# EQUIVALENT CONDITIONS FOR P-NILPOTENCE 

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#### Abstract

In the first part of this paper we prove without using the transfer or characters the equivalence of some conditions, each of which would imply $p$-nilpotence of a finite group $G$. The implication of $p$-nilpotence also can be deduced without the transfer or characters if the group is $p$-constrained. For $p$-constrained groups we also prove an equivalent condition so that $\mathrm{O}^{q^{\prime}}(G) P$ should be $p$-nilpotent. We show an example that this result is not true for some non- $p$-constrained groups.

In the second part of the paper we prove a generalization of a theorem of Itô with the help of the knowledge of the irreducible characters of the minimal non-nilpotent groups.


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## 1 Definitions and known results

We know the remarkable theorems of Frobenius which tell that in Theorem 2.1 (i) and (iv) both imply that the finite group $G$ has a normal $p$-complement. All existing proofs of them use the transfer homomorhism or characters.

The well-studied minimal non-nilpotent groups, i.e. non-nilpotent groups, each of whose subgroups are nilpotent, sometimes are called Schmidt groups or $(p, q)$-groups. They can be described without using the transfer, see 5.1 Satz and 5.2 Satz in pp. 280-281 of [6]. Let $G$ be a minimal non-nilpotent group. Then it can be proved without using the transfer or characters, that

1. $G$ is solvable.
2. $|G|$ is divisible only by 2 primes, say $|G|=p^{a} q^{b}$.
3. $G^{\prime}=P \in \operatorname{Syl}_{p}(G)$.
4. If $p>2$, then $\exp (P)=p$; if $p=2$, then $\exp (P) \leq 4 ; \exp (\mathrm{Z}(P))=p$ in all cases.
5. $P$ is either abelian or $P^{\prime}=\Phi(P)=\mathrm{Z}(P) \leq \mathrm{Z}(G)$.
6. If $Q \in \operatorname{Syl}_{q}(G)$, then $Q$ is cyclic; and if $Q=\langle x\rangle$, then $\left\langle x^{q}\right\rangle \leq \mathrm{Z}(G)$.
7. If $P$ is abelian, then $Q$ acts irreducibly on $P$; if $P$ is nonabelian, then $Q$ acts irreducibly on $P / \mathrm{Z}(P)$. If $P$ is abelian, then $P$ is of exponent $p$; and if $P$ is nonabelian, then $P / \mathrm{Z}(P)$ is also of exponent $p$. So, they can be considered as vector spaces over $\operatorname{GF}(p)$. Their dimension is $\mathrm{o}(p)(\bmod (q))$, which is even in the nonabelian case.
8. $G$ is generated by its Sylow $q$-subgroups.

These groups are non- $p$-nilpotent. It can also be proved using the transfer or characters that a group is $p$-nilpotent if and only if, it does not contain such a subgroup.
We shall use the following:
Notation 11. We shall write $(p, q) \not \leq G$ if the group $G$ does not contain a $(p, q)$-group, otherwise we write $(p, q) \leq G$.

Let us recall the definition of Thompson-ordering:
Definition 12. Let $G$ be a finite group. Let $\mathcal{P}$ be a property of subgroups of $G$. Let $\mathcal{A}=\left\{A \mid p\right.$-subgroup of $G, \mathrm{~N}_{G}(A)<G, A p$-group, $\mathrm{N}_{G}(A)$ has the property $\mathcal{P}\}$. We tell for $A_{1}, A_{2} \in \mathcal{A}$ that $A_{1}$ is smaller than $A_{2}$ in the Thompson-ordering if either $\left|\mathrm{N}_{G}\left(A_{1}\right)\right|_{p}<\left|\mathrm{N}_{G}\left(A_{2}\right)\right|_{p}$, or $\left|\mathrm{N}_{G}\left(A_{1}\right)\right|_{p}=$ $\left.\left|\mathrm{N}_{G}\left(A_{2}\right)\right|\right|_{p}$ and $\left|A_{1}\right|<\left|A_{2}\right|$.

Definition 13. Let $P \in \operatorname{Syl}_{p}(G)$. A subgroup $U \leq P$ is called strongly closed if for every $u \in U$ if $u^{x} \in P$ then $u^{x} \in U$.

## 2 Main results

The aim of this paper is to prove that the equivalence of the following four conditions can be proved without the use of transfer or characters:

Theorem 21. Let $G$ be a finite group, let $P \in \operatorname{Syl}_{p}(G)$. Then the following are equivalent:
(i) If $x, y \in P$ are $G$-conjugate, then they are conjugate already in $P$;
(ii) If $x, y \in P$ of order $p$ or 4 and are $G$-conjugate, then they are conjugate already in $P$;
(iii) $(p, q) \not \leq G$ for every prime $q \neq p$;
(iv) For every p-subgroup $U \leq G, \mathrm{~N}_{G}(U) / \mathrm{C}_{G}(U)$ is a p-group.

As a corollary we get:
Theorem 22. Let $G$ be a p-constrained group, $P \in \operatorname{Syl}_{p}(G)$. If any of the conditions of the above theorem holds for $G$, then one can deduce without using the transfer that $G$ has a normal p-complement.
As an application of Theorem 2.1 we prove also the following:
Theorem 23. Let $G$ be a finite group, let $p \neq q$ be primes with $p, q \in \pi(G)$. Let $P \in \operatorname{Syl}_{p}(G)$ and let $\mathrm{O}^{q^{\prime}}(G)$ denote the subgroup of $G$ generated by the $q$-elements of $G$. If $G$ is $p$-constrained then the following are equivalent:
(i) $(p, q) \not \leq G$.
(ii) $\mathrm{O}^{q^{\prime}}(G) P$ has a normal $p$-complement.

Another application of Theorem 2.2 is to prove without using the transfer the following generalization of a theorem of Itô:

Theorem 24. Let $G$ be a finite $p$-constrained group, let $P \in \operatorname{Syl}_{p}(G)$. Let us suppose that $\chi$ is a character of $G$ satisfying the following conditions:
a) $\chi(1) \leq 2 p-2$,
$\beta$ ) for every subgroup $H \leq G, \chi_{H}$ does not have a constituent of degree $p$,
ү) $\operatorname{Ker}(\chi)=1$.
Then one of the following two possibilities holds:
(i) $P$ is abelian and $P$ is normal in $G$;
(ii) $p$ is a Fermat-prime and one of the constituents of $\chi$ has degree at least $p-1$.
The inequality in $\alpha$ ) is sharp. There is a solvable group $G_{0}$ having a character $\chi_{0} \in \operatorname{Char}\left(G_{0}\right)$ satisfying $\beta$ ) and $\gamma$ ) with degree $\chi_{0}(1)=2 p-1$, such that for this pair the assertion of the Theorem does not hold.

## 3 Preliminary lemmas

In the proof of Theorem 2.1 we will need the following lemma, which is Lemma 2 in [4].
Lemma 31. Let $G$ be a group with $H \in \operatorname{Hall}_{\pi}(G)$ and with the property that every $\pi$-subgroup $Y$ of $G$ can be conjugated into $H$. Let $\mathcal{K}$ be a class of elements of $H$, which is closed under conjugation inside $H$ with elements of $G$ such that if two elements of $\mathcal{K}$ are conjugate in $G$ then they are already conjugate in $H$. Then if $G_{1} \triangleleft G$ and $\left|G: G_{1}\right|=q$, where $q \in \pi$, then for $H_{1}=H \cap G_{1}$ it holds that each pair of elements of $H_{1} \cap \mathcal{K}$ that are conjugate in $G_{1}$ are already conjugate in $H_{1}$.

For the proof of Theorem 2.1 we will also need the following lemma, which generalizes both Lemma 5 in [4] and Lemma 3.2 in [3].

Lemma 32. Let $q \in \pi(G) \backslash\{p\}$ be a fixed prime, $P \in \operatorname{Syl}_{p}(G), U<P$ abelian and strongly closed in $P$. Then if $(p, q) \not \leq \mathrm{N}_{G}(U)$ then $(p, q) \not \leq G$, as well.

Proof. Let $G$ be a counterexample of minimal order.
First we prove that we may assume that $\mathrm{O}_{p}(G)=1$.
Let $\mathrm{O}_{p}(G)>1$ and let $B \triangleleft G$ be a $p$-subgroup. Let $\bar{G}=G / B$, and the images of $U$ and $P$ in this factor group let $\bar{U}$ and $\bar{P}$, respectively. Then $\bar{P} \in \operatorname{Syl}_{p}(\bar{G})$ and the triple $(\bar{G}, \bar{P}, \bar{U})$ satisfies the conditions set for $(G, P, U)$. To see this we have to show only that $(p, q) \nsubseteq \mathrm{N}_{G}(U)$ implies $(p, q) \nsubseteq \mathrm{N}_{\bar{G}}(\bar{U})$. Let $M$ be the inverse image of $\mathrm{N}_{\bar{G}}(\bar{U})$ in $G$. Here $M<G$, since if $\bar{U} \triangleleft \bar{G}$ then $U^{G} \leq P$, and as $U \leq P$ is strongly closed, $U^{G}=U$ would follow. This would imply $(p, q) \not \leq \mathrm{N}_{G}(U)=G$, which is a contradiction. So $M<G$. But $P \leq M$ and $\mathrm{N}_{G}(U)=\mathrm{N}_{M}(U)$. The triple $(M, P, U)$ satisfies the conditions of the Lemma. By induction $(p, q) \not \leq M$. By [2], $(p, q) \not \leq \bar{M}=\mathrm{N}_{\bar{G}}(\bar{U})$, as well. Hence the conditions of the Lemma are satisfied by the triple $(\bar{G}, \bar{P}, \bar{U})$ and by induction $(p, q) \not \subset \bar{G}$.

Let $V$ be a $(p, q)$-group in $G$. Then $V^{\prime}=V_{p} \in \operatorname{Syl}_{p}(V)$ and by the above result, its image $\bar{V}$ in $\bar{G}$ is nilpotent. Hence $V_{p} \leq B$. There are two cases:
(i) $U \cap \mathrm{O}_{p}(G)=1$,
(ii) $U \cap \mathrm{O}_{p}(G) \neq 1$.

Ad (i): We know that $U \triangleleft P$ and we may assume that $V_{p} \leq P$, by replacing $V$ with a suitable conjugate of it. Hence $\left[U, V_{p}\right] \leq U$. On the other hand as $V_{p} \leq \mathrm{O}_{p}(G),\left[U, V_{p}\right] \leq \mathrm{O}_{p}(G) \cap U=1$.
Ad (ii): If $U \cap \mathrm{O}_{p}(G) \neq 1$, then we may choose $B$ equal to it, because $U \cap \mathrm{O}_{p}(G)$ is normal in $G$ as $U$ is strongly closed. Hence $V_{p} \leq B \leq U$ by the above results. But $U$ is abelian, so $\left[V_{p}, U\right]=1$ in this case, too.
Hence in both cases (i) and (ii) $\left[V_{p}, U\right]=1$. Let $N=\mathrm{N}_{G}\left(V_{p}\right)$. We claim that $N=G$. If $N<G$ then if we choose $S \in \operatorname{Syl}_{p}(N)$ with the property $U \leq S$, then the triple ( $N, S, U$ ) satisfies the conditions of the Lemma. So by induction $(p, q) \not \leq N$. This contradicts the fact that $V \leq N$. Thus $V_{p}$ is normal in $G$. Let $C=\mathrm{C}_{G}\left(V_{p}\right)$. Then $C \triangleleft G$. Let us choose $Q \in \operatorname{Syl}_{q}(G)$ so that it should contain a Sylow $q$-subgroup of $V$. Set $L=C Q$. Then $P \cap L=P \cap C \in \operatorname{Syl}_{p}(L)$ and $P \cap C \triangleleft P$. Then the triple ( $L, P \cap L, U$ ) satisfies the conditions of the Lemma. Hence if $L<G$ then by induction $(p, q) \not \leq L$, which is impossible as $V \leq L$. Thus $G=L=C Q$. Since $P=P \cap L=P \cap C, P \leq C$ and as $C \triangleleft G$, so by the Frattini argument we have that $G=C \mathrm{~N}_{G}(P)$. As $U$ is a strongly closed subgroup of $P$, $\mathrm{N}_{G}(P) \leq \mathrm{N}_{G}(U)$ and thus $(p, q) \not \leq \mathrm{N}_{G}(P)$. Since $V_{p} \leq \mathrm{C}_{G}(P) \leq \mathrm{N}_{G}(P)$ and $V_{p} \triangleleft G$, hence $V_{p} \triangleleft \mathrm{~N}_{G}(P)$. Thus $\left|\mathrm{N}_{G}(P): C \cap \mathrm{~N}_{G}(P)\right| \not \equiv 0(q)$. But then, since $\left|\mathrm{N}_{G}(P): C \cap \mathrm{~N}_{G}(P)\right|=\left|C \mathrm{~N}_{G}(P): C\right|=|G: C|=|C Q: C|=|Q: C \cap Q|$, $Q \leq C=G$ follows. This is a contradiction, since $V \leq G=\mathrm{C}_{G}\left(V_{p}\right)$.
End of the proof: Let $\mathcal{A}=\left\{A \mid \mathrm{N}_{G}(A)<G, A\right.$ p-group, $\left.(p, q) \leq \mathrm{N}_{G}(A)\right\}$. Let $K$ be a maximal element of $\mathcal{A}$ for the Thompson-ordering. Then $\left|\mathrm{N}_{G}(K)\right|_{p}$ is maximal and among those with this property $K$ is also of maximal order. As $\mathcal{A} \neq \emptyset$, so such $K$ exists. Then $K \leq P_{1}$ for a suitable Sylow $p$-subgroup $P_{1}$ of $G$. Let $x \in G$ such that $P_{1}^{x}=P$. Then $K^{x} \leq P$. Let $\mathrm{Z}\left(P_{1}\right) \leq R \in \operatorname{Syl}_{p}\left(\mathrm{~N}_{G}(K)\right)$. Then $\mathrm{Z}(P) \leq R^{x} \in \operatorname{Syl}_{p}\left(\mathrm{~N}_{G}\left(K^{x}\right)\right)$. Let $R^{x} \leq P_{2} \in \operatorname{Syl}_{p}(G)$. Choose $t \in G$ so that $P_{2}^{t}=P$. Thus $R^{x t} \leq P$. Since $U \triangleleft P, \mathrm{Z}(P) \cap U \neq 1$, thus $R^{x} \cap U \neq 1$, as well. As $U$ is stongly closed in $P$, $\left(R^{x} \cap U\right)^{t} \leq R^{x t} \cap U$ and $R^{x t} \cap U$ is strongly closed in $R^{x t}$. It is enough to prove that the triple ( $\mathrm{N}_{G}\left(K^{x t}\right), R^{x t}, R^{x t} \cap U$ ) satisfies the conditions of the Lemma. When we prove this, then $(p, q) \not \approx \mathrm{N}_{G}\left(K^{x t}\right)$ follows, contradicting our assumption. It is enough to prove that $(p, q) \nsucceq \mathrm{N}_{G}\left(R^{x t} \cap U\right)$. If $|R|=|P|$ then we have that the triple $\left(\mathrm{N}_{G}\left(K^{x t}\right), P, U\right)$ satisfies the conditions of the Lemma, and since $\mathrm{N}_{G}\left(K^{x t}\right)<G$, by induction we get that $(p, q) \not \subset \mathrm{N}_{G}\left(K^{x t}\right)$, contradicting the choice of $K$. Thus $|R|<|P|$. Then $\mathrm{N}_{P}\left(R^{x t}\right)>R^{x t}$, and since $R^{x t} \cap U$ is strongly closed in $R^{x t}, \mathrm{~N}_{G}\left(R^{x t}\right) \leq \mathrm{N}_{G}\left(R^{x t} \cap U\right)$.

So $\left|\mathrm{N}_{G}\left(R^{x t} \cap U\right)\right|_{p}>\left|\mathrm{N}_{G}\left(K^{x t}\right)\right|_{p}=|R|$. As $\mathrm{N}_{G}\left(R^{x t} \cap U\right) \neq G$, thus $(p, q) \not \leq \mathrm{N}_{G}\left(R^{x t} \cap U\right)$, by the maximality of $K$ in the Thompson ordering. The proof is complete.

For the proof of Theorem 2.4 we will need the description of irreducible characters of minimal non-nilpotent groups. As we did not find any reference to it in the literature, for the sake of selfcontainedness we include it here.

Lemma 33. Let $G$ be a $(p, q)$-group, $P \in \operatorname{Syl}_{p}(G), Q \in \operatorname{Syl}_{q}(G),|Q|=q^{n}$. Then $G$ has exactly $q^{n}$ linear characters.
(i) If $P$ is abelian, then all other characters in $\operatorname{Irr}(G)$ are of degree $q$. They are induced from nontrivial characters of the unique index $q$ subgroup of $G$. There are $(|P|-1) q^{n-2}$ such characters.
(ii) If $P$ is extraspecial, then $|P|=p^{2 m+1}$, where $2 m \equiv o(p)(\bmod (q))$. $P / \mathrm{Z}(P) Q$ is a $(p, q)$-group of type (i). So it has $\left(p^{2 m}-1\right) q^{n-2}$ irreducible characters of degree $q$. The $p-1$ irreducible characters of degree $p^{m}$ of $P$ can be extended to $G$ giving $(p-1) q^{n}$ irreducible characters of degree $p^{m}$.
(iii) If $P$ is special and nonabelian, then if $|\mathrm{Z}(P)|=p^{k}$ then $\mathrm{Z}(P)$ has $\frac{p^{k}-1}{p-1}$ maximal subgroups. By factoring with one of them we get a $(p, q)$ group of type (ii). The union of inverse images of these characters give $\operatorname{Irr}(G)$.

Proof. As $G^{\prime}=P,\left|G: G^{\prime}\right|=q^{n}$, so $G$ has exactly $q^{n}$ linear characters. Let $H=P\left\langle x^{q}\right\rangle$. Then $|G: H|=q$ and $H$ is normal in $G$.
Ad (i): If $P$ is abelian, then so is $H$, so if $\chi \in \operatorname{Irr}(G)$ nonlinear, then $\chi_{H}=$ $\sigma_{1}+\ldots+\sigma_{q}$ and $\chi=\sigma_{i}^{G}$ for $i=1, \ldots, q$. So $\chi(1)=q$ and $\chi$ is induced from exactly $q$ linear characters of $H$. As $|G|=|P| q^{n}=$ $q^{n}+q^{2}(|P|-1) q^{n-2}$, we get that each nontrivial character of $H$ that does not contain $P$ in its kernel is induced to $\operatorname{Irr}(G)$.
Ad (ii): If $P$ is extraspecial, then $|P|=p^{2 m+1}$. As $Q$ acts irreducibly on $P / \mathrm{Z}(P)$, by Lemma 3.10 in Chapter II. of [6] we get that $2 m=$ $\mathrm{o}(p)(\bmod (q))$. The $p-1$ faithful irreducible characters of $P$ are of degree $p^{m}$, they are 0 outside $\mathrm{Z}(P)$, so they are $G$-invariant, and as $(|P|,|G: P|)=1$, they can be extended to $G$. By Gallagher's theorem, se e.g. [7], they can be extended in $q^{n}$ ways. This way we get $(p-1) q^{n}$ irreducible characters of $G$. By taking into consideration those of degree 1 and $q$ the sum of squares of the degrees gives:

$$
q^{n}+q^{2}\left(p^{2 m}-1\right) q^{n-2}+p^{2 m}(p-1) q^{n}=q^{n} p^{2 m+1}=|G|,
$$

so we determined all $\operatorname{Irr}(G)$.
Ad (iii): We calculate the sum of squares of the irreducible characters we produced so far: $q^{n}+q^{2}\left(p^{2 m}-1\right) q^{n-2}+p^{2 m} \frac{p^{k}-1}{p-1}(p-1) q^{n}=q^{n} p^{2 m+k}=$ $|G|$, so we produced the whole $\operatorname{Irr}(G)$.

## 4 Proofs of the main results

Now we prove Theorem 2.1:
Proof of Thorem 2.1. (i) $\rightarrow$ (ii) is trivial.
(ii) $\rightarrow$ (iii): We use induction on $|G|$. The following argument is similar to one in the last part of the proof of Theorem 1 in [4]. For the sake of selfcontainedness, we repeat it here.

Let $A \triangleleft P$ be an abelian normal subgroup in $P$ such that $\exp (A) \leq p$ if $p>2$, and $\exp (A) \leq 4$ if $p=2$, and $A$ is maximal with these properties.
(a) If $A \leq \mathrm{Z}(P)$ :
then according to Alperin's theorem $[1], \Omega_{j}(P) \leq \mathrm{Z}(P)$, where $j=1$ if $p>2$ and $j=2$ if $p=2$. Then $A$ is strongly closed in $P$, as if we take two elements $a$ and $a^{x}$ of order $p$ or of order 4 in $A$ and $P$, then they are conjugate in $P$, and as $A$ is normal in $P$, we get that $a^{x} \in A$, too. Let $N=\mathrm{N}_{G}(A)$. If $N<G$, then as $P \in \operatorname{Syl}_{p}(N)$, by induction we get that $(p, q) \not \leq N$, and by Lemma $3.2,(p, q) \nsubseteq G$. So we may assume that $\mathrm{N}_{G}(A)=G$. Then $P \leq \mathrm{C}_{G}(A) \triangleleft G$. As for each $a \in A^{\#}$ for the conjugacy class $\mathrm{K}_{G}(a)$ of $a$ in $G$ and for the conjugacy class $\mathrm{K}_{P}(a)$ in $P$ it holds that $\mathrm{K}_{G}(a)=\mathrm{K}_{G}(a) \cap P=\mathrm{K}_{P}(a)=a$, as $A \triangleleft G$ and $A \leq \mathrm{Z}(P)$, so $\left|G: \mathrm{C}_{G}(a)\right|=\left|P: \mathrm{C}_{P}(a)\right|=1$, and we get that $A \leq \mathrm{Z}(G)$. Thus, if $p>2$, then $\Omega_{1}(P) \leq A \leq \mathrm{Z}(G)$; if $p=2$, then $\Omega_{2}(P) \leq \mathrm{Z}(G)$; and this means that $G \nsupseteq(p, q)$ in cases $p>2$ and $p=2$, either, for every prime divisor $q \neq p$ of $G$.
(b) If $A \not \subset \mathrm{Z}(P)$ :
then $A \cap \mathrm{Z}(P)<A$. Thus $A / A \cap \mathrm{Z}(P)$ contains a central subgroup of $P / A \cap \mathrm{Z}(P)$ of order $p$. Let $A_{1}$ be its inverse image in $A$. According to our assumption, $A_{1}=\langle A \cap \mathrm{Z}(P), x\rangle$, where $o(x)=p$ or $o(x)=4$ and $x \notin \mathrm{Z}(P) . A_{1}$ is strongly closed in $P$ as if $a_{1} \in A_{1}$ and $a_{1}^{u} \in P$, then by assumption $a_{1}^{u}$ is conjugate to $a_{1}$ in $P$. But as $A_{1} \triangleleft P, a_{1}^{u} \in A_{1}$. If
$N_{1}=\mathrm{N}_{G}\left(A_{1}\right)<G$, then as $P \leq N_{1}$, by induction we have that $(p, q) \not \leq N_{1}$. Then, by Lemma $3.2,(p, q) \not \leq G$. So, we may assume that $N_{1}=G$. Then $\mathrm{C}_{G}\left(A_{1}\right) \triangleleft G$ and $A \cap \mathrm{Z}(P) \leq \mathrm{Z}(G)$, as if $a \in A \cap \mathrm{Z}(P)$ and $g \in G$ then $a^{g} \in A_{1} \leq P$, so by assumption there is a $u \in P$ such that $a^{g}=a^{u}$ and as $a \in A \cap \mathrm{Z}(P) a^{u}=a$. So, $\left|G: \mathrm{C}_{G}(a)\right|=1$ and thus $A \cap \mathrm{Z}(P) \leq \mathrm{Z}(G)$ and $\mathrm{C}_{G}\left(A_{1}\right)=\mathrm{C}_{G}(x)$. Let $g \in G$, then $x^{g} \in A_{1} \leq P$, so there exists an $u \in P$ such that $x^{g}=x^{u}$, therefore the conjugacy classes $\mathrm{K}_{G}(x)$ and $\mathrm{K}_{P}(x)$ coincide and thus $\left|G: \mathrm{C}_{G}\left(A_{1}\right)\right|=\left|G: \mathrm{C}_{G}(x)\right|=\left|P: \mathrm{C}_{P}(x)\right|>1$. So, $\mathrm{C}_{G}\left(A_{1}\right)$ is a proper normal subgroup of $p$-power index in $G$ and it is contained in some normal subgroup $G_{1}$ of index $p$. By Lemma 3.1, applied for $H=P \in \operatorname{Syl}_{p}(G)$ if we take $\mathcal{K}$ to be the set of elements of $P$ of order $p$ or 4 , then induction gives that $(p, q) \not \leq G_{1}$ for every prime $q \neq p$. As a $(p, q)$-group is generated by its Sylow $q$-subgroups, $(p, q) \not \leq G$, either.
(iii) $\rightarrow$ (iv): Let $T$ be a $p$-subgroup of $G$. Let $N=\mathrm{N}_{G}(T)$. Let $q \in$ $\pi(N) \backslash\{p\}, Q \in \operatorname{Syl}_{q}(N)$. If $[Q, T] \neq 1$, then $(p, q) \leq Q T$, which cannot happen by assumption. Hence (iv) follows.
(iv) $\rightarrow$ (i): The proof is similar to the second part of Lemma 5 in [4]. For the sake of selfcontainedness we repeat it here.

Let $a, b \in P$ such that $a=b^{x}$ for some $x \in G$. By the thereom of Alperin, see e.g. Chapter 7, Theorem 2.6 in [5], there exist Sylow $p$-subgroups $Q_{1}, \ldots, Q_{n}$ of $G$, elements $x_{1}, \ldots, x_{n}$ with $x_{j} \in \mathrm{~N}_{G}\left(P \cap Q_{j}\right)$, and $y \in \mathrm{~N}_{G}(P)$ such that $b \in P \cap Q_{1}, b^{x_{1} \ldots x_{j-1}} \in P \cap Q_{j}, x=x_{1} \ldots x_{n} y$ and $\mathrm{N}_{P}\left(P \cap Q_{j}\right) \in \operatorname{Syl}_{p}\left(N_{G}\left(P \cap Q_{j}\right)\right)$ for $j=1, \ldots, n$. Let $N_{j}=\mathrm{N}_{G}\left(P \cap Q_{j}\right)$, $C_{j}=\mathrm{C}_{G}\left(P \cap Q_{j}\right), P_{j}=\mathrm{N}_{P}\left(P \cap Q_{j}\right), j=1, \ldots, n$.

So $N_{j}=C_{j}\left(P \cap N_{j}\right)$ and hence $x_{j}=y_{j} z_{j}$, where $y_{j} \in C_{j}$ and $z_{j} \in P \cap N_{j}$. It is easy to see that that $a=b^{x}=b^{z_{1} \ldots z_{n} y} . \mathrm{N}_{G}(P)=\mathrm{C}_{G}(P) P$, so $y=c z_{n+1}$, where $c \in \mathrm{C}_{G}(P), z_{n+1} \in P$. Thus $a=b^{x}=b^{z_{1} \ldots z_{n+1}}$, which means that $a$ and $b$ are conjugate in $P$. The proof is complete.
Now we prove Theorem 2.2:
Proof of Theorem 2.2. Let $H=\mathrm{O}_{p^{\prime}, p}(G), R=H \cap P$. As $G$ is $p$-constrained, $\mathrm{C}_{G}(R) \leq H$. By the Frattini argument, $G=\mathrm{O}_{p^{\prime}}(G) \mathrm{N}_{G}(R)$. Let $q \neq p$ prime, $Q \in \operatorname{Syl}_{q}\left(\mathrm{~N}_{G}(R)\right)$. Then $Q R=Q \times R$, as $(p, q) \not \leq Q R$. Hence $Q \leq \mathrm{C}_{G}(R) \leq H$, and so $Q \leq \mathrm{O}_{p^{\prime}}(G)$. Thus $G=\mathrm{O}_{p^{\prime}}(G) P$.

Now we prove Theorem 2.3:
Proof of Theorem 2.3. (i) $\rightarrow$ (ii): Repeating the argument of the previous proof one gets that $\mathrm{O}^{q^{\prime}}(G) \leq \mathrm{O}_{p^{\prime}}(G)$. As $\mathrm{O}^{q^{\prime}}(G) \triangleleft G$ and it is a $p^{\prime}$-group, so $\mathrm{O}^{q^{\prime}}(G) P$ is a subgroup of $G$ having normal $p$-complement.
(ii) $\rightarrow$ (i): If $\mathrm{O}^{q^{\prime}}(G) P$ has a normal $p$-complement, then $\mathrm{O}^{q^{\prime}}(G)$ is a $p^{\prime}$ subgroup. If $U$ is a $(p, q)$-group in $G$, then $U \leq \mathrm{O}^{q^{\prime}}(U) \leq \mathrm{O}^{q^{\prime}}(G)$. As $\mathrm{O}^{q^{\prime}}(G)$ is a $p^{\prime}$-subgroup, this is a contradiction. So, the proof is complete.
Remark 41. In Theorem 2.3 the condition that $G$ is $p$-constrained cannot be omitted. Take $G=A_{5}, P \in \operatorname{Syl}_{5}(G)$. Then $(5,3) \npreceq G$, since 25 does not divide $|G|$. As $G$ is simple, $G=\mathrm{O}^{3^{\prime}}(G)=\mathrm{O}^{3^{\prime}}(G) P$.

Now we prove Theorem 2.4:
Proof of Theorem 2.4. We use induction on $|G|$. We assume that $G$ is a finite $p$-constrained group of minimal order that has character $\chi \in \operatorname{Char}(G)$ satisfying the conditions in our Theorem such that the assertion is not yet known. We may assume that (ii) is false for $G$, and we have to prove that for $G$ (i) holds. As (ii) does not hold for $G$, so it cannot hold for any proper subgroup $H$ of $G$. So by induction (i) holds for all such $H$. Using $\alpha$ ) and $\beta$ ) we can deduce that every constituent of $\chi_{P}$ is linear, and thus $P^{\prime} \leq \operatorname{Ker}(\chi)$. By $\gamma$ ), $P$ is abelian. As $G$ is $p$-constrained, so $P^{\prime}=1$ implies that $G$ is $p$-solvable. By the choice of $G$ we get immidiately that $\pi(G)=\{p, q\}$ for a suitable prime $q \neq p$. This gives that $G$ is solvable, hence $G$ is also $q$-constrained.

We have to prove that $P \triangleleft G$. As $\pi(G)=\{p, q\}$, this means that $G$ is $q$-nilpotent. As $G$ is $q$-constrained, our Theorem 2.1 (iii) and Theorem 2.2 implies, (even without the use of the transfer), that either $P \triangleleft G$ or $G$ is a $(q, p)$-group. To finish the proof, it is enough to show that the second possibility cannot occur. Assume that $G$ is a $(q, p)$-group. Let $m$ be $\mathrm{o}(q)(\bmod (p))$. Using $\beta)$ an appeal to Lemma 33. shows that $\chi$ can have a nonlinear constituent only in the case when $m$ is even, say $m=2 a$. The degree of a nonlinear irreducible constituent of $\chi$ is then $q^{a}$. Since $q^{a}+1 \equiv 0$ $(\bmod (p)), q^{a}=p l-1$, for a suitable natural number $l$. From this one deduces that either $q^{a} \geq 2 p-1$ or $q^{a}=p-1$ and $q=2$. If $q^{a}=p-1, p=2$ cannot occur. As (ii) is not true in $G$, case $q^{a}=p-1$ cannot hold, either. So $\chi$ has only linear constituents. But then $G^{\prime} \leq \operatorname{Ker}(\chi)$, so by $\left.\gamma\right) G$ has to be abelian in this case, contradicting the assumption that $G$ is a ( $q, p$ )-group. This completes the proof.
Now we give an example showing that in $\alpha$ ) $2 p-2$ cannot be replaced by $2 p-1$.

Let $p$ and $q=2 p-1$ be primes, where $p \geq 7$. E.g. $p=7$ and $q=13$. Let $G_{0}$ be a ( $q, p$ )-group of order $q^{3} p$ with extraspecial Sylow $q$-subgroup. In this case $\mathrm{o}(q)(\bmod p)=2$, so such a group exists. Then $Q_{0}=G_{0}^{\prime} \in \operatorname{Syl}_{q}\left(G_{0}\right)$,
and $G_{0}$ has a character $\chi_{0}$ of degree $q$ which is irreducible and faithful. So $\alpha$ ) is not satisfied for $\chi_{0}$, as $\chi_{0}(1)=2 p-1$. Since $\chi_{0}$ is faithful $\gamma$ ) holds. As all proper subgroups of $G_{0}$, except for $Q_{0}$, are abelian, and $\chi_{Q_{0}}$ is irreducible, for $\chi_{0} \beta$ ) also holds. (i) is not true for $G_{0}$. But $p$ is not a Fermat prime either, as then $p=2^{2^{k}}+1$ would hold and $q=2 p-1=2^{2^{k}+1}+1 \equiv 0(\bmod (3))$ so (ii) cannot hold, either.

Remark 42. This theorem extends a well-known result of N. Itô ([8], see also [7]).

It can be deduced from our statement if we replace $2 p-2$ by $p-1$ in $\alpha$ ) and we assume also $\gamma$ ). Then $\beta$ ) is automatically satisfied. If $P$ is not normal in $G$, then $\chi(1)=p-1$ and $\chi \in \operatorname{Irr}(G)$ also holds.

On the other hand the assumption $\beta$ ) is vital for our proof as if case (i) holds, then, by a theorem of N . Itô (see e.g. [7]), $(\chi(1), p)=1$ holds for every irreducible character $\chi \in \operatorname{Irr}(G)$.

Remark 43. The conditions of our Theorem 2.4 however do not guarantee that if $P$ is not normal in $G$, then $\chi$ should be irreducible. Let us take $p=2^{2^{k}}+1$ to be a Fermat-prime. Let $G$ be a $(2, p)$-group of order $2^{2^{k+1}+1} p$ with extraspecial Sylow 2-subgroup and Sylow $p$-subgroup $P$ of order $p$. Then $G$ has a faithful irreducible character $\chi$ of degree $p-1$. Let us choose a character $\sigma \in \operatorname{Char}(G)$ with $p-1<\sigma(1) \leq 2 p-2$, and $(\sigma, \chi)=1$ and all other constituents of $\sigma$ are choosen to be linear. Then $\sigma$ satisfies $\alpha), \beta$ ) and $\gamma$ ), but $\sigma$ is not irreducible.

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