# A FACTORIZATION OF ELEMENTS IN $P S L(2, F)$, WHERE $F=\mathbb{Q}, \mathbb{R}$ 

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#### Abstract

Let $G$ be a group and $K_{n}=\{g \in G: o(g)=n\}$. It is prowed: (i) if $F=\mathbb{R}, n \geq 4$, then $\operatorname{PSL}(2, F)=K_{n}^{2}$; (ii) if $F=\mathbb{Q}, \mathbb{R}, n=\infty$, then $\operatorname{PSL}(2, F)=K_{n}^{2}$; (iii) if $F=\mathbb{R}$, then $P S L(2, F)=K_{3}^{3}$; (iv) if $F=\mathbb{Q}, \mathbb{R}$, then $\operatorname{PSL}(2, F)=K_{2}^{3} \cup E, E \notin K_{2}^{3}$, where $E$ denotes the unit matrix; (v) if $F=\mathbb{Q}$, then $P S L(2, F) \neq K_{3}^{3}$. Keywords: factorization of linear groups, linear groups, matrix representations of groups, sets of elements of the same order in groups. 1991 Mathematics Subject Classification: 20G20, 11E57, 15A23, 20G15.


Let $G$ be a group and $K_{n}=K_{n}(G)=\{g \in G: o(g)=n\}$. Let $S L(m, F)$ and $P S L(m, F)$ be a special linear or projective specjal linear (resp.) groups of degree $m$ over a field $F$. Many papers have been devoted to the powers of the set $K_{2}$ (see [3] - [9]) but only few papers have been written about the powers of the set $K_{n}$ for $n>2$ (see [1] - [3]). In the papers [3] and [5], it has been proved that if $F$ is an algebraically closed field, then $P S L(3, F)=K_{n} K_{n}$ for $n>2$ and $P S L(3, F)=K_{2}^{4}$ for any $F$. Note that we do not identify $K_{2}$ with the set of involutions. In the paper [7], it has been proved that if $F=\mathbb{Q}, \mathbb{R}$, where $\mathbb{Q}$ denotes the field of rational numbers and $\mathbb{R}$ denotes the field of real numbers, then $P S L(2, F)=K_{n}^{4}$.

In this paper we will prove the following properties:
(i) if $F=\mathbb{R}, n \geq 4$, then $\operatorname{PSL}(2, F)=K_{n}^{2}$;
(ii) if $F=\mathbb{Q}, \mathbb{R}, n=\infty$, then $P S L(2, F)=K_{n}^{2}$;
(iii) if $F=\mathbb{R}$, then $\operatorname{PSL}(2, F)=K_{3}^{3}$;
(iv) if $F=\mathbb{Q}$ or $\mathbb{R}$, then $P S L(2, F)=K_{2}^{3} \cup E, E \notin K_{2}^{3}$, where E denotes the unit matrix;
(v) if $F=\mathbb{Q}$, then $P S L(2, F) \neq K_{3}^{3}$.

Recall, that $\operatorname{PSL}(2, \mathbb{C})=K_{n}^{2}$, where $\mathbb{C}$ denotes the field of complex numbers (see [2]).

We begin with some lemmas.
Lemma 1. Let $F$ be any field. In $S L(2, F)$, each non-scalar matrix is similar to a matrix of the form $\left[\begin{array}{cc}0 & r \\ -r^{-1} & s\end{array}\right]=D$. The order of $D$ depends only on $s$.

If $F=\mathbb{R}$, then
a) the order of the matrix $D \in S L(2, \mathbb{R})$ is $n>2$ iff $s=2 \cos \frac{2 k \pi}{n}$ and $(k, n)=1 ;$
b) the order of the matrix $D \in P S L(2, \mathbb{R})$ is $n>2$ iff $s=2 \cos \frac{k \pi}{n}$ and $(k, n)=1$ or $s=2 \cos \frac{2 k \pi}{n},(k, n)=1$.
If $F=\mathbb{Q}$ or $\mathbb{R}$ and $|s|>2$, then the order of $D$ is $\infty$.
Proof. If $F$ is any field, then for each $A=\left[\begin{array}{ll}a & b \\ c & d\end{array}\right] \in S L(2, F)$ there exists a matrix

$$
X=\left[\begin{array}{cc}
x & y \\
\frac{1}{r}(x a+c y) & \frac{1}{r}(b x+y d)
\end{array}\right] \text { and } D=\left[\begin{array}{cc}
0 & r \\
-r^{-1} & s
\end{array}\right]
$$

such that $A=X^{-1} D X$ and $s=a+d$. The condition $\operatorname{det} X=1$ holds since the equation $\frac{x}{r}(b x+y d)-\frac{y}{r}(x a+c y)=1$ has a solution in $r, x, y$.

If $F$ is any field, then we can find that

$$
D^{n}=\left[\begin{array}{cc}
\varphi_{n-2}(s) & r \psi_{n-1}(s) \\
-r^{-1} \psi_{n-1}(s) & \omega_{n}(s)
\end{array}\right]
$$

where $\varphi_{n-2}, \psi_{n-1}, \omega_{n}$ are polynomials in $s$ which means that the order of $D$ depends only on $s$.

In the case $F=\mathbb{R}$, it is easy to notice that the order of any matrix $A$ over $\mathbb{R}$ is the same as over $F=\mathbb{C}$. Thus if $-2<s<2$, we can put $s=2 \cos \varphi$, and then the matrix $D$ is similar to the diagonal matrix $\left[\begin{array}{cc}e^{i \varphi} & 0 \\ 0 & e^{-i \varphi}\end{array}\right]$ over $\mathbb{C}$. Hence, the rest of the proof follows from obvious properties of the group of the $n$-th roots of unity. If $|s|>2$, then the order of $D$ is $\infty$.

Lemma 2 (see [5]). If $V=\operatorname{diag}\left(v_{1}, \ldots, v_{m}\right), W=\operatorname{diag}\left(w_{1}, \ldots, w_{m}\right), v_{i} \neq v_{j}$, $w_{i} \neq w_{j}$ for $i \neq j$ and $V, W \in S L(m, F)$, then $S L(m, F)=C_{V} C_{W} \cup Z$, where $C_{V}$ denotes the conjugacy class of $V$ and $Z$ denotes the center of $S L(m, F)$.

Lemma 3. If

$$
N_{i}=\left[\begin{array}{cc}
0 & w_{i} \\
-w_{i}^{-1} & 0
\end{array}\right], \quad T_{i}=\left[\begin{array}{cc}
0 & 1 \\
-1 & x_{i}
\end{array}\right], \quad N_{i}, T_{i} \in S L(2, F),
$$

then the trace $\operatorname{tr}\left(N_{1}^{T_{1}} N_{2}^{T_{2}} N_{3}^{T_{3}}\right)=s$ is any arbitrary element of $F$, where $\left(N_{i}^{T i}=T_{i}^{-1} N_{i} T_{i}\right)$.

Proof. If we put $x_{1}=x_{2}=0$, then $s=-w_{3} w_{1}^{-1} w_{2}^{-1}\left(w_{1}^{2}+w_{2}^{2}\right) x_{3}$. Thus $s$ is directly proportional to $x_{3}$ and $s$ can be any arbitrary element of $F$.

Lemma 4. If

$$
M_{i}=\left[\begin{array}{cc}
0 & w_{i} \\
-w_{i}^{-1} & d_{i}
\end{array}\right](i=1,2,3), \quad T_{i}=\left[\begin{array}{cc}
0 & 1 \\
-1 & x_{i}
\end{array}\right], \quad d_{i} \neq 0,
$$

and $M_{i}, T_{i} \in S L(2, F)$, then there are $w_{i}$ such that the trace $\operatorname{tr}\left(M_{1} M_{2}^{T_{2}} M_{3}^{T_{3}}\right)=$ $s$ is any arbitrary element of $F$.

Proof. A calculation shows that if we take $w_{2}=-d_{2} d_{1}^{-1} w_{1}^{-1}, x_{3}=x_{2}+$ $d_{3} w_{3}^{-1}$ and $\left(w_{1} w_{3} d_{1}\right)^{2} \neq d_{2}^{2}$, then $s=x_{2}\left(d_{1} d_{2}^{-1} w_{3}-d_{1} d_{2}^{-1} w_{1}^{-1} w_{3}^{-1}\right)+d_{1} d_{3} d_{2}^{-1}$, so $s$ varies as a linear function of $x_{2}$.

Lemma 5. Let $M_{i}=\left[\begin{array}{cc}0 & w_{i} \\ -w_{i}^{-1} & d_{i}\end{array}\right], S_{i}=\left[\begin{array}{cc}0 & y_{i} \\ -y_{i}^{-1} & x_{i}\end{array}\right]$, over $\mathbb{R}$. Then

$$
\begin{align*}
& s=\operatorname{tr}\left(M_{1}^{S_{1}} M_{2}^{S_{2}}\right)=-w_{1} w_{2}\left(\frac{x_{1} y_{1}-x_{2} y_{2}}{y_{2} y_{1}}\right)^{2}+  \tag{1}\\
& \left(x_{1} y_{1}-x_{2} y_{2}\right)\left(\frac{d_{1} w_{2}}{y_{2}^{2}}-\frac{w_{1} d_{2}}{y_{1}^{2}}\right)-\left(\frac{w_{2}}{w_{1}}\right)\left(\frac{y_{1}}{y_{2}}\right)^{2}-\left(\frac{w_{1}}{w_{2}}\right)\left(\frac{y_{2}}{y_{1}}\right)+d_{1} d_{2} .
\end{align*}
$$

achieves the minimum

$$
s_{\min }=\frac{1}{2} \sqrt{\left(4-d_{1}^{2}\right)\left(4-d_{2}^{2}\right)}+\frac{1}{2} d_{1} d_{2}
$$

and the maximum value

$$
s_{\max }=-\frac{1}{2} \sqrt{\left(4-d_{1}^{2}\right)\left(4-d_{2}^{2}\right)}+\frac{1}{2} d_{1} d_{2}
$$

for $w_{1} w_{2}<0$ and $w_{1} w_{2}>0$, respectively.
Proof. If we consider the trace $s$ as a function of $x_{1}, x_{2}$, then the condition

$$
\begin{equation*}
\frac{\partial s}{\partial x_{1}}=\frac{\partial s}{\partial x_{2}}=0 \tag{2}
\end{equation*}
$$

is equivalent to the condition

$$
\begin{equation*}
2\left(x_{1} y_{1}-x_{2} y_{2}\right)=\frac{d_{1}}{d_{2}} y_{1}^{2}-\frac{d_{2}}{d_{1}} y_{2}^{2} \tag{3}
\end{equation*}
$$

Since $\quad \frac{\partial^{2} s}{\partial x_{1}^{2}}=-\frac{2 w_{1} w_{2}}{y_{2}^{2}}, \quad \frac{\partial^{2} s}{\partial x_{2}^{2}}=-\frac{2 w_{1} w_{2}}{y_{1}^{2}}, \quad \frac{\partial^{2} s}{\partial x_{1} \partial x_{2}}=-\frac{2 w_{1} w_{2}}{y_{1} y_{2}}, \quad$ therefore

$$
\begin{equation*}
s\left(x_{1}+h, x_{2}+k\right)-s\left(x_{1}, x_{2}\right)=-\frac{2 w_{1} w_{2}}{y_{1}^{2} y_{2}^{2}}\left(y_{1}-y_{2} k\right)^{2} \tag{4}
\end{equation*}
$$

Hence, $s\left(x_{1}, x_{2}\right)$ achieves the minimum and the maximum value for $w_{1} w_{2}<0$ and $w_{1} w_{2}>0$, respectively. The value of the trace $s$, at the surface (3) equals

$$
\begin{equation*}
\frac{1}{4} x\left(d_{2}^{2}-4\right)+\frac{1}{4}\left(d_{1}^{2}-4\right) \frac{1}{x}+\frac{1}{2} d_{1} d_{2} \tag{5}
\end{equation*}
$$

where $x=\frac{w_{1}}{w_{2}}\left(\frac{y_{1}}{y_{2}}\right)^{2}$.
The function (5) in $x$ and, as a result, also s achieves the minimum $s_{\text {min }}$ and the maximum $s_{\max }$ value for

$$
\frac{w_{1}}{w_{2}}=-\left(\frac{y_{1}}{y_{2}}\right)^{2} \sqrt{\frac{d_{1}^{2}-4}{d_{2}^{2}-4}} \text { and } \frac{w_{1}}{w_{2}}=\left(\frac{y_{1}}{y_{2}}\right)^{2} \sqrt{\frac{d_{1}^{2}-4}{d_{2}^{2}-4}}
$$

respectively.
Lemma 6. If $F=\mathbb{R}$, then the non-scalar matrix $A=\left[\begin{array}{ll}a & b \\ c & d\end{array}\right] \in S L(2, F)$ and $D=\left[\begin{array}{cc}0 & r \\ -r^{-1} & s\end{array}\right]$, are similar in $S L(2, F)$ provided $s=a+d, b r \geq 0$ or $-c r \geq 0$.

Proof. We have $X A X^{-1}=D$, where

$$
X=\left[\begin{array}{cc}
x & y \\
\frac{1}{r}(a x+c y) & \frac{1}{r}(b x+y d)
\end{array}\right], \operatorname{det} X \neq 0 .
$$

The condition $X \in S L(2, F)$ is equivalent to the solvability of the equation

$$
\begin{equation*}
b x^{2}+x y(d-a)-c y^{2}-r=0 \text { in } x \text { or } y . \tag{6}
\end{equation*}
$$

The descriminant $\Delta=y^{2}\left(s^{2}-4\right)+4 b r$ or $\Delta=x^{2}\left(s^{2}-4\right)-4 c r$, respectively, must be a non negative element of $F$.

By the assumption $b r \geq 0$ or $-c r \geq 0$, we can chose so small $y$ or $x$ such that $\Delta \geq 0$ for any $a, d \in \mathbb{R}$.

Lemma 7. Let $s=\operatorname{tr}\left(M_{1}^{S_{1}} M_{2}^{S_{2}}\right)$ be defined by (1) and let $n$ be the order of $M_{i}$. Then:

$$
\begin{aligned}
& \text { if } n=2 \text {, then }-\infty<s \leq-2 \text { or } 2 \leq s<\infty \text {; } \\
& \text { if } n=3 \text {, then }-\infty<s \leq-1 \text { or } 1 \leq s<\infty \text {; } \\
& \text { if } n \geq 4 \text {, then }-\infty<s<\infty \text {. }
\end{aligned}
$$

Proof. For $d_{1}=2 \cos \frac{\pi}{n}$ and $d_{2}=2 \cos \frac{\pi(n-1)}{n}$ the trace $s$ achieves the minimum

$$
\begin{equation*}
s_{\min }=-2 \cos \frac{2 \pi}{n} \tag{7}
\end{equation*}
$$

and for $d_{1}=2 \cos \frac{\pi}{n}$ and $d_{2}=2 \cos \frac{\pi}{n}$, the trace $s$ achieves the maximum value

$$
\begin{equation*}
s_{\max }=2 \cos \frac{2 \pi}{n} \tag{8}
\end{equation*}
$$

by Lemma 5 . The rest of the proof follows from (7), (8) and definition (1) of $s$.

Lemma 8 (see [4]). Let $G$ be a group. An element $g \in K_{2}^{m}(m \geq 2)$ if and only if there is an element $x \in K_{2}^{m-1}, x \neq g^{-1}$ such that $(g x)^{2}=1$.

Theorem 1. $\operatorname{PSL}(2, \mathbb{R})=K_{n}^{2}$, for $n \geq 4$.
Proof. Let

$$
M_{i}=\left[\begin{array}{cc}
0 & w_{i} \\
-w_{i}^{-1} & d_{i}
\end{array}\right], \quad T_{1}=\left[\begin{array}{cc}
0 & 1 \\
-1 & d_{i}
\end{array}\right], \quad T_{2}=\left[\begin{array}{cc}
0 & 1 \\
-1 & 0
\end{array}\right]
$$

and $d_{i}=2 \cos \frac{\pi j}{n}, i=1,2 ;(j, n)=1$. From (1) for $x_{2}=0, y_{1}=y_{2}=1$, it results that

$$
\begin{equation*}
s=-w_{1} w_{2} x_{1}^{2}+\left(w_{2} d_{1}-w_{1} d_{2}\right) x_{1}-\frac{w_{1}}{w_{2}}-\frac{w_{2}}{w_{1}}+d_{1} d_{2} \tag{9}
\end{equation*}
$$

The function (9) in $x_{1}$ achieve the same minimum and maximum value as the function (1). For this reason, the trace $\operatorname{tr}\left(M_{1}^{S_{1}} M_{2}^{S_{2}}\right)=s$ fulfills the condition of Lemma 7. The matrix

$$
M_{1}^{T_{1}} M_{2}^{T_{2}}=\left[\begin{array}{ll}
a & b \\
c & d
\end{array}\right], \quad \text { where } b=-\frac{w_{1}}{w_{2}} x_{1}+\frac{d_{1}}{w_{2}}, \quad c=w_{1}\left(-d_{2}-w_{2} x_{1}\right)
$$

is similar in $G L(2, \mathbb{R})$ to the matrix

$$
D=\left[\begin{array}{cc}
0 & r \\
-r^{-1} & s
\end{array}\right], s=a+d \text { for any } r \neq 0
$$

By Lemma 6, these matrices are similar in $S L(2, \mathbb{R})$ provided

$$
\begin{equation*}
r c \leq 0 \text { or } r b \geq 0 \tag{10}
\end{equation*}
$$

From Lemma 7, it results that the equation (9) is solvable in $x_{1}$ and

$$
x_{1}^{\prime}=\frac{w_{2} d_{1}-w_{1} d_{2}+\sqrt{\triangle}}{2 w_{1} w_{2}}, x_{1}^{\prime \prime}=\frac{w_{2} d_{1}-w_{1} d_{2}-\sqrt{\triangle}}{2 w_{1} w_{2}}
$$

where $\triangle=\left(w_{2} d_{1}+w_{1} d_{2}\right)^{2}-4 w_{1}^{2}-4 w_{2}^{2}-4 w_{1} w_{2} s$.
If we put $x_{1}=x_{1}^{\prime}$, then

$$
b=\frac{1}{2 w_{2}^{2}}\left(w_{2} d_{1}+w_{1} d_{2}-\sqrt{\triangle}\right) \text { and } c=-\frac{1}{2}\left(w_{2} d_{1}+w_{1} d_{2}+\sqrt{\triangle}\right)
$$

Note that $\triangle\left(-w_{1},-w_{2}\right)=\triangle\left(w_{1}, w_{2}\right)$. Hence, if $r>0$, then the signs of $w_{1}$ and $w_{2}$ can be chosen such that $w_{2} d_{1}+w_{1} d_{2}>0$, thus $c r<0$; if $r<0$, then the signs of $w_{1}$ and $w_{2}$ can be chosen such that $w_{2} d_{1}+w_{1} d_{2}<0$, thus $b r>0$. If $w_{2} d_{1}+w_{1} d_{2}=0$, then $c<0$ and $b<0$, thus for $r>0, r c<0$ and for $r<0, r b>0$. Hence, condition (10) holds in all cases. Thus $M_{1}^{T_{1}} M_{2}^{T_{2}}$ and $D$ are similar in $S L(2, \mathbb{R})$, by Lemma 6 . Hence, matrices conjugate to $D$ run over all non-scalar matrices of $\operatorname{PSL}(2, \mathbb{R})$, by Lemma 1. Our set of matrices contains together with the matrix $L=\left[\begin{array}{cc}0 & r \\ -r^{-1} & d_{i}\end{array}\right]$ also $L^{-1}$, so $E=L L^{-1} \in K_{n}^{2}$. Therefore, $K_{n}^{2}=P S L(2, \mathbb{R})$.

Theorem 2. a) $\operatorname{PSL}(2, \mathbb{R})=K_{3}^{3}$,
b) $\operatorname{PSL}(2, \mathbb{Q}) \neq K_{3}^{3}$,
c) $\operatorname{PSL}(2, F) \neq K_{2}^{3}$ for $F=\mathbb{Q}$ and $\mathbb{R}$.

Proof. Let $M_{i}=\left[\begin{array}{cc}0 & z \\ -z^{-1} & d_{i}\end{array}\right]$, where $d_{i}=2 \cos \frac{\pi j}{n}, i=1,2 ;(j, n)=1$ and $M_{i}, T_{i}$ as in Lemma 3 or 4 . If we take $r=x^{2} b+x y(d-a)-c y^{2}, x, y \in F$, then the matrices $M_{1}^{T_{1}} M_{2}^{T_{2}}=\left[\begin{array}{ll}a & b \\ c & d\end{array}\right]$ and $D=\left[\begin{array}{cc}0 & r \\ -r^{-1} & s\end{array}\right], s=a+d$ are similar in $S L(2, F)$.

Consider the matrix

$$
M_{i} D=\left[\begin{array}{cc}
0 & z \\
-z^{-1} & d_{i}
\end{array}\right] \cdot\left[\begin{array}{cc}
0 & r \\
-r^{-1} & s
\end{array}\right]=\left[\begin{array}{cc}
-z r^{-1} & z s \\
-d_{i} r^{-1} & -r z^{-1}+d_{i} s
\end{array}\right] .
$$

By Lemma 3 or 4 the trace $\operatorname{tr}\left(M_{i} D\right)=t$ runs over all of $F$, according to $n=2$ or $n=3$. The matrix $M_{i} D$ is similar in the gerneral linear group $G L(2, F)$ to the matrix $C=\left[\begin{array}{cc}0 & m \\ -m^{-1} & t\end{array}\right]$. The similarity of $M_{i} D$ and $C$ in $S L(2, F)$ is equivalent to the condition

$$
\begin{equation*}
x^{2}\left(t^{2}-4\right)+4 \frac{m d_{i}}{r} \geq 0 \tag{11}
\end{equation*}
$$

by Lemma 6 .
Since $d_{i}= \pm 1$ for $n=3$, it is possible to chose $d_{i}$ and $x$ such that the condition (11) holds in $\mathbb{R}$. Hence, by the Lemma 1, matrices conjugate to $C$ run over all non-scalar matrices of $P S L(2, F)$. By Lemma 7, $K_{3}^{2}$ contains the matrix $B=\left[\begin{array}{cc}0 & b \\ -b^{-1} & d_{i}\end{array}\right] \in K_{3}$, where $\mathrm{d}_{i}=2 \cos \frac{\pi j}{3},(j, 3)=1$. The set $K_{3}$ together with $B$ contains also $B^{-1}$. Hence $E=B B^{-1} \in K_{3}^{3}$. Therefore $\operatorname{PSL}(2, \mathbb{R})=K_{3}^{3}$.

If $F=\mathbb{Q}$, then the condition (11) cannot hold for $t=2$ and for any arbitrary $m \in \mathbb{Q}$. Hence $\operatorname{PSL}(2, \mathbb{R}) \neq K_{3}^{3}$.

If $n=2$, then $d_{i}=0$ and the condition (11) cannot hold for $|t|<2$ even for $F=\mathbb{R}$. Hence $\operatorname{PSL}(2, F) \neq K_{3}^{3}$ for $F=\mathbb{Q}, \mathbb{R}$. The part b) of Theorem 2 follows.

The statement c) results from Lemma 8. Indeed, the set of non-scalar matrices of $K_{2}^{2} \subset \operatorname{PSL}(2, F)$ consist of matrices

$$
X=\left[\begin{array}{cc}
0 & x  \tag{12}\\
-x^{-1} & 0
\end{array}\right] \cdot\left[\begin{array}{cc}
y & z \\
-z^{-1}\left(1+y^{2}\right) & -y
\end{array}\right] \in K_{2}^{2}
$$

and their conjugates. The conditions $(X G)^{2}=E, G= \pm E, X \neq G$ are equivalent to

$$
\begin{equation*}
x^{2}\left(y^{2}+1\right)+z^{2}=0 ; \quad x, y, z \neq 0 \tag{13}
\end{equation*}
$$

which cannot be fulfilled over $\mathbb{Q}$ and $\mathbb{R}$. Hence $E \notin K_{2}^{3}$, by Lemma 8 .
Theorem 3. If $F=\mathbb{Q}$ or $\mathbb{R}$ and $n=\infty$, then $S L(2, F)=K_{n}^{2}$ and $\operatorname{PSL}(2, F)=K_{n}^{2}$.

Proof. Among matrices of order $n=\infty$ in $\operatorname{PSL}(2, F)$ there are matrices of the form

$$
A_{i}=\left[\begin{array}{cc}
0 & a_{i} \\
-a_{i}^{-1} & d_{i}+d_{i}^{-1}
\end{array}\right]
$$

with distinct eigenvalues $d_{i}, d_{i}^{-1}$, where $d_{i} \neq 0$. Observe that $o\left(A_{i}\right)=$ $o\left(-A_{i}\right)=o\left(A_{i}^{-1}\right)=\infty$ and $K_{n}^{2}=\bigcup_{i, j} C_{A_{i}} C_{A_{j}}$. Lemma 2 implies that $K_{n}^{2} \cup Z=$ $S L(2, F)$ but $E \in C_{A_{i}} C_{A_{i}^{-1}}$ and $-E \in C_{A_{i}} C_{\left(-A_{i}\right)}$, so $K_{n}^{2}=S L(2, F)$.

The equality $K_{\infty}^{2}=P S L(2, F)$ can be proved similarly.
From Theorems 1, 2, 3, all proporties (i) - (v) follow immediately.

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