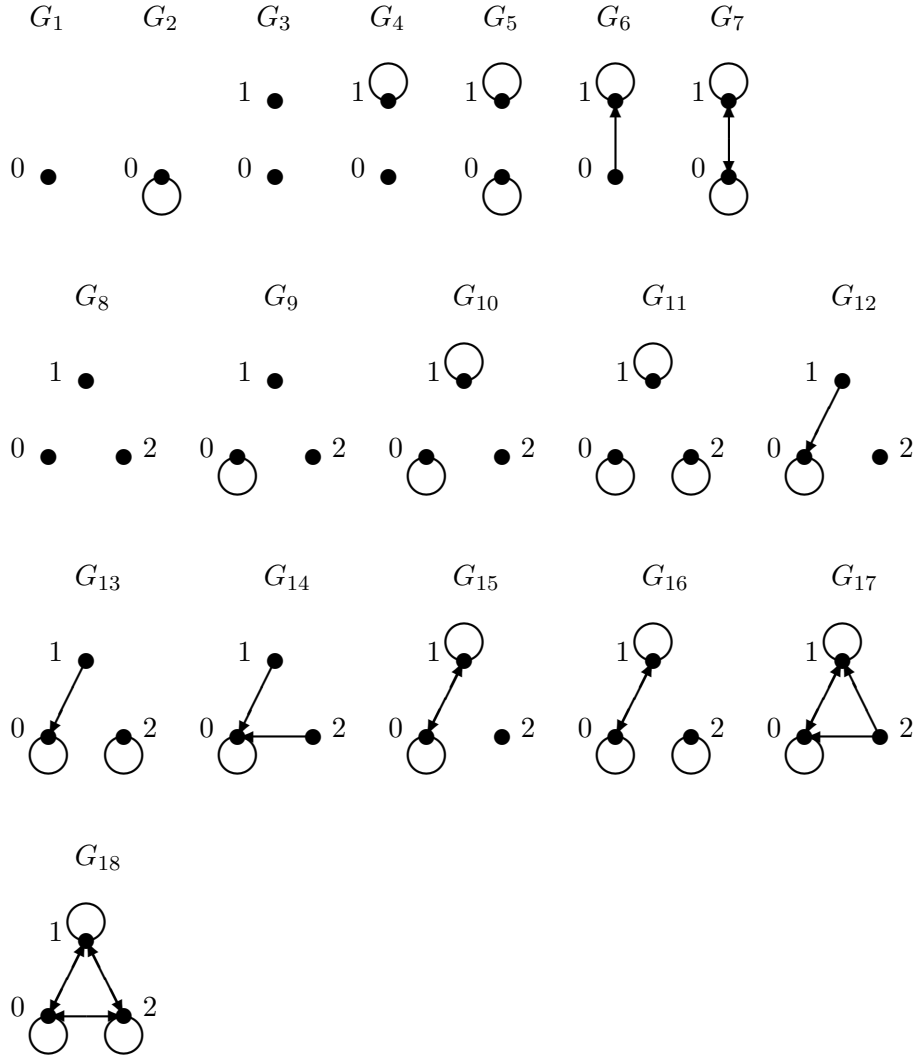
(iii)  $\Rightarrow$  (i): Let  $G = (V, E)$  be a graph which satisfies (iii), let  $s$  and  $t$  be non-trivial terms such that  $s := (xy)z, t := x(yz)$ . Suppose that  $h : V(s) \rightarrow V$  is a homomorphism from  $G(s)$  into  $G$ . Since  $(x, y), (x, z) \in E(s)$  we have  $(h(x), h(y)), (h(x), h(z)) \in E$ . By  $N(h(x))$  is a complete graph, we have  $(h(y), h(z)) \in E$ , therefore  $h$  is a homomorphism from  $G(t)$  into  $G$ . By the same way, if  $h$  is a homomorphism from  $G(t)$  into  $G$  and since  $E$  is transitive, then we have  $h$  is a homomorphism from  $G(s)$  into  $G$ . Hence by Proposition 2.1, we get  $A(G)$  satisfies  $s \approx t$ . ■

For the identity  $(xy)z \approx x(yz)$  we have, if two of the variables  $x, y, z$  coincide, then we get as special cases the identities,  $(xx)y \approx x(xy), (xy)y \approx$

$x(yy)$  and  $x(yx) \approx (xy)x$ . By Proposition 2.2 we can conclude that for any graph  $G = (V, E)$  if it has an associative graph algebra, then

- (i) if  $(u, v) \in E$ , then  $v$  has a loop,
- (ii) if  $(u, v) \in E$ , then  $u$  has a loop iff  $(v, u) \in E$ .

From this and Proposition 2.2 give the list of all graphs  $G = (V, E)$  which  $|V| \leq 3$  and  $G$  has an associative graph algebra.



**Remark.** Let  $\mathcal{B}$  be the set of all graphs in the list. Then a graph  $G$  has an associative graph algebra if and only if each induced subgraph  $H$  of  $G$  with  $|V(H)| \leq 3$  is isomorphic to an element of  $\mathcal{B}$ .

### 3. IDENTITIES IN ASSOCIATIVE GRAPH ALGEBRAS

Graph identities were characterized in [3] by the following proposition:

**Proposition 3.1.** *A non-trivial equation  $s \approx t$  is an identity in the class of all graph algebras iff either both terms  $s$  and  $t$  are trivial or none of them is trivial,  $G(s) = G(t)$  and  $L(s) = L(t)$ . ■*

Further they proved:

**Proposition 3.2.** *Let  $G = (V, E)$  be a graph and let  $h : X \longrightarrow V \cup \{\infty\}$  be an evaluation of the variables. Consider the canonical extension of  $h$  to the set of all terms. Then there holds: if  $t$  is a trivial term then  $h(t) = \infty$ . Otherwise, if  $h : G(t) \longrightarrow G$  is a homomorphism of graphs, then  $h(t) = h(L(t))$ , and if  $h$  is not a homomorphism of graphs, then  $h(t) = \infty$ . ■*

By Proposition 3.2 we have the following lemma:

**Lemma 3.1.** *Let  $G = (V, E)$  be a graph, let  $t$  be a term and let  $h : X \longrightarrow V \cup \{\infty\}$  be an evaluation of the variables.*

- (i) *If  $h : G(t) \longrightarrow G$  has the property that the subgraph of  $G$  induced by  $h(V(t))$  is complete, then  $h(t) = h(L(t))$ .*
- (ii) *If  $h : G(t) \longrightarrow G$  has the property that the subgraph of  $G$  induced by  $h(V(t))$  is disconnected, then  $h(t) = \infty$ .*

**Proof.** (i) Since the subgraph of  $G$  induced by  $h(V(t))$  is complete,  $h$  is a homomorphism of graphs. By Proposition 3.2, we have  $h(t) = h(L(t))$ .

(ii) Since the subgraph of  $G$  induced by  $h(V(t))$  is disconnected and  $G(t)$  is a rooted graph which is connected, there is an edge  $(u, v)$  of  $G(t)$  such that  $(h(u), h(v))$  is not an edge of the subgraph of  $G$  induced by  $h(V(t))$ . Hence,  $h$  is not a homomorphism. By Proposition 3.2, we have  $h(t) = \infty$ . ■

Now we apply our results to characterize all identities in the class of all associative graph algebras. Clearly, if  $s$  and  $t$  are trivial, then  $s \approx t$  is an identity in the class of all associative graph algebras and  $x \approx x, x \in X$  is an identity in the class of all associative graph algebras too. So we consider the case that  $s$  and  $t$  are non-trivial and different from variables. Then all identities in associative graph algebras are characterized by the following theorem:

**Theorem 3.1.** *Let  $s$  and  $t$  be non-trivial terms and let  $x_0 = L(s)$ . Then  $s \approx t$  is an identity in the class of all associative graph algebras if and only if the following conditions are satisfied:*

- (i)  $L(s) = L(t)$ ,
- (ii)  $V(s) = V(t)$ ,
- (iii) *The variable  $x_0$  occurs in  $s$  more than once if and only if  $x_0$  occurs in  $t$  more than once.*

**Proof.** Suppose that  $s \approx t$  is an identity in the class of all associative graph algebras.

If  $V(s) \neq V(t)$ , then we can suppose that there exists  $x$  such that  $x \in V(s)$  and  $x \notin V(t)$ . Consider the graph  $G_6$  in the list and  $h : V(s) \cup V(t) \rightarrow V(G_6)$  is the restriction of an evaluation of the variables such that  $h(x) = 0$ ,  $h(y) = 1$  for all other  $y \in V(s) \cup V(t)$ . Obviously  $A(G)$  is not satisfies  $s \approx t$ .

Let  $G = (V, E)$  be a complete graph with  $V = V(s) = V(t)$  and let an identity function  $h : V(s) \rightarrow V$  be the restriction of an evaluation of the variables. Since  $G$  is a complete graph, by Lemma 3.1, we have  $L(s) = h(L(s)) = h(s) = h(t) = h(L(t)) = L(t)$ .

Let the variable  $x_0$  occur in  $s$  more than once and occur in  $t$  only once. Then  $(x, x_0) \in E(s)$  for some  $x \in V(s)$  and  $(y, x_0) \notin E(t)$  for all  $y \in V(t)$ . Consider the graph  $G_6$  in the list again and  $h : V(s) \rightarrow V(G_6)$  such that  $h(x_0) = 0$ , and  $h(y) = 1$  for all other  $y \in V(s)$ . We see that  $h(s) = \infty$  and  $h(t) = 0$ . Hence,  $A(G_6)$  does not satisfy  $s \approx t$ .

Conversely, suppose that  $s$  and  $t$  are non-trivial terms and satisfy (i), (ii) and (iii). Let  $G = (V, E)$  be a graph which has an associative graph algebra and let  $h : X \rightarrow V \cup \{\infty\}$  be an evaluation of the variables. Consider the restriction function of  $h$  on  $V(s)$  and  $V(t)$ . Since  $V(s) = V(t)$ , we have that the subgraphs of  $G$ , which are induced by  $h(V(s))$  and  $h(V(t))$ , are the same subgraph of  $G$ , say it is the subgraph  $H$ .

If  $H$  is a complete subgraph of  $G$ , then  $h(s) = L(s) = x_0 = L(t) = h(t)$ .

If  $H$  is a disconnected subgraph of  $G$ , then  $h(s) = \infty = h(t)$ .

For the case  $H$  is a connected subgraph and not complete, by Proposition 2.2 we can prove that for any  $v_1, v_2 \in V(H)$ ,  $N(v_1) = N(v_2)$  and since  $H$  is not complete, therefore there exists at least one element  $v' \in V(H)$  such that  $v' \notin N(v')$ .

If there exist  $v_1, v_2 \in V(H)$  such that  $v_1 \neq v_2$ ,  $v_1 \notin N(v_1)$  and  $v_2 \notin N(v_2)$ , then  $h(s) = \infty = h(t)$ .

If there exists only one element  $v' \in V(H)$  such that  $v' \notin N(v')$ , then

- a)  $h(s) = \infty = h(t)$  if there exist  $x \neq x_0$ ,  $h(x) = v'$  or  $h(x_0) = v'$ ,  $h(x) \neq v'$  for all  $x \neq x_0$  and  $x_0$  occurs in  $s$  more than once,
- b)  $h(s) = v' = h(t)$  if  $h(x_0) = v'$ ,  $h(x) \neq v'$  for all  $x \neq x_0$  and  $x_0$  occurs in  $s$  only once.

Hence we have  $A(G)$  satisfies  $s \approx t$ . ■

#### 4. HYPERIDENTITIES IN ASSOCIATIVE GRAPH ALGEBRAS

Let  $\mathcal{AG}$  be the class of all associative graph algebras and let  $Id\mathcal{AG}$  be the set of all identities satisfied in  $\mathcal{AG}$ . Now we want to precisise the concept of a hypersubstitution for graph algebras.

**Definition 4.1.** A mapping  $\sigma : \{f, \infty\} \rightarrow W_\tau(X_2)$ , where  $f$  is the operation symbol corresponding to the binary operation of a graph algebra, is called *graph hypersubstitution* if  $\sigma(\infty) = \infty$  and  $\sigma(f) = s \in W_\tau(X_2)$ . The graph hypersubstitution with  $\sigma(f) = s$  is denoted by  $\sigma_s$ .

**Definition 4.2.** An identity  $s \approx t$  is an *associative graph hyperidentity* iff for all graph hypersubstitutions  $\sigma$ , the equations  $\hat{\sigma}[s] \approx \hat{\sigma}[t]$  are identities in  $\mathcal{AG}$ .

Clearly, if  $s$  and  $t$  are trivial terms, then  $s \approx t$  is an associative graph hyperidentity. If we want to check that  $s \approx t$  ( $s, t$  are non-trivial) is an associative graph hyperidentity, then we can restrict ourselves to a (small) subset of  $Hyp\mathcal{G}$  – the set of all graph hypersubstitutions.

In [4] the following relation between hypersubstitutions was defined:



**Definition 4.3.** Two graph hypersubstitutions  $\sigma_1, \sigma_2$  are called  $\mathcal{AG}$ -equivalent iff  $\sigma_1(f) \approx \sigma_2(f)$  is an identity in  $\mathcal{AG}$ . In this case, we write  $\sigma_1 \sim_{\mathcal{AG}} \sigma_2$ .

In [1] (see also [4]) was proved:

**Lemma 4.1.** *If  $\hat{\sigma}_1[s] \approx \hat{\sigma}_1[t] \in Id\mathcal{AG}$  and  $\sigma_1 \sim_{\mathcal{AG}} \sigma_2$ , then  $\hat{\sigma}_2[s] \approx \hat{\sigma}_2[t] \in Id\mathcal{AG}$ .* ■

Therefore, it is enough to consider the quotient set  $Hyp\mathcal{G} / \sim_{\mathcal{AG}}$ .

In [6] was shown that any non-trivial term  $t$  over the class of graph algebras has a uniquely determined normal form term  $NF(t)$  and there is an algorithm to construct the normal form term to a given term  $t$ . Now we want to describe how to construct the normal form term. Let  $t$  be a non-trivial term. The *normal form term* of  $t$  is the term  $NF(t)$  constructed by the following algorithm:

- (i) Construct  $G(t) = (V(t), E(t))$ .
- (ii) Construct for every  $x \in V(t)$  the list  $l_x = (x_{i_1}, \dots, x_{i_{k(x)}})$  of all out-neighbours (i.e.  $(x, x_{i_j}) \in E(t), 1 \leq j \leq k(x)$ ) ordered by increasing indices  $i_1 \leq \dots \leq i_{k(x)}$  and let  $s_x$  be the term  $((x_{i_1})x_{i_2}) \dots x_{i_{k(x)}}$ .
- (iii) Starting with  $x := L(t), Z := V(t), s := L(t)$ , choose the variable  $x_i \in Z \cap V(s)$  with the least index  $i$ , substitute the first occurrence of  $x_i$  by the term  $s_{x_i}$ , denote the resulting term again by  $s$  and put  $Z := Z \setminus \{x_i\}$ . While  $Z \neq \emptyset$ , continue this procedure. The resulting term is the normal form  $NF(t)$ .

The algorithm stops after a finite number of steps, since  $G(t)$  is a rooted graph. Without difficulties one shows  $G(NF(t)) = G(t), L(NF(t)) = L(t)$ .

In [2] was defined:

**Definition 4.4.** The graph hypersubstitution  $\sigma_{NF(t)}$ , is called *normal form graph hypersubstitution*. Here  $NF(t)$  is the normal form of the binary term  $t$ .

Since for any binary term  $t$  the rooted graphs of  $t$  and  $NF(t)$  are the same, we have  $t \approx NF(t) \in Id\mathcal{AG}$ . Then for any graph hypersubstitution  $\sigma_t$  with  $\sigma_t(f) = t \in W_\tau(X_2)$ , one obtains  $\sigma_t \sim_{\mathcal{AG}} \sigma_{NF(t)}$ .

In [2] all rooted graphs with at most two vertices were considered. Then we formed the corresponding binary terms and used the algorithm to construct normal form terms. The result is given in the following table.

normal form term	graph hypers.	normal form term	graph hypers.
$x_1x_2$	$\sigma_0$	$x_1$	$\sigma_1$
$x_2$	$\sigma_2$	$x_1x_1$	$\sigma_3$
$x_2x_2$	$\sigma_4$	$x_2x_1$	$\sigma_5$
$(x_1x_1)x_2$	$\sigma_6$	$(x_2x_1)x_2$	$\sigma_7$
$x_1(x_2x_2)$	$\sigma_8$	$x_2(x_1x_1)$	$\sigma_9$
$(x_1x_1)(x_2x_2)$	$\sigma_{10}$	$(x_2(x_1x_1))x_2$	$\sigma_{11}$
$x_1(x_2x_1)$	$\sigma_{12}$	$x_2(x_1x_2)$	$\sigma_{13}$
$(x_1x_1)(x_2x_1)$	$\sigma_{14}$	$x_2((x_1x_1)x_2)$	$\sigma_{15}$
$x_1((x_2x_1)x_2)$	$\sigma_{16}$	$(x_2(x_1x_2))x_2$	$\sigma_{17}$
$(x_1x_1)((x_2x_1)x_2)$	$\sigma_{18}$	$(x_2((x_1x_1)x_2))x_2$	$\sigma_{19}$

By Theorem 3.1, we have the following relations:

- (1)  $\sigma_0 \sim_{\mathcal{AG}} \sigma_8$ ,
- (2)  $\sigma_5 \sim_{\mathcal{AG}} \sigma_9$ ,
- (3)  $\sigma_6 \sim_{\mathcal{AG}} \sigma_{10} \sim_{\mathcal{AG}} \sigma_{12} \sim_{\mathcal{AG}} \sigma_{14} \sim_{\mathcal{AG}} \sigma_{16} \sim_{\mathcal{AG}} \sigma_{18}$ ,
- (4)  $\sigma_7 \sim_{\mathcal{AG}} \sigma_{11} \sim_{\mathcal{AG}} \sigma_{13} \sim_{\mathcal{AG}} \sigma_{15} \sim_{\mathcal{AG}} \sigma_{17} \sim_{\mathcal{AG}} \sigma_{19}$ .

Let

$$M_{\mathcal{AG}} = \{\sigma_0, \sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5, \sigma_6, \sigma_7\}.$$

We defined the product of two normal form graph hypersubstitutions in  $M_{\mathcal{AG}}$  as follows.

**Definition 4.5.** The *product*  $\sigma_{1N} \circ_N \sigma_{2N}$  of two normal form graph hypersubstitutions is defined by  $(\sigma_{1N} \circ_N \sigma_{2N})(f) = NF(\hat{\sigma}_{1N}[\sigma_{2N}(f)])$ .

The following table gives the multiplication of  $M_{\mathcal{AG}}$ .

$\circ_N$	$\sigma_0$	$\sigma_1$	$\sigma_2$	$\sigma_3$	$\sigma_4$	$\sigma_5$	$\sigma_6$	$\sigma_7$
$\sigma_0$	$\sigma_0$	$\sigma_1$	$\sigma_2$	$\sigma_3$	$\sigma_4$	$\sigma_5$	$\sigma_6$	$\sigma_7$
$\sigma_1$	$\sigma_1$	$\sigma_1$	$\sigma_2$	$\sigma_1$	$\sigma_2$	$\sigma_2$	$\sigma_1$	$\sigma_2$
$\sigma_2$	$\sigma_2$	$\sigma_1$	$\sigma_2$	$\sigma_1$	$\sigma_2$	$\sigma_1$	$\sigma_2$	$\sigma_2$
$\sigma_3$	$\sigma_3$	$\sigma_1$	$\sigma_2$	$\sigma_3$	$\sigma_4$	$\sigma_4$	$\sigma_3$	$\sigma_4$
$\sigma_4$	$\sigma_4$	$\sigma_1$	$\sigma_2$	$\sigma_3$	$\sigma_4$	$\sigma_3$	$\sigma_4$	$\sigma_4$
$\sigma_5$	$\sigma_5$	$\sigma_1$	$\sigma_2$	$\sigma_3$	$\sigma_4$	$\sigma_0$	$\sigma_5$	$\sigma_7$
$\sigma_6$	$\sigma_6$	$\sigma_1$	$\sigma_2$	$\sigma_3$	$\sigma_4$	$\sigma_7$	$\sigma_6$	$\sigma_7$
$\sigma_7$	$\sigma_7$	$\sigma_1$	$\sigma_2$	$\sigma_3$	$\sigma_4$	$\sigma_6$	$\sigma_7$	$\sigma_7$

In [2] the leftmost normal form graph hypersubstitution was defined.

**Definition 4.6.** A graph hypersubstitution  $\sigma$  is called *leftmost* if

$$L(\sigma(f)) = x_1.$$

The set  $M_L$  of all leftmost normal form graph hypersubstitution in  $M_{\mathcal{AG}}$  contains exactly the following elements:

$$M_L = \{\sigma_0, \sigma_1, \sigma_3, \sigma_6\}.$$

In [5] the concept of a proper hypersubstitution of a class of algebras was introduced.

**Definition 4.7.** A hypersubstitution  $\sigma$  is called *proper with respect to a class  $\mathcal{K}$  of algebras* if  $\hat{\sigma}[s] \approx \hat{\sigma}[t] \in Id\mathcal{K}$  for all  $s \approx t \in Id\mathcal{K}$ .

A graph hypersubstitution with the property that  $\sigma(f)$  contains both variables  $x_1$  and  $x_2$  is called regular. It is easy to check that the set of all regular graph hypersubstitutions form a groupoid  $M_{reg}$ .

We want to prove that  $M_L$  is the set of all proper normal form graph hypersubstitutions with respect to  $\mathcal{AG}$ .

In [2] the following lemma was proved:

**Lemma 4.2.** *For each non-trivial term  $s, (s \neq x \in X_2)$ ,*

$$E(\hat{\sigma}_6[s]) = E(s) \cup \{(u, u) \mid (u, v) \in E(s), u, v \in X_2\}.$$

Then we obtain:

**Theorem 4.2.**  *$M_L$  is the set of all proper graph hypersubstitutions.*

**Proof.** If  $s \approx t \in Id\mathcal{AG}$  and  $s, t$  are trivial terms, then for every graph hypersubstitution  $\sigma \in M_{\mathcal{AG}}$  the terms  $\hat{\sigma}[s]$  and  $\hat{\sigma}[t]$  are also trivial and thus  $\hat{\sigma}[s] \approx \hat{\sigma}[t] \in Id\mathcal{AG}$ .

By the same way we see that  $\hat{\sigma}[s] \approx \hat{\sigma}[t] \in Id\mathcal{AG}$  for every  $\sigma \in M_{\mathcal{AG}}$ , if  $s = t = x$ .

Now, assume that  $s$  and  $t$  are non-trivial terms, different from variables and  $s \approx t \in Id\mathcal{AG}$ . Then (i)  $L(s) = L(t)$  (let  $L(s) = L(t) = x'$ ), (ii)  $V(s) = V(t)$  and (iii) the variable  $x'$  occurs in  $s$  more than once if and only if  $x'$  occurs in  $t$  more than once.

At first assume that  $\sigma \in \{\sigma_1, \sigma_3\}$ . Then for  $\sigma_1$  we have  $\hat{\sigma}_1[s] = \hat{\sigma}_1[t] = x'$  and for  $\sigma_3$  we get  $\hat{\sigma}_3[s] = \hat{\sigma}_3[t] = x'x'$ .

For  $\sigma_6$  we obtain  $L(\hat{\sigma}_6[s]) = L(s) = L(t) = L(\hat{\sigma}_6[t])$ . Since  $\sigma_6$  is regular, we have  $V(s) = V(\hat{\sigma}_6[s])$  and  $V(t) = V(\hat{\sigma}_6[t])$ . Because of  $V(s) = V(t)$  we have  $V(\hat{\sigma}_6[s]) = V(\hat{\sigma}_6[t])$ .

By Lemma 4.2, we have

$$E(\hat{\sigma}_6[s]) = E(s) \cup \{(u, u) \mid (u, v) \in E(s)\} \text{ and}$$

$$E(\hat{\sigma}_6[t]) = E(t) \cup \{(u, u) \mid (u, v) \in E(t)\}.$$

Since  $L(s) = L(t) = x'$ , we obtain  $(x', x_i) \in E(s)$  and  $(x', x_j) \in E(t)$  for some  $i, j = 1, 2$ , hence  $(x', x') \in E(\hat{\sigma}_6[s])$  and  $(x', x') \in E(\hat{\sigma}_6[t])$  thus  $x'$  occurs in  $E(\hat{\sigma}_6[s])$  and  $E(\hat{\sigma}_6[t])$  more than once.

By Theorem 3.1, we have  $\hat{\sigma}_6[s] \approx \hat{\sigma}_6[t] \in Id\mathcal{AG}$

For any  $\sigma \notin M_L$  i.e.  $L(\sigma(f)) = x_2$ , we give an identity  $s \approx t$  in  $\mathcal{AG}$  and show that  $\hat{\sigma}[s] \approx \hat{\sigma}[t] \notin Id\mathcal{AG}$ .

Let  $s = (x_1x_2)x_1$ ,  $t = ((x_1x_2)x_1)x_2$ . By Theorem 3.1 we get  $s \approx t \in Id\mathcal{AG}$ .

If  $\sigma \in \{\sigma_2, \sigma_4, \sigma_5, \sigma_7\}$ , then  $L(\sigma(f)) = x_2$ . We see that  $L(\hat{\sigma}[s]) = x_1$  and  $L(\hat{\sigma}[t]) = x_2$  for all  $\sigma \in \{\sigma_2, \sigma_4, \sigma_5, \sigma_7\}$ ; thus  $\hat{\sigma}[s] \approx \hat{\sigma}[t] \notin Id\mathcal{AG}$ . ■

Now we apply our results to characterize all hyperidentities in associative graph algebras. Clearly, if  $s$  and  $t$  are trivial, then  $s \approx t$  is a hyperidentity in  $\mathcal{AG}$  and  $x \approx x, x \in X$ , is a hyperidentity in  $\mathcal{AG}$  too. So we consider the case that  $s$  and  $t$  are non-trivial and different from variables.

In [2] the concept of a *dual term*  $s^d$  of the non-trivial term  $s$  was defined in the following way:

If  $s = x \in X$ , then  $s^d = x$ , if  $s = t_1 t_2$ , then  $s^d = t_2^d t_1^d$ . The dual term  $s^d$  can be obtained by application of the graph hypersubstitution  $\sigma_5$ , i.e.,  $\hat{\sigma}_5[s] = s^d$ .

**Theorem 4.2.** *An identity  $s \approx t$  in  $\mathcal{AG}$ , where  $s, t$  are non-trivial and  $s \neq x, t \neq x$ , is a hyperidentity in  $\mathcal{AG}$  if and only if the dual equation  $s^d \approx t^d$  is also an identity in  $\mathcal{AG}$ .*

**Proof.** If  $s \approx t$  is a hyperidentity in  $\mathcal{AG}$ , then  $\hat{\sigma}_5[s] \approx \hat{\sigma}_5[t]$  is an identity in  $\mathcal{AG}$ , i.e.  $s^d \approx t^d$  is an identity in  $\mathcal{AG}$ .

Assume that  $s \approx t$  is an identity in  $\mathcal{AG}$  and that  $s^d \approx t^d$  is an identity in  $\mathcal{AG}$  too. We have to prove that  $s \approx t$  is closed under all graph hypersubstitutions from  $M_{\mathcal{AG}}$ .

If  $\sigma$  is proper, then  $\hat{\sigma}[s] \approx \hat{\sigma}[t] \in Id\mathcal{AG}$ , i.e. if  $\sigma \in M_L$  we are ready. By assumption  $\hat{\sigma}_5[s] = s^d \approx t^d = \hat{\sigma}_5[t]$  is an identity in  $\mathcal{AG}$ . Because of  $\sigma_1 \circ_N \sigma_5 = \sigma_2, \sigma_3 \circ_N \sigma_5 = \sigma_4, \sigma_6 \circ_N \sigma_5 = \sigma_7$  and  $\hat{\sigma}[\hat{\sigma}_5[t']] = \hat{\sigma}[t'^d]$  for all  $\sigma \in M_{\mathcal{AG}}, t' \in W_\tau(X_2)$ . Then we have  $\hat{\sigma}_2[s] \approx \hat{\sigma}_2[t], \hat{\sigma}_4[s] \approx \hat{\sigma}_4[t], \hat{\sigma}_7[s] \approx \hat{\sigma}_7[t]$  are identities in  $\mathcal{AG}$ . ■

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