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BOOLEAN MATRICES . . . NEITHER BOOLEAN NOR MATRICES

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Abstract

Boolean matrices, the incidence matrices of a graph, are known not to be the (universal) matrices of a Boolean algebra. Here, we also show that their usual composition cannot make them the matrices of any algebra. Yet, later on, we "show" that it can. This seeming paradox comes from the hidden intrusion of a widespread set-theoretical (mis) definition and notation and denies its harmlessness. A minor modification of this standard definition might fix it.

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1. Preliminaries

1.1. Introduction

When one compares a general mathematical notion with a particular one, one does not expect any difficulty in proving that the latter is a case of the former. On the contrary, the general notion of universal matrix (or generalized matrix of [5], p. 140) and the particular one of Boolean matrix show such a difficulty: the proof coexists with its disproof.

Of course, we are not going to present any mathematical inconsistency. We will merely exhibit a misuse of the general notion, a misuse that still shows some curious features. We will find that it occurs quite naturally and it is prompted by a very well-known definition of standard Set Theory. This might reopen an old problem of this theory.

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1.2. Boolean matrices

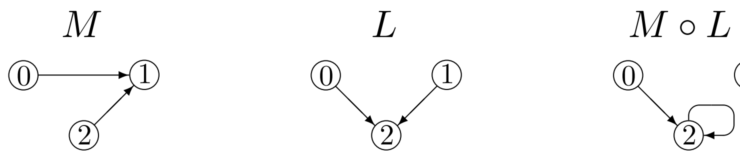
Recall that, given a finite set of *vertices*, $n = \{0, \ldots, n-1\}$ (see 11.1 of [9]), we can identify a simple (directed) graph in it by its *incidence matrix*, i.e. by an $n \times n$ array M with Boolean entries $0, 1 \in 2$. In fact, when $M_{i,j}$ denotes its entry at the i-th row and j-column, we can set $M_{i,j} = 1$ if there is an arc (or oriented edge) from i to j and $M_{i,j} = 0$ otherwise. Such a matrix merely is the characteristic function of the relation in n, corresponding to our graph.

Given M and another graph with matrix L, we know that the graph composition of the former with the latter has the matrix $M \circ L$ defined for all $i, j \in n$ by

$$(M \circ L)_{i,j} = \bigvee_{\ell \in n} M_{i,\ell} L_{\ell,j} \ . \tag{0}$$

Up to the replacement of the (iterated) Boolean join $\bigvee$ for the summation $\sum$, this is the same matrix product we perform in a finite dimensional vector space.

Example:



Furthermore, we know that the unit of our composition has the usual Kronecker matrix (Definition 1.4 will motivate why we are avoiding the term "identity matrix", usual in Linear Algebra)

$$b_{i,j} = \begin{cases} 1 & \text{when} \quad i = j \ , \\ 0 & \text{otherwise.} \end{cases}$$

(Therefore, for a fixed n, our matrices form a monoid with respect to this matrix product and this unit.) Hence, one often says that we are dealing with *"Boolean matrices"*, i.e. with square arrays with entries from the two-element Boolean algebra, as it was considered in [16] (see also [7]).

Yet, this does not entail that Boolean matrices are the "matrices" of a Boolean algebra (once one has introduced a formal notion of a matrix that is enough general for using it also on such algebras). Then, let us check it.

1.3. Remark (Generalized matrices)

The definition in the next subsection will recall the notion of a matrix we are going to use for such a check. It is not a widely known notion, owing to some historical reasons that we outline here. It is a case of the "generalized matrices" of [5], the earliest published formal notion that generalized the vector-space matrices by *algebraic* means.

At that time, this was a fairly unusual proposal. Some authors working in the field of applied Category Theory were trying to generalize matrices by *categorical* means, as one could see in I.3.6 of [8]. (See also the so-called "matricial theories", introduced by C.C. Elgot in [4], cf. [2] and [3].)

However, generalized matrices hardly were a readily acceptable subject even in Universal Algebra. A century ago, A.N. Whitehead implicitely stated the necessity of studying them when introduced his idea of universality in the introduction of a book [17], that actually was about Linear Algebra. Yet, in the eighty years elapsed from [17] to [5], the only relevant work came from the analyst and geometer K. Menger, who partly generalized the product of a matrix times a vector (details in 2.2 of [14]) by his "superassociative systems with selectors".

Perhaps, some technical reason (together with a possible disregard of vector spaces as we will explain in the next subsection) was underlying such a reluctance. In fact, the subsequent theoretical developments employed tools outside the conventional ones of Universal Algebra and of Category Theory (details in 2.5 of [14]).

We are not going to use such developments, but for a characterization, that however we will recall in subsection 1.6. Hence, if a reader is interested also to see how generalized matrices preserve most of the matrix properties known from Linear Algebra, he/she should refer to [14] (for linear properties), to [10] (for non linear properties) and to [13] (for other types of Cayley-Hamilton properties). (As the journal of [10] retyped the paper without Author's approval, there are many misprints. Most of them are corrected in [11]. Anyway, the Author will send interested people his paper.)

1.4. Definitions

We consider a fixed (universal) algebra and we denote by $\mathcal{E}$ the set of all its endomorphisms. Hence, when A denotes the carrier of this algebra,

$$\mathcal{E} \subseteq A^A \ .$$

We also consider another set X (often a finite one: $X = n = \{0, 1, \ldots, n-1\}$) and the set A^X of the families of algebra elements with index set X. Let b: $X \to A$ be one of such families and consider the function $\boldsymbol{r}_b$: $\mathcal{E} \to A^X$ that provides each endomorphism h: $A \to A$ with the family $\boldsymbol{r}_b(h) = h{\cdot}b : X \to A$ of the endomorphic images $h(b_x)$ of the elements of b. If we have got a bijection

(1) $$\boldsymbol{r}_b : \mathcal{E} \rightarrowtail\!\!\twoheadrightarrow A^X \ ,$$

then we say that A^X is the set of *(universal square) matrices* of the algebra, that we are considering, with respect to the *base* b. Hence, a matrix is any family M: $X \to A$ and we can represent it by some (and single) endomorphism, $M = \boldsymbol{r}_b(h)$. In such a case, we also say that the image $\circ$: $A^X \times A^X \to A^X$ under $\boldsymbol{r}_b$ of the functional composition on $\mathcal{E} \subseteq A^A$ is the *matrix product*, namely we formally define $\circ$ by

$$\boldsymbol{r}_b(k) \circ \boldsymbol{r}_b(h) = \boldsymbol{r}_b(k \cdot h) \ \text{ for all } h, k \in \mathcal{E} \ .$$

Trivially its unit is the matrix b (corresponding to the identical endomorphism).

For a familiar example, take any usual finite-dimensional vector space (defined on a finite power K^n of a field K) and consider its endomorphisms. Choose as base elements $b_x = b(x)$ (for $x \in X$) the ones forming the Kronecker matrix, then their endomorphic images $h(b_x)$ are the column vectors of the matrix identifying any endomorphism by its system of transition equations, while the above product turns out to be the familiar one “rows times columns”.

Notice also that our definitions work in a vector space even when one chooses another base, though with a different matrix product. (Hence, the “identity” matrix can differ from the Kronecker matrix.) They also do even when one isomorphically replaces the very carrier (consider polar representations). This invariance agrees with the “generalized conception of space” stressed by A.N. Whitehead in [17] and hints that universal matrices were within the reach of anybody (who *focused on* vector spaces) in spite of the eighty years before [5].

The matrices of a universal algebra provide several heterogeneous objects rising from the applications with a mathematically uniform and precise formalization [12], [15]. Hence, one would guess that our very natural “Boolean matrices” are a case of universal matrices.

1.5. Remark (Not Boolean)

In spite of their name, Boolean matrices (i.e. graphs) are known not to be the matrices of (free) Boolean algebras. One might easily get it by merely considering the cardinalities of the carrier sets involved: when n is the number of generators (base elements) such a Boolean algebra has 2^{2^n} elements, whereas the carrier of a possible algebra for graphs has the 2^n binary vectors that can be a column of an incidence matrix. Anyway, one knows that the latter matrices consist of other networks, the synchronous sequential autonomous circuits (or the families of Karnaugh maps or Veitch diagrams that define them), e.g. see 3.1 and 3.2 in [10]. Hence, one would expect that other algebras, possibly simpler ones, are the “universal spaces” for our simple graphs.

Unfortunately, the definition in subsection 1.2 of a Boolean matrix does not mention any algebra to check wheter it fulfil the definition in 1.4 of a universal matrix. We can only guess that $X = n$ and that the algebra elements are Boolean n-tuples, $A = 2^n$. This merely enables us to think of our L and M in formula (0) as the above families in A^X, once we consider them as functions, $L, M\colon n \to 2^n$, instead of $L, M\colon n \times n \to 2$. (Also, we can write $b(x)_j = b_{x,j}$.)

Hence, in order to avoid to guess algebras by trial and error, we have to resort to some characterization of universal matrices that does not involve any algebra (while it will provide us with some in the affirmative case). In our case (of tentative set of matrices, matrix product and unit) a simple characterization is the one that we recall below from [14]. Its underlying idea is the usual one of relating matrix products with units by certain axioms. (Yet, now the axioms will not be equational.)

1.6. Characterization of universal matrices

Let $\boldsymbol{k}\colon A \to A^X$ denote the (constant generating) function defined by

$$\boldsymbol{k}_a(\ell) = a \text{ for all } a \in A \text{ and } \ell \in X \ . \tag{2}$$

Recalled theorem. (See 1.7 (M3) in [14].) *For arbitrary sets X and A, A^X is a set of universal matrices as in subsection 1.3 iff there is a binary operation $\circ\colon A^X \times A^X \to A^X$, such that, for some family $b\colon X \to A$,*

$$M \circ \boldsymbol{k}_{b(x)} = \boldsymbol{k}_{M(x)} \ , \tag{3}$$

$$b \circ \boldsymbol{k}_a = \boldsymbol{k}_a \text{ and} \tag{4}$$

$$(M \circ L) \circ \boldsymbol{k}_a = M \circ (L \circ \boldsymbol{k}_a) \ , \tag{5}$$

for all $x \in X$, $L, M\colon X \to A$ and all constants $\boldsymbol{k}_a\colon X \to A$.

As it was proved in 1.7 of [14], when these three conditions are fulfilled, we can always define an algebra by an (algebraically) unusual construction starting from $\circ$. This algebra satisfies the requirements of subsection 1.4, namely A^X is the set of its matrices and $\circ$ is its matrix product, that has b as unit. For Boolean matrices, when $\circ$ is the one defined in (0) and b is the Kronecker matrix, such an algebra will be the one that we were looking for in subsection 1.5.

(The above mentioned construction is unusual, because it too involves function $\boldsymbol{k}$, as our three conditions do. This function is a set-theoretical counterpart of a non trivial, yet very simple, *combinator,* i.e. a functional operator in the sense of Combinatory Logic [6]. Though $\boldsymbol{k}$ does not play a big rôle in (conventional) Universal Algebra, its corresponding combinator is one of the just two basic combinators that allows [6] to build *all possible* functional operators.)

We only mention this combinatory construction, because we are not going to use it. Yet, we stress that it cannot miss a (free) algebra, if any, for our matrices. (On the contrary, as shown in 2.4 of [14], the seemingly more general approach of Category Theory in [8] resulted in a categorical construction with a narrow extent: it is unable to go from the very category **Mat** in [1] of usual matrices to their vector spaces!) Therefore, this characterization cannot fail to answer our starting question about the matricial nature of "Boolean matrices".

1.7. Remark (Analytic monoids)

The three conditions of the previous characterization are related with the (old) problem of generalizing matrices, recalled in subsection 1.3. We might wonder whether they also relate to well-known structures of Universal Algebra or, at least, we can reword them in such a way. It is not so. We are going to show that they do have an exact algebraic rewording, that still does not relates to them.

One can well consider these three conditions as the axioms defining certain mathematical structures. This is what [14] does in 2.1, where it calls them *analytic monoids* and shows that they can serve to define all abstract monoids (up to an isomorphism, $A^1 \simeq A$), see also [15]. Hence, from the algebraic point of view abstract and analytic monoids are equivalent. Yet, such an equivalence does not hold from a different point of view: the one where we care of the index (or "dimension") set X, as done by (3). In fact, not all abstract monoids, on a carrier structured as a power A^X, are analytic.

Consider the (modulo 4) addition on $4 = \{0,1,2,3\}$, that trivially determines a monoid. Then, consider the "binary-lexicographic" bijection $j\colon 4 \to 2^2$, such that $j(2i+k) = \{\langle 0,i\rangle, \langle 1,k\rangle\}$, for all $i,k = 0,1$. The images $\circ$ and b under j respectively of the addition and of 0 define a monoid on $A^X = 2^2$, that fails to satisfy (3). In fact, this axiom fails whenever it occurs that $M = j(1), j(2)$ and $x = 0,1$. (The constant families of the right hand side of (2) can only be either $j(0)$ or $j(3)$, not $j(1)$ or $j(2)$.)

This counterexample shows that analytic monoids are mathematical structures stronger than abstract monoids under the latter point of view. (Still, [15] and 2.1 in [14] also show that in a sense the former monoids are *more elementary* than the latter.) This strength difference and the above equivalence rule out any possibility of reconciling the two corresponding points of view. One might says that analytic monoids neither are monoids nor universal algebras: one would call them *monoids intertwined with dimensions.*

Outside Universal Algebra, on the contrary, our three axioms are related with well-known mathematical structures. 2.2 in [14] shows that each axiom is equivalent to one of the axioms for Menger's systems mentioned in subsection 1.3, once one considers a proper functional equation, that restricts the range of the possible products $\circ$. Interestingly, such a functional equation (together with the rule that converts the axioms) involves our combinator $\boldsymbol{k}$ again, as in subsection 1.6.

2. The "paradox"

2.1. A disproof

Let us check whether the seeming matrix product $\circ$ for Boolean matrices, given by (0), satisfies the above characterization of universal matrices. Take a matrix M such that $M_{j,j} = 0$ and $\bigvee_{\ell\in n} M_{i,\ell} = 1$ for some $i,j \in X = n$. Replace the tentative matrix product of (0) in (3). Then, by (2) we get

$$(M\circ\boldsymbol{k}_{b(x)})_{i,j} = \bigvee_{\ell\in n} M_{i,\ell}(\boldsymbol{k}_{b(x)})_{\ell,j} = \bigvee_{\ell\in n} M_{i,\ell}b(x)_j = b_{x,j}\bigvee_{\ell\in n} M_{i,\ell} = 1 \text{ for } x = j\,,$$

since b is the Kronecker matrix. On the other hand:

$$(\boldsymbol{k}_{M(x)})_{i,j} = (\boldsymbol{k}_{M(x)}(i))_j = M_{x,j} = 0 \text{ for } x = j\ .$$

Hence, (3), the first of our axioms, fails. Our very natural graph composition does not form an analytic monoid. There is not a "universal space" for graphs.

2.2. A "proof"

The case of our graph composition in subsection 1.2 – on the contrary – looks to be a safe bet as far as our expectation in subsection 1.5 is concerned, e.g. because it is very close to the one of the "algebraic theory" of relations in I.3.5 of [8]. In short, define an algebra on our $A = 2^n$ by the complete Boolean join. Recall that one can do it by the null n-tuple $\mathbf{0}$, considered as a nullary operation, and by the binary join $\vee\colon A \times A \to A$, where

$$(f \vee g)_i = \begin{cases} 0 & \text{when } f_i = g_i = 0\ , \\ 1 & \text{otherwise,} \end{cases}$$

for all n-tuples $f, g\colon n \to 2$. Then, directly check (1). Namely, check that the set $\mathcal{E}$ of all endomorphisms of this algebra and our Kronecker matrix b satisfy the bijection requirements of $\boldsymbol{r}_b : \mathcal{E} \rightarrowtail\!\!\!\twoheadrightarrow A^n$.

To get the injectivity, consider any $h, k \in \mathcal{E}$ and all $a \in A$. If $\boldsymbol{r}_b(h) = \boldsymbol{r}_b(k)$, i.e. if $h(b_i) = k(b_i)$ for all $i \in n$, then $h(a) = k(a)$, i.e. $h = k$. In fact, when $\bigvee$ also denotes the iterations of this join $\vee$, we can write $a = \bigvee_{i\in I} b_i$, where $I = \{i \mid a_i = 1\}$ and get it by the homomorphic condition as in a vector space, $h(a) = h(\bigvee_{i\in I} b_i) = \bigvee_{i\in I} h(b_i) = \bigvee_{i\in I} k(b_i) = k(a)$.

To get the ontoness on A^n, let us take any $M\colon n \to A$ and define a function $\eta_M\colon A \to A$ by

$$\eta_M(a) = \bigvee_{i\in I} M_i \tag{6}$$

for all $a \in A$ with the above I. Then, for the null n-tuple $a = \mathbf{0}$ we clearly get $\eta_M\,(\mathbf{0}) = \mathbf{0}$ and, for a join $a = a' \vee a''$, $\eta_M(a) = \eta_M(a') \vee \eta_M(a'')$, namely we defined an $\eta\colon A^X \to \mathcal{E}$. Furthermore, $\boldsymbol{r}_b(\eta_M) = M$ for all $M\colon n \to A$, because, for all $i \in X = n$, we get $(\boldsymbol{r}_b(\eta_M))_i = (\eta_M \cdot b)_i = \eta_M(b_i) = \bigvee_{j\in J_i} M_j = M_i$, where $J_i = \{j \mid b_{i,j} = 1\}$. (From this ontoness proof, we also get that the two-sided inverse of $\boldsymbol{r}_b$ is η.)

Our complete Boolean join turns out to be the "universal space" for graphs, contrary to subsection 2.1. The reader, wishing to check what happened by him/herself, might stop reading here.

2.3. What happened

As a first check, let us find which is the matrix product $\diamond$ that has to follow from the definition of universal matrices in subsection 1.4 after the "proof" in subsection 2.2. Given $h, k \in \mathcal{E}$, we set $L = h \cdot b$ and $M = k \cdot b$ and get $M \diamond L = (k \cdot h) \cdot b = k \cdot L$. Hence, since k is a homomorphism, for all $i, j \in n$, $(M \diamond L)_{i,j} = (k(L_i))_j = (k(\bigvee_{\ell\in J_i} b_\ell))_j = (\bigvee_{\ell\in J_i} k(b_\ell))_j = (\bigvee_{\ell\in J_i} M_\ell)_j =$

$\bigvee_{\ell\in n} M_{\ell,j}L_{i,\ell}$, where now $J_i = \{\ell \mid L_{i,\ell} = 1\}$. Therefore, we found that $M \diamond L \neq M \circ L$, since from (0)

$$M \circ L = L \diamond M \ . \tag{7}$$

This might spare the characterization in subsection 1.6 at the expense of the definition in subsection 1.4. In fact, this universal definition looks unable to grasp the very elementary notion of a graph (or relational) composition in subsection 1.2 since it yields its (silly) converse. Hence, we could even have got something worst than a wrong characterization: a wrong definition. We need a further check.

Let us check our Boolean matrices against our universal ones. Since we can rewrite (6) as $\eta_M(a) = \bigvee_{j\in n} a_j M_j$, the endomorphism η_M is given by the system of equations $a'_i = (\eta_M(a))_i = \bigvee_{j\in n} a_j M_{j,i}$ for $i \in n$. This is the system $a'_i = \bigvee_{j\in n} a_j M^\top_{i,j}$ for $i \in n$, when $M^\top$ denotes the transposed of M. Hence, the universal matrices for the algebra in subsection 2.2 are the transposed of the Boolean ones in subsection 1.2 and transposition explains (7), because $M \circ L = (L^\top \circ M^\top)^\top$. When in (0) we took our (wrong) $\circ$, we were thinking about wrong universal matrices. In a sense, $\circ$ was not wrong: it was mending a preceding misunderstanding.

(Besides, consider any $k \in \mathcal{E}$, i.e. an η_M as in (6) for some $M\colon n \to A$. We can think of any its argument $a\colon n \to 2$ as the characteristic function of a subset of vertices, $s \subseteq n$, viz. $a(i) = 1$ iff $i \in s$. The corresponding set s', given by $a' = \eta_M(a)\colon n \to 2$, by (6) has the property that $j \in s'$ iff $M_{i,j} = 1$ for some $i \in s$, viz. *s' is the graph image* of s. Now, given another graph with an L as in subsection 1.2, s' goes onto an s'' corresponding to $\eta_L(a')$. Hence, our graph composition of subsection 1.2 sends s to an s'' corresponding to $\eta_L(\eta_M(a)) = \eta_{(L\diamond M)}(a)$. In (7) the sensible graph composition is the very $L \diamond M$. It is routine to check that $\diamond$ satisfies the three axioms (3), (4) and (5) of the theorem in subsection 1.6.)

2.4. Why it happened

When one handles graphs in the usual way, one cannot easily perceive the two misdefinitions (wrong matrices and wrong $\circ$) in subsection 1.2, because they annul each other. Our $\circ$ practically coincides with relational composition, which is a very elementary set-theoretical notion. On the contrary, the definition of a matrix product in subsection 1.4 intrudes another notion between the two ill-defined ones: the one of functional composition (from the endomorphism monoid). The contradiction between subsection 2.1 and subsection 2.2 stems from this intrusion.

Functional composition can cause such troubles, because standard Set Theory (see 4.7 in [9]) defines and denotes it as the converse of (a restriction of) relational composition. Such a reversal allows one to preserve the order while one writes $(f \cdot g)(x) = f(g(x))$ and mends the historical (mis) notation of functional application, where one writes the application argument at the right.

2.5. A proposal

One could well avoid such troubles (as well as possible other ones) by avoiding the above reversal, that looks illogical even without any paradox: one either merely flips the notation for functional application or gives up the above order preservation. Unfortunately, this way turned out to be linguistically naive. Several authors (e.g. [8]) in the fields of Algebra and Category Theory did it, yet it did not spread. Logical propriety did not win. Perhaps, the standard definition was considered harmless.

The present "paradox" shows that something has to be done. A modest trick: keep the standard notation and avoid any logical reversal. When we merely write "$f \cdot g$ denotes the *composition of* g *and* f", we still see functional composition as a restriction of the relational one (a motivation for stating in subsection 1.4 that $M \circ L$ is the matrix product *of* L *and* M). We might well extend this trick to relational composition and denote it by $\cdot$ again.

This minor mending should work with mathematically conscious readers. (With a wider readership, one should only manage to bypass the problem.) In such a case, the virus, born several centuries ago with the standard notation for application, will be kept dormant (at least for a while, of course).

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