Discussiones Mathematicae General Algebra and Applications 20 (2000) 159–167

A FACTORIZATION OF ELEMENTS IN PSL(2, F), WHERE $F = \mathbb{Q}, \mathbb{R}$

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Abstract

Let G be a group and $K_n = \{g \in G : o(g) = n\}$. It is prowed: (i) if $F = \mathbb{R}$, $n \geq 4$, then $PSL(2, F) = K_n^2$; (ii) if $F = \mathbb{Q}$, \mathbb{R} , $n = \infty$, then $PSL(2, F) = K_n^2$; (iii) if $F = \mathbb{R}$, then $PSL(2, F) = K_3^3$; (iv) if $F = \mathbb{Q}$, \mathbb{R} , then $PSL(2, F) = K_2^3 \cup E$, $E \notin K_2^3$, where E denotes the unit matrix; (v) if $F = \mathbb{Q}$, then $PSL(2, F) \neq K_3^3$.

Keywords: factorization of linear groups, linear groups, matrix representations of groups, sets of elements of the same order in groups.

1991 Mathematics Subject Classification: 20G20, 11E57, 15A23, 20G15.

Let G be a group and $K_n = K_n(G) = \{g \in G : o(g) = n\}$. Let SL(m, F) and PSL(m, F) be a special linear or projective specjal linear (resp.) groups of degree m over a field F. Many papers have been devoted to the powers of the set K_2 (see [3] - [9]) but only few papers have been written about the powers of the set K_n for n > 2 (see [1] - [3]). In the papers [3] and [5], it has been proved that if F is an algebraically closed field, then $PSL(3, F) = K_nK_n$ for n > 2 and $PSL(3, F) = K_2^4$ for any F. Note that we do not identify K_2 with the set of involutions. In the paper [7], it has been proved that if $F = \mathbb{Q}$, \mathbb{R} , where \mathbb{Q} denotes the field of rational numbers and \mathbb{R} denotes the field of real numbers, then $PSL(2, F) = K_n^4$.

In this paper we will prove the following properties:

- (i) if $F = \mathbb{R}$, $n \ge 4$, then $PSL(2, F) = K_n^2$;
- (ii) if $F = \mathbb{Q}, \mathbb{R}, n = \infty$, then $PSL(2, F) = K_n^2$;
- (iii) if $F = \mathbb{R}$, then $PSL(2, F) = K_3^3$;

- (iv) if $F = \mathbb{Q}$ or \mathbb{R} , then $PSL(2, F) = K_2^3 \cup E, E \notin K_2^3$, where E denotes the unit matrix;
- (v) if $F = \mathbb{Q}$, then $PSL(2, F) \neq K_3^3$.

Recall, that $PSL(2,\mathbb{C}) = K_n^2$, where \mathbb{C} denotes the field of complex numbers (see [2]).

We begin with some lemmas.

Lemma 1. Let F be any field. In SL(2, F), each non-scalar matrix is similar to a matrix of the form $\begin{bmatrix} 0 & r \\ -r^{-1} & s \end{bmatrix} = D$. The order of D depends only on s.

- If $F = \mathbb{R}$, then
- a) the order of the matrix $D \in SL(2,\mathbb{R})$ is n > 2 iff $s = 2\cos\frac{2k\pi}{n}$ and (k,n) = 1;
- b) the order of the matrix $D \in PSL(2,\mathbb{R})$ is n > 2 iff $s = 2\cos\frac{k\pi}{n}$ and (k,n) = 1 or $s = 2\cos\frac{2k\pi}{n}, (k,n) = 1$.

If $F = \mathbb{Q}$ or \mathbb{R} and |s| > 2, then the order of D is ∞ .

Proof. If F is any field, then for each $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SL(2, F)$ there exists a matrix

$$X = \begin{bmatrix} x & y \\ \frac{1}{r}(xa+cy) & \frac{1}{r}(bx+yd) \end{bmatrix} \text{ and } D = \begin{bmatrix} 0 & r \\ -r^{-1} & s \end{bmatrix}$$

such that $A = X^{-1}DX$ and s = a + d. The condition det X = 1 holds since the equation $\frac{x}{r}(bx + yd) - \frac{y}{r}(xa + cy) = 1$ has a solution in r, x, y.

If F is any field, then we can find that

$$D^{n} = \begin{bmatrix} \varphi_{n-2}(s) & r\psi_{n-1}(s) \\ -r^{-1}\psi_{n-1}(s) & \omega_{n}(s) \end{bmatrix},$$

where $\varphi_{n-2}, \psi_{n-1}, \omega_n$ are polynomials in s which means that the order of D depends only on s.

In the case $F = \mathbb{R}$, it is easy to notice that the order of any matrix A over \mathbb{R} is the same as over $F = \mathbb{C}$. Thus if -2 < s < 2, we can put $s = 2\cos\varphi$, and then the matrix D is similar to the diagonal matrix $\begin{bmatrix} e^{i\varphi} & 0\\ 0 & e^{-i\varphi} \end{bmatrix}$ over \mathbb{C} . Hence, the rest of the proof follows from obvious properties of the group of the *n*-th roots of unity. If |s| > 2, then the order of D is ∞ .

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Lemma 2 (see [5]). If $V = \text{diag}(v_1, ..., v_m)$, $W = \text{diag}(w_1, ..., w_m)$, $v_i \neq v_j$, $w_i \neq w_j$ for $i \neq j$ and $V, W \in SL(m, F)$, then $SL(m, F) = C_V C_W \cup Z$, where C_V denotes the conjugacy class of V and Z denotes the center of SL(m, F).

Lemma 3. If

$$N_i = \begin{bmatrix} 0 & w_i \\ -w_i^{-1} & 0 \end{bmatrix}, \quad T_i = \begin{bmatrix} 0 & 1 \\ -1 & x_i \end{bmatrix}, \quad N_i, T_i \in SL(2, F),$$

then the trace $\operatorname{tr}(N_1^{T_1}N_2^{T_2}N_3^{T_3}) = s$ is any arbitrary element of F, where $(N_i^{T_i} = T_i^{-1}N_iT_i)$.

Proof. If we put $x_1 = x_2 = 0$, then $s = -w_3w_1^{-1}w_2^{-1}(w_1^2 + w_2^2)x_3$. Thus s is directly proportional to x_3 and s can be any arbitrary element of F.

Lemma 4. If

$$M_{i} = \begin{bmatrix} 0 & w_{i} \\ -w_{i}^{-1} & d_{i} \end{bmatrix} (i = 1, 2, 3), \ T_{i} = \begin{bmatrix} 0 & 1 \\ -1 & x_{i} \end{bmatrix}, \ d_{i} \neq 0,$$

and $M_i, T_i \in SL(2, F)$, then there are w_i such that the trace $tr(M_1M_2^{T_2}M_3^{T_3}) = s$ is any arbitrary element of F.

Proof. A calculation shows that if we take $w_2 = -d_2d_1^{-1}w_1^{-1}, x_3 = x_2 + d_3w_3^{-1}$ and $(w_1w_3d_1)^2 \neq d_2^2$, then $s = x_2(d_1d_2^{-1}w_3 - d_1d_2^{-1}w_1^{-1}w_3^{-1}) + d_1d_3d_2^{-1}$, so s varies as a linear function of x_2 .

Lemma 5. Let $M_i = \begin{bmatrix} 0 & w_i \\ -w_i^{-1} & d_i \end{bmatrix}$, $S_i = \begin{bmatrix} 0 & y_i \\ -y_i^{-1} & x_i \end{bmatrix}$, over \mathbb{R} . Then

(1)

$$s = \operatorname{tr}(M_1^{S_1} M_2^{S_2}) = -w_1 w_2 \left(\frac{x_1 y_1 - x_2 y_2}{y_2 y_1}\right)^2 + (x_1 y_1 - x_2 y_2) \left(\frac{d_1 w_2}{y_2^2} - \frac{w_1 d_2}{y_1^2}\right) - \left(\frac{w_2}{w_1}\right) \left(\frac{y_1}{y_2}\right)^2 - \left(\frac{w_1}{w_2}\right) \left(\frac{y_2}{y_1}\right) + d_1 d_2.$$

achieves the minimum

$$s_{\min} = \frac{1}{2}\sqrt{(4-d_1^2)(4-d_2^2)} + \frac{1}{2}d_1d_2$$

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and the maximum value

$$s_{\max} = -\frac{1}{2}\sqrt{(4-d_1^2)(4-d_2^2)} + \frac{1}{2}d_1d_2$$

for $w_1w_2 < 0$ and $w_1w_2 > 0$, respectively.

Proof. If we consider the trace s as a function of x_1, x_2 , then the condition

(2)
$$\frac{\partial s}{\partial x_1} = \frac{\partial s}{\partial x_2} = 0$$

is equivalent to the condition

(3)
$$2(x_1y_1 - x_2y_2) = \frac{d_1}{d_2}y_1^2 - \frac{d_2}{d_1}y_2^2.$$

Since $\frac{\partial^2 s}{\partial x_1^2} = -\frac{2w_1w_2}{y_2^2}$, $\frac{\partial^2 s}{\partial x_2^2} = -\frac{2w_1w_2}{y_1^2}$, $\frac{\partial^2 s}{\partial x_1\partial x_2} = -\frac{2w_1w_2}{y_1y_2}$, therefore

(4)
$$s(x_1+h, x_2+k) - s(x_1, x_2) = -\frac{2w_1w_2}{y_1^2y_2^2}(y_1-y_2k)^2$$

Hence, $s(x_1, x_2)$ achieves the minimum and the maximum value for $w_1 w_2 < 0$ and $w_1w_2 > 0$, respectively. The value of the trace s, at the surface (3) equals

(5)
$$\frac{1}{4}x(d_2^2 - 4) + \frac{1}{4}(d_1^2 - 4)\frac{1}{x} + \frac{1}{2}d_1d_2$$

where $x = \frac{w_1}{w_2} (\frac{y_1}{y_2})^2$. The function (5) in x and, as a result, also s achieves the minimum s_{\min} and the maximum s_{\max} value for

$$\frac{w_1}{w_2} = -(\frac{y_1}{y_2})^2 \sqrt{\frac{d_1^2 - 4}{d_2^2 - 4}}$$
 and $\frac{w_1}{w_2} = (\frac{y_1}{y_2})^2 \sqrt{\frac{d_1^2 - 4}{d_2^2 - 4}}$,

respectively.

Lemma 6. If $F = \mathbb{R}$, then the non-scalar matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SL(2, F)$ and $D = \begin{bmatrix} 0 & r \\ -r^{-1} & s \end{bmatrix}$, are similar in SL(2, F) provided s = a + d, $br \ge 0$ or $-cr \ge 0$.

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Proof. We have $XAX^{-1} = D$, where

$$X = \begin{bmatrix} x & y\\ \frac{1}{r}(ax+cy) & \frac{1}{r}(bx+yd) \end{bmatrix}, \quad \det X \neq 0.$$

The condition $X \in SL(2, F)$ is equivalent to the solvability of the equation

(6)
$$bx^2 + xy(d-a) - cy^2 - r = 0$$
 in x or y.

The descriminant $\Delta = y^2(s^2 - 4) + 4br$ or $\Delta = x^2(s^2 - 4) - 4cr$, respectively, must be a non negative element of F.

By the assumption $br \ge 0$ or $-cr \ge 0$, we can chose so small y or x such that $\Delta \ge 0$ for any $a, d \in \mathbb{R}$.

Lemma 7. Let $s = tr(M_1^{S_1}M_2^{S_2})$ be defined by (1) and let n be the order of M_i . Then:

 $\begin{array}{l} \mbox{if $n=2$, then $-\infty< s\leq -2$ or $2\leq s<\infty$;} \\ \mbox{if $n=3$, then $-\infty< s\leq -1$ or $1\leq s<\infty$;} \\ \mbox{if $n\geq 4$, then $-\infty< s<\infty$.} \end{array}$

Proof. For $d_1 = 2\cos\frac{\pi}{n}$ and $d_2 = 2\cos\frac{\pi(n-1)}{n}$ the trace s achieves the minimum 2π

(7)
$$s_{\min} = -2\cos\frac{2\pi}{n}$$

and for $d_1 = 2\cos\frac{\pi}{n}$ and $d_2 = 2\cos\frac{\pi}{n}$, the trace *s* achieves the maximum value

(8)
$$s_{\max} = 2\cos\frac{2\pi}{n},$$

by Lemma 5. The rest of the proof follows from (7), (8) and definition (1) of s.

Lemma 8 (see [4]). Let G be a group. An element $g \in K_2^m$ $(m \ge 2)$ if and only if there is an element $x \in K_2^{m-1}$, $x \ne g^{-1}$ such that $(gx)^2 = 1$.

Theorem 1. $PSL(2, \mathbb{R}) = K_n^2$, for $n \ge 4$.

Proof. Let

$$M_i = \begin{bmatrix} 0 & w_i \\ -w_i^{-1} & d_i \end{bmatrix}, \quad T_1 = \begin{bmatrix} 0 & 1 \\ -1 & d_i \end{bmatrix}, \quad T_2 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

and $d_i = 2\cos\frac{\pi j}{n}, i = 1, 2; (j, n) = 1$. From (1) for $x_2 = 0, y_1 = y_2 = 1$, it results that

(9)
$$s = -w_1w_2x_1^2 + (w_2d_1 - w_1d_2)x_1 - \frac{w_1}{w_2} - \frac{w_2}{w_1} + d_1d_2.$$

The function (9) in x_1 achieve the same minimum and maximum value as the function (1). For this reason, the trace $tr(M_1^{S_1}M_2^{S_2}) = s$ fulfills the condition of Lemma 7. The matrix

$$M_1^{T_1}M_2^{T_2} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \text{ where } b = -\frac{w_1}{w_2}x_1 + \frac{d_1}{w_2}, \ c = w_1(-d_2 - w_2x_1)$$

is similar in $GL(2,\mathbb{R})$ to the matrix

$$D = \begin{bmatrix} 0 & r \\ -r^{-1} & s \end{bmatrix}, \ s = a + d \text{ for any } r \neq 0$$

By Lemma 6, these matrices are similar in $SL(2,\mathbb{R})$ provided

(10)
$$rc \leq 0 \text{ or } rb \geq 0.$$

From Lemma 7, it results that the equation (9) is solvable in x_1 and

$$x_1' = \frac{w_2 d_1 - w_1 d_2 + \sqrt{\Delta}}{2w_1 w_2}, \ \ x_1'' = \frac{w_2 d_1 - w_1 d_2 - \sqrt{\Delta}}{2w_1 w_2},$$

where $\triangle = (w_2d_1 + w_1d_2)^2 - 4w_1^2 - 4w_2^2 - 4w_1w_2s$. If we put $x_1 = x_1'$, then

$$b = \frac{1}{2w_2^2} \left(w_2 d_1 + w_1 d_2 - \sqrt{\Delta} \right)$$
 and $c = -\frac{1}{2} \left(w_2 d_1 + w_1 d_2 + \sqrt{\Delta} \right).$

Note that $\triangle(-w_1, -w_2) = \triangle(w_1, w_2)$. Hence, if r > 0, then the signs of w_1 and w_2 can be chosen such that $w_2d_1 + w_1d_2 > 0$, thus cr < 0; if r < 0, then the signs of w_1 and w_2 can be chosen such that $w_2d_1 + w_1d_2 < 0$, thus br > 0. If $w_2d_1 + w_1d_2 = 0$, then c < 0 and b < 0, thus for r > 0, rc < 0 and for r < 0, rb > 0. Hence, condition (10) holds in all cases. Thus $M_1^{T_1} M_2^{T_2}$ and D are similar in $SL(2,\mathbb{R})$, by Lemma 6. Hence, matrices conjugate to D run over all non-scalar matrices of $PSL(2,\mathbb{R})$, by Lemma 1. Our set of matrices contains together with the matrix $L = \begin{bmatrix} 0 & r \\ -r^{-1} & d_i \end{bmatrix}$ also L^{-1} , so $E = LL^{-1} \in K_n^2$. Therefore, $K_n^2 = PSL(2, \mathbb{R})$.

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Theorem 2. a) $PSL(2, \mathbb{R}) = K_3^3$, b) $PSL(2, \mathbb{Q}) \neq K_3^3$, c) $PSL(2, F) \neq K_2^3$ for $F = \mathbb{Q}$ and \mathbb{R} .

Proof. Let $M_i = \begin{bmatrix} 0 & z \\ -z^{-1} & d_i \end{bmatrix}$, where $d_i = 2\cos\frac{\pi j}{n}, i = 1, 2; (j, n) = 1$ and M_i, T_i as in Lemma 3 or 4. If we take $r = x^2b + xy(d-a) - cy^2, x, y \in F$, then the matrices $M_1^{T_1}M_2^{T_2} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $D = \begin{bmatrix} 0 & r \\ -r^{-1} & s \end{bmatrix}$, s = a + d are similar in SL(2, F).

Consider the matrix

$$M_i D = \begin{bmatrix} 0 & z \\ -z^{-1} & d_i \end{bmatrix} \cdot \begin{bmatrix} 0 & r \\ -r^{-1} & s \end{bmatrix} = \begin{bmatrix} -zr^{-1} & zs \\ -d_ir^{-1} & -rz^{-1} + d_is \end{bmatrix}.$$

By Lemma 3 or 4 the trace $\operatorname{tr}(M_iD) = t$ runs over all of F, according to n = 2 or n = 3. The matrix M_iD is similar in the gerneral linear group GL(2,F) to the matrix $C = \begin{bmatrix} 0 & m \\ -m^{-1} & t \end{bmatrix}$. The similarity of M_iD and C in SL(2,F) is equivalent to the condition

(11)
$$x^2(t^2 - 4) + 4\frac{md_i}{r} \ge 0,$$

by Lemma 6.

Since $d_i = \pm 1$ for n = 3, it is possible to chose d_i and x such that the condition (11) holds in \mathbb{R} . Hence, by the Lemma 1, matrices conjugate to C run over all non-scalar matrices of PSL(2, F). By Lemma 7, K_3^2 contains the matrix $B = \begin{bmatrix} 0 & b \\ -b^{-1} & d_i \end{bmatrix} \in K_3$, where $d_i = 2\cos\frac{\pi j}{3}, (j,3) = 1$. The set K_3 together with B contains also B^{-1} . Hence $E = BB^{-1} \in K_3^3$. Therefore $PSL(2, \mathbb{R}) = K_3^3$.

If $F = \mathbb{Q}$, then the condition (11) cannot hold for t = 2 and for any arbitrary $m \in \mathbb{Q}$. Hence $PSL(2, \mathbb{R}) \neq K_3^3$.

If n = 2, then $d_i = 0$ and the condition (11) cannot hold for |t| < 2even for $F = \mathbb{R}$. Hence $PSL(2, F) \neq K_3^3$ for $F = \mathbb{Q}$, \mathbb{R} . The part b) of Theorem 2 follows.

The statement c) results from Lemma 8. Indeed, the set of non-scalar matrices of $K_2^2 \subset PSL(2,F)$ consist of matrices

(12)
$$X = \begin{bmatrix} 0 & x \\ -x^{-1} & 0 \end{bmatrix} \cdot \begin{bmatrix} y & z \\ -z^{-1}(1+y^2) & -y \end{bmatrix} \in K_2^2$$

and their conjugates. The conditions $(XG)^2 = E$, $G = \pm E$, $X \neq G$ are equivalent to

(13)
$$x^2(y^2+1) + z^2 = 0; \ x, y, z \neq 0,$$

which cannot be fulfilled over \mathbb{Q} and \mathbb{R} . Hence $E \notin K_2^3$, by Lemma 8.

Theorem 3. If $F = \mathbb{Q}$ or \mathbb{R} and $n = \infty$, then $SL(2, F) = K_n^2$ and $PSL(2, F) = K_n^2$.

Proof. Among matrices of order $n = \infty$ in PSL(2, F) there are matrices of the form

$$A_i = \begin{bmatrix} 0 & a_i \\ -a_i^{-1} & d_i + d_i^{-1} \end{bmatrix},$$

with distinct eigenvalues d_i, d_i^{-1} , where $d_i \neq 0$. Observe that $o(A_i) = o(-A_i) = o(A_i^{-1}) = \infty$ and $K_n^2 = \bigcup_{i,j} C_{A_i} C_{A_j}$. Lemma 2 implies that $K_n^2 \cup Z = SL(2, F)$ but $E \in C_{A_i} C_{A_i^{-1}}$ and $-E \in C_{A_i} C_{(-A_i)}$, so $K_n^2 = SL(2, F)$.

The equality $K_{\infty}^2 = PSL(2, F)$ can be proved similarly.

From Theorems 1, 2, 3, all proporties (i) – (v) follow immediately.

References

- J. Ambrosiewicz, On the property W for the multiplicative group of the quaternions algebra, Studia Univ. Babeş-Bolyai Math. 25 (1980), no. 2, p. 2–3.
- [2] J. Ambrosiewicz, The property W² for the multiplicative group of the quaternions field, Studia Univ. Babeş-Bolyai Math. 29 (1984), 63–67.
- [3] J. Ambrosiewicz, On the square of sets of the group SL(3, F), PSL(3, F), Demonstratio Math. **18** (1985), 963–968.
- [4] J. Ambrosiewicz, On square of sets of linear groups, Rend. Sem. Mat. Univ. Padova 75 (1986), 253–256.
- [5] J. Ambrosiewicz, Powers of sets in linear group, Demonstratio Math. 23 (1990), 395–403.
- [6] J. Ambrosiewicz, Square of set of elements of order two in orthogonal groups, Publ. Math. Debrecen 41 (1992), 189–198.
- [7] J. Ambrosiewicz, If K is a real field then cn(PSL(2, K)) = 4, Demonstratio Math. **29** (1996), 783–785.

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- [8] E.W. Ellers, Bireflectionality in classical groups, Canad. J. Math. 29 (1977), 1157–1162.
- [9] E.W. Ellers, R. Frank, and W. Nolte, Bireflectionality of the weak orthogonal and the weak sympletic groups, J. Algebra 88 (1984), 63–67.

Received 15 January 1997 Revised 12 July 1999 Revised 15 November 1999