

ON UNIQUE FACTORIZATION SEMILATTICES

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Abstract

The class of unique factorization semilattices (UFSs) contains important examples of semilattices such as free semilattices and the semilattices of idempotents of free inverse monoids. Their structural properties allow an efficient study, among other things, of their principal ideals. A general construction of UFSs from arbitrary posets is presented and some categorical properties are derived. The problem of embedding arbitrary semilattices into UFSs is considered and complete characterizations are obtained for particular classes of semilattices. The study of the Munn semigroup for regular UFSs is developed and a complete characterization is accomplished with respect to being E -unitary.

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1 Preliminaries

The general terminology and notation are those of Howie [1]. We denote by \mathbf{N} the set $\{0, 1, 2, \dots\}$ of all nonnegative integers.

We define a *semilattice* to be a commutative semigroup consisting of idempotents. Given a semilattice E , the *natural partial order* on E is defined by

$$e \leq f \quad \Leftrightarrow \quad e = ef.$$

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For this partial order, the product ef is clearly the meet of the elements e and f . Hence we can view a semilattice as a poset where the meet of two elements is always defined. Conversely, given a poset (E, \leq) where the meet $e \wedge f$ of two elements $e, f \in E$ is always defined, we can view (E, \wedge) as a semilattice. It follows that every semilattice has a semigroup structure and an ordered structure, and each one of them is completely determined by the other. Obviously, the word “semilattice” refers originally to the ordered structure.

A semigroup S is said to be *inverse* if, for every $a \in S$, there exists a unique $b \in S$ such that

$$aba = a \quad \text{and} \quad bab = b.$$

Given an inverse semigroup S , the subset of all idempotents of S constitutes a semilattice $E(S)$, usually called the *semilattice of idempotents* of S .

Let E be a semilattice and $e \in E$. We say that e is *irreducible* if

$$e = fg \quad \Rightarrow \quad e = f \quad \text{or} \quad e = g$$

for all $f, g \in E$. We say that e is *prime* if

$$e \geq fg \quad \Rightarrow \quad e \geq f \quad \text{or} \quad e \geq g$$

for all $f, g \in E$. The subset of all irreducible elements of E is denoted by $\text{Irr}(E)$.

Lemma 11 ([2], Lemma 2.1). *Let E be a semilattice and $e \in E$ be prime. Then e is irreducible.* ■

The semilattice E is called a *unique factorization semilattice* (UFS) if:

- (i) E is generated by $\text{Irr}(E)$;
- (ii) Every irreducible of E is prime.

We say that $e = f_1 \dots f_n$ is a *reduced irreducible factorization* of e if $f_1, \dots, f_n \in \text{Irr}(E)$ and $f_i \not\geq f_j$ whenever $i \neq j$. If E is generated by $\text{Irr}(E)$, then every $e \in E$ admits a reduced irreducible factorization. The next result justifies the expression “unique factorization”.

Lemma 12 ([2], Lemma 2.2). *Let E be a UFS.*

- (i) *Let $e_1 \dots e_n = f_1 \dots f_m$ with all $e_i, f_j \in \text{Irr}(E)$. Then, for every $i \in \{1, \dots, n\}$, there exists $j \in \{1, \dots, m\}$ such that $e_i \geq f_j$.*
- (ii) *Every $e \in E$ admits a unique reduced irreducible factorization.* ■

Given a poset X and $A \subseteq X$, we say that A is an *ideal* of X if $a \in A$ and $x \leq a$ together imply $x \in A$. Dually, A is a *dual-ideal* of X if $a \in A$ and $x \geq a$ together imply $x \in A$. Given $Y \subseteq X$, we define

$$\begin{aligned} Y \downarrow &= \{x \in X \mid x \leq y \text{ for some } y \in Y\}, \\ Y \uparrow &= \{x \in X \mid x \geq y \text{ for some } y \in Y\}. \end{aligned}$$

Clearly, $Y \downarrow$ (respectively, $Y \uparrow$) is the smallest ideal (respectively, dual-ideal) of X containing Y . If $A = Y \downarrow$ for some finite subset Y of X , we say that A is a *finitely generated* ideal. If $A = x \downarrow$ for some $x \in X$, we say that A is a *principal* ideal. We have similar definitions for dual-ideals. If E is a semilattice and $e \in E$, note that

$$e \downarrow = Ee = \{fe \mid f \in E\}.$$

Clearly, Ee is a subsemilattice of E .

In the case of a UFS, the structure of principal ideals is particularly amenable.

Lemma 13 ([2], Lemma 2.3). *Let E be a UFS and $e \in E$.*

- (i) $\text{Irr}(Ee) = (\text{Irr}(E))e$.
- (ii) Ee is a UFS. ■

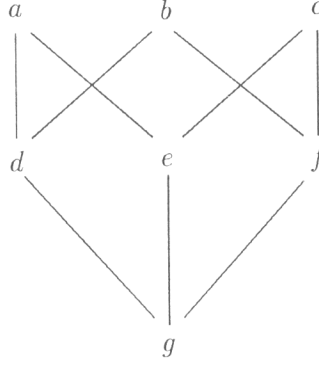
Principal dual-ideals are also subsemilattices and show a simple structure.

Lemma 14. *Let E be a UFS and $e \in E$.*

- (i) $\text{Irr}(e \uparrow) = \text{Irr}(E) \cap e \uparrow$.
- (ii) $e \uparrow$ is a UFS.

Proof. Straightforward. ■

In general, (finitely generated) dual-ideals are not subsemilattices, and (finitely generated) ideals need not be UFSs, as the next example shows.



Straightforward verification shows that the semilattice depicted above is a UFS. However, the ideal $\{d, e, f, g\}$ is not.

We shall consider several particular subclasses of both posets and semilattices throughout this work, and we must introduce further terminology and notation.

Let X be a poset. We say that X is *upper finite* if $x \uparrow$ is finite for every $x \in X$. Given $x, y \in X$, we write $x \succ y$ if $x > y$ and there is no $z \in X$ such that $x > z > y$. We define also

$$\hat{x} = \{y \in X \mid y \succ x\}$$

and

$$[x, y] = \{z \in X \mid x \leq z \leq y\}.$$

The poset X is said to be *connected* if, given $x, y \in X$, $[x, y]$ contains no infinite chain. Clearly, every upper finite poset is connected.

Next we introduce two important examples of UFSs.

The *free semilattice* on a nonempty set X may be described as the set $F(X)$ of all finite nonempty subsets of X , endowed with the union operation. It is common practice to identify the singleton subset $\{x\}$ of $F(X)$ with the element $x \in X$. Clearly, the natural partial order is described by

$$A \leq B \iff A \supseteq B$$

and $\text{Irr}(F(X))$ consists of all the singleton sets. Proving that $F(X)$ is a connected upper finite UFS is a simple exercise.

Let $FG(X)$ denote the free group on a set X , viewed as the set of all reduced words on the double alphabet $X \cup X^{-1}$. A subset $A \subseteq FG(X)$ is *prefix-closed*

if it contains all the prefixes of its elements. The semilattice of idempotents of the *free inverse monoid* on a set X may be described as the set $I(X)$ of all finite nonempty prefix-closed subsets of $FG(X)$, endowed with the union operation. The natural partial order is also described by $A \leq B \Leftrightarrow A \supseteq B$.

Lemma 15 ([2], Lemma 2.5). *Let X be a set.*

- (i) $A \in I(X)$ is irreducible if and only if A is the set of prefixes of a single word $u \in FG(X)$.
- (ii) $I(X)$ is a connected upper finite UFS. ■

A *semilattice presentation* is a formal expression of the form $\mathcal{S}\langle X \mid R \rangle$, where X is a nonempty set and $R \subseteq F(X) \times F(X)$. The semilattice defined by the above presentation is the quotient $F(X)/R^\sharp$, where R^\sharp denotes the congruence on $F(X)$ generated by the relation R . If a semilattice E is isomorphic to $F(X)/R^\sharp$, we say that $\mathcal{S}\langle X \mid R \rangle$ is a presentation of E .

2 Constructing UFSs out of posets

In this section we introduce a general construction of UFSs.

Let (X, \leq) be a poset, which we denote simply by X whenever possible. We write

$$R(X) = \{(x, Y) \in F(X) \times F(X) \mid x \in Y \subseteq x \uparrow\},$$

where $x \uparrow = \{y \in X \mid y \geq x\}$. Let $U(X)$ be the semilattice defined by the presentation $\mathcal{S}\langle X \mid R(X) \rangle$. Given $A, B \in F(X)$, we write $A \geq_X B$ if for every $a \in A$ there exists $b \in B$ such that $a \geq b$, and we write $A \rho_X B$ if $A \geq_X B$ and $B \geq_X A$.

Lemma 21. *Let X be a poset.*

- (i) $\rho_X = (R(X))^\sharp$.
- (ii) *The natural partial order of $U(X)$ is given by*

$$(A \rho_X) \leq (B \rho_X) \Leftrightarrow A \leq_X B.$$

Proof. (i) Clearly, ρ_X is a reflexive and symmetric relation on $F(X)$, and transitivity follows from the fact that \geq_X is itself transitive. Clearly,

$A \rho_X B$ implies $(A \cup C) \rho_X (B \cup C)$ for every $C \in F(X)$, hence ρ_X is a congruence on $F(X)$. The inclusion $R(X) \subseteq \rho_X$ is easily verified and thus $(R(X))^\sharp \subseteq \rho_X$.

To prove the converse inclusion, assume that $A \rho_X B$ with $A, B \in F(X)$. Let $\rho = (R(X))^\sharp$. We show that $A \rho (A \cup B)$. By symmetry, we get also $B \rho (A \cup B)$ and so $A \rho B$ as required.

Let

$$C = \cup_{a \in A} ((A \cup B) \cap (a \uparrow)) = (A \cup B) \cap (\cup_{a \in A} (a \uparrow)).$$

Since $a \rho ((A \cup B) \cap (a \uparrow))$ for every $a \in A$ and ρ is a congruence, we obtain $A \rho C$. Since $B \geq_X A$ by hypothesis, it follows that $B \subseteq \cup_{a \in A} (a \uparrow)$ and so $B \subseteq C$. Hence $(A \cup B) \rho (C \cup B) = C \rho A$.

(ii) We have

$$(A \rho_X) \leq (B \rho_X) \Leftrightarrow A \rho_X = (A \cup B) \rho_X \Leftrightarrow (A \leq_X (A \cup B) \wedge (A \cup B) \leq_X A).$$

Since the condition $(A \cup B) \leq_X A$ holds trivially and $A \leq_X (A \cup B)$ if and only if $A \leq_X B$, the result follows. \blacksquare

From now on, we shall omit the subscript X and even the congruence symbol ρ whenever possible.

Before presenting our next result, we note that, given a semilattice E , we can view $\text{Irr}(E)$ as a poset simply by considering the restriction of the natural partial order of E to $\text{Irr}(E)$.

Lemma 22. *Let X be a poset.*

$$(i) \text{ Irr}(U(X)) = \{x \rho \mid x \in X\}.$$

$$(ii) \text{ Irr}(U(X)) \simeq X \text{ as posets.}$$

$$(iii) U(X) \text{ is a UFS.}$$

Proof. (i) Let $A \in F(X)$ be such that $A \rho \in \text{Irr}(U(X))$. We can assume that the cardinal of A is least possible. Suppose that $|A| > 1$ and let $a \in A$. Then we can write $A \rho = (a \rho)((A - \{a\}) \rho)$ and $A \rho$ irreducible yields $A \rho = a \rho$ or $A \rho = (A - \{a\}) \rho$, in either case contradicting the minimality of A . Hence $|A| = 1$ and so $\text{Irr}(U(X)) \subseteq \{x \rho \mid x \in X\}$.

Now let $x \in X$. Assume that $x \rho \geq (A \rho)(B \rho) = (A \cup B) \rho$ for some $A, B \in F(X)$. We have $x \geq_X A \cup B$ by Lemma 21(ii) and so there exists some $y \in A \cup B$ such that $x \geq y$ in X . It follows that $x \geq_X A$ or $x \geq_X B$,

and so $x\rho \geq A\rho$ or $x\rho \geq B\rho$, by the same reference. Thus $x\rho$ is prime. By Lemma 11, we obtain $\{x\rho \mid x \in X\} \subseteq \text{Irr}(U(X))$, as required.

(ii) The mapping

$$\varphi : X \rightarrow \text{Irr}(U(X)), \quad x \mapsto x\rho$$

is well defined by part (i) and is surjective.

Let $x, y \in X$. By Lemma 21(ii), we have

$$x \geq y \Leftrightarrow x \geq_X y \Leftrightarrow x\rho \geq y\rho \Leftrightarrow x\varphi \geq y\varphi.$$

Since the relation on X is a partial order, this implies that φ is also injective and thus an isomorphism of posets.

(iii) In part (i), we have proved that every irreducible of $U(X)$ is prime, and it is immediate that these irreducibles generate $U(X)$. ■

In view of the preceding lemma, we proceed to identify X with $\text{Irr}(U(X))$.

Lemma 23. *If X is a poset, E a semilattice and $\varphi : X \rightarrow E$ an order-preserving map, then there exists a unique semilattice homomorphism $\Phi : U(X) \rightarrow E$ such that $\Phi|_X = \varphi$.*

Proof. Since $F(X)$ is the free semilattice on X , there is a unique semilattice homomorphism $\bar{\varphi} : F(X) \rightarrow E$ such that $\bar{\varphi}|_X = \varphi$. Let $x \in Y$ where Y is a finite nonempty subset of $x\uparrow$. We have

$$Y\bar{\varphi} = \left(\bigcup_{y \in Y} y \right) \bar{\varphi} = \prod_{y \in Y} y\bar{\varphi} = \prod_{y \in Y} y\varphi.$$

Since $x \in Y$, it follows that $Y\bar{\varphi} \leq x\bar{\varphi}$. Conversely, $Y \subseteq x\uparrow$ together with φ being order-preserving yield $x\bar{\varphi} \leq y\bar{\varphi}$ for every $y \in Y$ and so $x\bar{\varphi} \leq Y\bar{\varphi}$. Thus $x\bar{\varphi} = Y\bar{\varphi}$ and the homomorphism $\bar{\varphi}$ induces a quotient homomorphism

$$\Phi : U(X) \rightarrow E, \quad A\rho \mapsto A\bar{\varphi}.$$

Clearly, $(x\rho)\Phi = x\bar{\varphi} = x\varphi$ for every $x \in X$. Uniqueness follows from the fact that X generates $U(X)$. ■

Lemma 24. *If E is a UFS, then $E \simeq U(\text{Irr}(E))$.*

Proof. Let $X = \text{Irr}(E)$ viewed as a poset. By Lemma 23, the inclusion mapping $\iota : X \rightarrow E$ induces a semilattice homomorphism

$$\Phi : U(X) \rightarrow E, \quad A\rho \mapsto \prod_{a \in A} a\iota.$$

Since E is a UFS, X generates E and so Φ is surjective.

Let $A, B \in F(X)$ be such that $(A\rho)\Phi = (B\rho)\Phi$. Then $\prod_{a \in A} a\iota = \prod_{b \in B} b\iota$. Let $a \in A$. Since $A\iota \subseteq \text{Irr}(E)$ and E is a UFS, Lemma 12(i) implies that $a\iota \geq b\iota$ for some $b \in B$. Since ι is the inclusion mapping, we obtain $a \geq b$. Thus $A \geq_X B$. By symmetry, we get $B \geq_X A$ and so $A\rho = B\rho$. Hence Φ is injective and therefore an isomorphism. ■

From Lemmas 22 and 24 we get

Theorem 25. *Up to isomorphism, UFSs are semilattices of the form $U(X)$, where X is a poset.* ■

We can take this discussion into a categorical framework. Let **Pos** denote the category with posets as objects and order-preserving mappings as morphisms, and let **Ufs** denote the category with UFSs as objects and semilattice homomorphisms as morphisms. We define an operator U as follows. Given a poset X , $U(X)$ is the UFS defined as before. Let $\varphi : X \rightarrow Y$ be a morphism in **Pos**. Since Y embeds in $U(Y)$ by Lemma 22(ii), we can view φ as an order-preserving map from the poset X into the semilattice $U(Y)$. According to Lemma 23, we can denote by $U(\varphi)$ the unique semilattice homomorphism from $U(X)$ into $U(Y)$ that extends φ .

Proposition 26. *U is a faithful functor from **Pos** into **Ufs**.*

Proof. We prove that $U(\varphi\psi) = U(\varphi)U(\psi)$ for morphisms $\varphi : X \rightarrow Y$ and $\psi : Y \rightarrow Z$ in **Pos**. The remaining verifications are straightforward.

Clearly, both $U(\varphi\psi)$ and $U(\varphi)U(\psi)$ are semilattice homomorphisms from $U(X)$ into $U(Z)$. Since $U(X)$ is generated by X , it suffices to show that these two homomorphisms agree on X . Indeed, for every $x \in X$, we have

$$x(U(\varphi\psi)) = x\varphi\psi = x\varphi(U(\psi)) = (x(U(\varphi)))U(\psi) = x(U(\varphi)U(\psi))$$

and so $U(\varphi\psi) = U(\varphi)U(\psi)$. ■

In the next examples, we compute $U(X)$ for some particular posets X .

Example 27. Let X be a nonempty set partially ordered by the identity relation. Then $U(X) = F(X)$.

Proof. In this case, we have $x \uparrow = \{x\}$ for every $x \in X$ and so ρ is the identity relation on $F(X)$. ■

Example 28. Let X be a set and let $FG(X)$ be partially ordered by the prefix relation (that is, $v \leq w$ if and only if v is a prefix of w). Then $U(FG(X)) \simeq I(X)$.

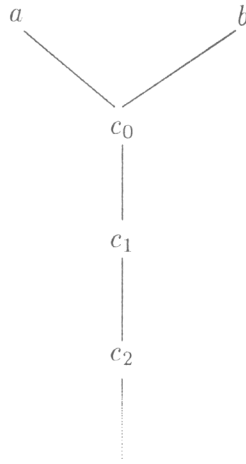
Proof. In view of Lemmas 15(ii) and 24, we only need to show that $\text{Irr}(I(X))$ and $FG(X)$ are isomorphic as posets, and this follows easily from Lemma 15(i). ■

Example 29. Let X be a chain. Then $U(X) \simeq X$. The converse implication is not true.

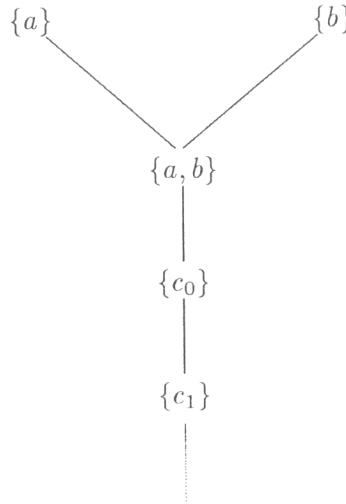
Proof. Since X is a chain, it can be viewed as a semilattice, and it is straightforward to see that all elements of X are prime. In particular, X is a UFS and so Lemma 24 yields

$$X \simeq U(\text{Irr}(X)) = U(X).$$

Now let X be the poset described by



It is a simple exercise to verify that $U(X)$ can be described by



Thus $U(X) \simeq X$ in this case, even though X is not a chain. ■

3 Embeddings into UFSs

The problem of determining necessary and sufficient conditions for a semilattice to be embeddable into a UFS remains an open problem. We can only present partial results. We note that the problem of embedding semilattices into upper finite UFSs is intimately connected with the problem of embedding semilattices into free inverse monoids, as the next result shows.

Proposition 31. *Let E be a semilattice. The following conditions are equivalent:*

- (i) E is embeddable into a countable upper finite UFS,
- (ii) E is embeddable into $I(X)$ for some countable set X ,
- (iii) E is embeddable into $F(X)$ for some countable set X .

Proof. (i) \Rightarrow (ii): By Theorem 5.2 of [2].

(ii) \Rightarrow (iii): Since $I(X)$ embeds naturally in $F(FG(X))$ and X countable implies $FG(X)$ countable.

(iii) \Rightarrow (i): If X is countable, then $F(X)$ is a countable upper finite UFS. ■

Given a semilattice E and $e \in E$, we say that a factorization $e = e_1 \dots e_n$ in E is *reduced* if $e_1 \dots e_{j-1} e_{j+1} \dots e_n > e$ for every $j \in \{1, \dots, n\}$. We

say that $e \in E$ has *finite order* k if k is the maximum length of a reduced factorization of e . Otherwise, e has *infinite order*.

Proposition 32. *Let E be a semilattice. If E embeds into a UFS, then every element of E has finite order.*

Proof. Suppose that $\varphi : E \rightarrow U$ is an embedding of E into a UFS U and let $e \in E$. We can write $e\varphi = u_1 \dots u_k$ for some $u_1, \dots, u_k \in \text{Irr}(U)$. Let $e = e_1 \dots e_n$ be a reduced factorization of e . Then

$$(e_1\varphi) \dots (e_n\varphi) = e\varphi = u_1 \dots u_k.$$

Since every u_j is prime, we have that

$$\forall j \in \{1, \dots, k\} \exists i_j \in \{1, \dots, n\} u_j \geq e_{i_j}\varphi.$$

It follows that

$$e\varphi \leq (e_{i_1}\varphi) \dots (e_{i_k}\varphi) \leq u_1 \dots u_k = e\varphi$$

and so

$$e\varphi = (e_{i_1}\varphi) \dots (e_{i_k}\varphi) = (e_{i_1} \dots e_{i_k})\varphi.$$

Since φ is injective, we get $e = e_{i_1} \dots e_{i_k}$. Now $e = e_1 \dots e_n$ being a reduced factorization of e yields $\{i_1, \dots, i_k\} = \{1, \dots, n\}$ and so $n \leq k$. Thus e has finite order at most k . ■

This result can be used to prove that some semilattices are not embeddable into UFSs.

Example 33. *Let X be an infinite set and let $P(X)$ denote the set of all subsets of X , endowed with the union operation. Then $P(X)$ is not embeddable into a UFS.*

Proof. By the preceding result, it suffices to show that the element $X \in P(X)$ has infinite order. Every partition of X into finitely many classes produces a reduced factorization of X . Since X is infinite, the number of such classes cannot be bounded and so X has infinite order. ■

We can obtain full characterizations by restricting the type of semilattice considered. We need a few preliminary results.

Lemma 34. *Let E be a semilattice, $e \in E$ and $F \subseteq E$ be such that:*

- (i) F is infinite,
- (ii) $e \not\geq f$ for every $f \in F$,
- (iii) $e \geq fg$ for all distinct $f, g \in F$.

Then E is not embeddable into a UFS.

Proof. Suppose that $\varphi : E \rightarrow U$ is an embedding of E into a UFS U . Let $e\varphi = u_1 \dots u_k$ with $u_1, \dots, u_k \in \text{Irr}(U)$. By (ii), we have that $e\varphi \not\geq f\varphi$. Thus, for every $f \in F$ there exists $i \in \{1, \dots, k\}$ such that $u_i \not\geq f\varphi$. By (i), there exist $f, g \in F$ distinct such that $u_i \not\geq f\varphi$ and $u_i \not\geq g\varphi$. However,

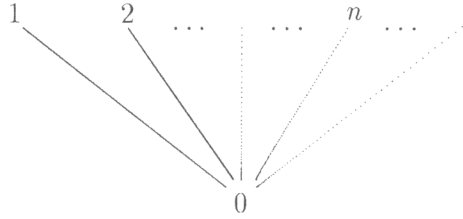
$$u_i \geq e\varphi \geq (fg)\varphi = (f\varphi)(g\varphi)$$

by (iii) and u_i prime yields $u_i \geq f\varphi$ or $u_i \geq g\varphi$, a contradiction. Therefore there is no embedding of E into a UFS. ■

Let $K_1 = \mathbf{N}$ be endowed with the binary operation

$$m \star n = \begin{cases} m & \text{if } m = n \\ 0 & \text{otherwise} \end{cases}$$

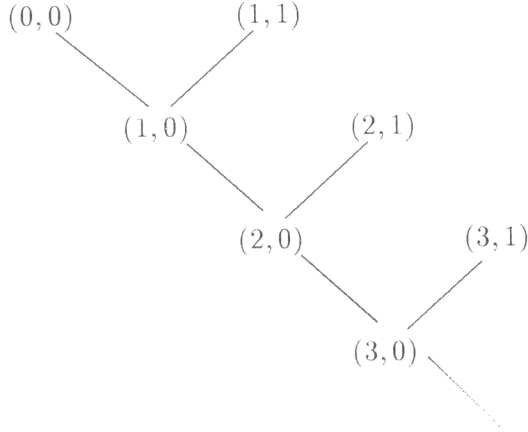
Then K_1 is a semilattice which admits the following graphical description.



Let also $K_2 = (\mathbf{N} \times \{0, 1\}) - \{(0, 1)\}$ be endowed with the binary operation

$$(m, n)(k, l) = \begin{cases} (m, n) & \text{if } (m, n) = (k, l) \\ (\max\{m, k\}, \min\{n, l\}) & \text{otherwise} \end{cases}$$

Then K_2 is a semilattice which admits the following graphical description.



Corollary 35. *Neither K_1 nor K_2 are embeddable into a UFS.*

Proof. We only need to show that both K_1 and K_2 satisfy the conditions in Lemma 34. For K_1 , we take $e = 0$ and $F = \mathbf{N} - \{0\}$. For K_2 , we take $e = (0, 0)$ and $F = \{(n, 1) \mid n \geq 1\}$. ■

Given a semilattice E and $e \in E$, it is easy to see that the complement $E - Ee$ is a dual-ideal of E . Our next result provides a sufficient condition for embeddability into a UFS.

Lemma 36. *Let E be a semilattice such that $E - Ee$ is a finitely generated dual-ideal for every $e \in E$. Then E is embeddable into a UFS.*

Proof. First of all we remark that, given a finitely generated dual-ideal A of E , there is a subset Y of E such that $A = Y \uparrow$ and that has minimum cardinal with respect to this property. Let also $A = Y' \uparrow$ and take $y \in Y$. Since $y \in A = Y' \uparrow$, we have $y \geq y'$ for some $y' \in Y'$. Also $y' \in A = Y \uparrow$ yields $y' \geq z$ for some $z \in Y$. By minimality of Y , no two elements of Y can be comparable, hence $y \geq y' \geq z$ implies that $y = z$ and thus $y = y' \in Y'$. Therefore $Y \subseteq Y'$ and we may define $\beta(A)$ to be the unique generating set of A with minimum cardinal.

Our second remark is that, given a poset X , we can adjoin a (new) greatest element 1 to X and obtain a new poset X^1 .

Finally, we define a mapping $\varphi : E \rightarrow U(E^1)$ by

$$e\varphi = \begin{cases} (\beta(E - Ee))\rho & \text{if } Ee \neq E \\ 1\rho & \text{otherwise,} \end{cases}$$

where ρ denotes the congruence on $F(E^1)$ induced by E^1 . Next we show that φ is a homomorphism. Let $e, f \in E$. We assume that $Ee \neq E \neq Ef$, the remaining cases being straightforward. We must show that

$$(1) \quad (\beta(E - Ee) \cup \beta(E - Ef)) \rho \beta(E - Eef).$$

It is easy to see that, given a finite nonempty subset X of E^1 , we have

$$(2) \quad \beta(X \uparrow) \rho X.$$

If A and B are finitely generated nonempty dual-ideals of E^1 , then $\beta(A) \cup \beta(B)$ generates the dual-ideal $A \cup B$, hence (2) yields

$$(3) \quad \beta(A \cup B) \rho (\beta(A) \cup \beta(B)).$$

Clearly, $Ee \cap Ef = Eef$ and so $(E - Ee) \cup (E - Ef) = E - Eef$. Replacing A and B in (3) by $E - Ee$ and $E - Ef$, we obtain (1). Therefore φ is a homomorphism.

Now let $e\varphi = f\varphi$ hold for some $e, f \in E$. Since a semilattice has at most one identity element, we can assume that $E \neq Ee$. Suppose that $Ef = E$. Then $\beta(E - Ee) \rho 1$ and so $1 \leq g$ for every $g \in \beta(E - Ee)$, a contradiction, since $\beta(E - Ee) \subseteq E$ and $1 > x$ for every $x \in E$. Thus we may assume that $Ef \neq E$ and so

$$(4) \quad \beta(E - Ee) \rho \beta(E - Ef).$$

Suppose that $e \neq f$. Without loss of generality, we may assume that $e \not\geq f$. Hence $f \in E - Ee$ and so $f \geq x$ for some $x \in \beta(E - Ee)$. By (4), there exists some $y \in \beta(E - Ef)$ such that $y \leq x$. Now $y \leq x \leq f$ yields $y \in Ef$, contradicting $y \in \beta(E - Ef)$. Thus $e = f$ and φ is an embedding. \blacksquare

Before proceeding to apply the previous results to particular classes of semilattices, we need to establish a few consequences of connectedness.

Lemma 37. *Let X be a connected poset and let $x, y \in X$.*

(i) *Every chain contained in $[x, y]$ is contained in a maximal chain of $[x, y]$.*

(ii) *A maximal chain of $[x, y]$ is a subset $\{z_0, \dots, z_n\}$ such that*

$$x = z_0 \prec z_1 \prec \dots \prec z_n = y.$$

(iii) *If \hat{x} is finite for every $x \in X$, then $[x, y]$ is finite.*

Proof. (i) Let C denote a chain contained in $[x, y]$, and let \mathcal{A} be the set of all chains contained in $[x, y]$ and containing C , partially ordered by inclusion. Let $\mathcal{A}_0 \subseteq \mathcal{A}$ be a chain (of chains). A standard argument shows that $\cup \mathcal{A}_0 \in \mathcal{A}$ and so $\cup \mathcal{A}_0$ is an upper bound for \mathcal{A}_0 in \mathcal{A} . By Zorn's Lemma, \mathcal{A} has some maximal element, and such an element is obviously a maximal chain of $[x, y]$ containing C .

(ii) It is immediate that every subset of the considered form is a maximal chain of $[x, y]$.

Conversely, let C be a maximal chain of $[x, y]$. Since X is connected, C must be finite, and we can write it in the form

$$x = z_0 < z_1 < \dots < z_n = y.$$

By maximality, there are no $i \in \{1, \dots, n\}$ and $w \in X$ such that $z_{i-1} < w < z_i$. Hence $z_{i-1} \prec z_i$ for every $i \in \{1, \dots, n\}$ and so C has the required form.

(iii) Assume that \widehat{x} is finite for every $x \in X$. Suppose that $[x, y]$ is infinite. We define a sequence (z_n) in $[x, y]$ such that $z_n \uparrow \cap [x, y]$ is infinite for every $n \geq 0$. Let $z_0 = x$. By hypothesis, $z_0 \uparrow \cap [x, y] = [x, y]$ is infinite. Assume that z_{n-1} is defined and satisfies the required condition. We may write $\widehat{z_{n-1}} = \{w_1, \dots, w_k\}$ with $k \geq 0$. By parts (i) and (ii), we must have

$$z_{n-1} \uparrow \cap [x, y] = \{z_{n-1}\} \cup \left(\bigcup_{i=1}^k (w_i \uparrow \cap [x, y]) \right).$$

Since $z_{n-1} \uparrow \cap [x, y]$ is infinite by hypothesis, we may define $z_n = w_i$ for some $i \in \{1, \dots, k\}$ such that $w_i \uparrow \cap [x, y]$ is infinite.

Clearly, the sequence (z_n) so constructed satisfies

$$x = z_0 \prec z_1 \prec \dots < y,$$

contradicting the connectedness of X . Thus $[x, y]$ must be finite. ■

We say that a semilattice E is a *tree semilattice* if E is connected and Ee is a chain for every $e \in E$.

Theorem 38. *Let E be a tree semilattice. Then the following conditions are equivalent:*

- (i) E is embeddable into a UFS,
- (ii) Neither K_1 nor K_2 embed into E ,

(iii) $E - Ee$ is a finitely generated dual-ideal of E for every $e \in E$.

Proof. (i) \Rightarrow (ii): By Corollary 35, since the composition of embeddings is still an embedding.

(ii) \Rightarrow (iii): Suppose that the dual-ideal $E - Ee$ is not finitely generated for some $e \in E$. Since E is a tree semilattice, the ideal Ee is a chain. Moreover, E is connected and so it follows easily from Lemma 37 that the elements of Ee may be written in the form

$$e = e_0 \succ e_1 \succ e_2 \succ \dots,$$

the chain being either finite or countably infinite. It should be also clear that $E - Ee$ is generated by the set

$$X = \bigcup_{n \geq 0} \widehat{e}_n - Ee.$$

By the hypothesis, X must be infinite. Two cases may occur.

Suppose that \widehat{e}_n is infinite for some $n \geq 0$. Since the product of two distinct elements of \widehat{e}_n cannot lie above e_n , it follows that $\{e_n\} \cup \widehat{e}_n$ contains a subsemilattice of E isomorphic to K_1 .

Otherwise, we may assume that $\widehat{e}_n - Ee$ is nonempty for infinitely many positive values of n . We may also assume that these are $i_1 < i_2 < i_3 < \dots$ and we may choose $f_k \in \widehat{e}_{i_k} - Ee$ for every $k \geq 1$. Straightforward checking shows that

$$\{e_0, e_{i_1}, e_{i_2}, \dots\} \cup \{f_1, f_2, \dots\}$$

is a subsemilattice of E isomorphic to K_2 .

(iii) \Rightarrow (i): By Lemma 36. ■

We say that a semilattice E has a *tail* if there exists some $z \in E$ such that $E = Ez \cup (z \uparrow)$ and Ez is a chain.

Theorem 39. *Let E be a connected semilattice with a tail. Then the following conditions are equivalent:*

- (i) E is embeddable into a UFS,
- (ii) K_1 does not embed into E ,
- (iii) \widehat{e} is finite for every $e \in E$,
- (iv) $E - Ee$ is a finitely generated dual-ideal of E for every $e \in E$.

Proof. (i) \Rightarrow (ii): By Corollary 35, since the composition of embeddings is still an embedding.

(ii) \Rightarrow (iii): Straightforward.

(iii) \Rightarrow (iv): Let $z \in E$ be such that $E = Ez \cup (z \uparrow)$ and Ez is a chain. Let $e \in E$. If $e \in Ez$, then $E - Ee$ is generated by \hat{e} , which is finite by hypothesis, hence we may assume that $e > z$. By Lemma 37(iii), $[z, e]$ is finite. Given $f \in E - Ee$, we have $f > z$ and so $fe \in [z, e]$. Since E is connected, we conclude that $E - Ee$ is generated by

$$\bigcup_{f \in [z, e]} \hat{f} - [z, e],$$

a finite set.

(iv) \Rightarrow (i): By Lemma 36. ■

4 The Munn semigroup

The *Munn semigroup* of a semilattice E consists of the set T_E of all isomorphisms of the form $\varphi : Ee \rightarrow Ef$ ($e, f \in E$), endowed with the usual composition of partial transformations. Munn semigroups play a major role in the study of fundamental inverse semigroups. An inverse semigroup is *fundamental* if the unique idempotent-separating congruence on S is the identity congruence. Given an inverse semigroup S ,

- there is a homomorphism $\varphi : S \rightarrow T_{E(S)}$ whose kernel is the greatest idempotent-separating congruence on S ([1], Theorem 4.9);
- S is fundamental if and only if S is isomorphic to a full inverse sub-semigroup of $T_{E(S)}$ ([1], Theorem 4.10).

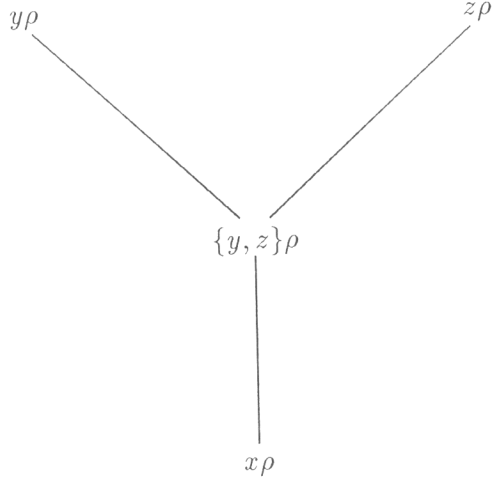
We shall discuss some properties of the Munn semigroup for a particular class of UFSs which includes our favourite semilattices $F(X)$ and $I(X)$.

A UFS E is said to be *regular* if E is upper finite and $\text{Irr}(E)$ is an dual-ideal of E . An alternative characterization can be provided in terms of $\text{Irr}(E)$ alone.

Lemma 41. *Let X be a poset. Then $U(X)$ is regular if and only if $x \uparrow$ is a finite chain for every $x \in X$.*

Proof. Suppose that $x \uparrow$ is infinite for some $x \in X$. It is immediate that the projection of $x \uparrow$ on $U(X)$ will provide infinitely many elements of $(x\rho) \uparrow$, hence $U(X)$ is not upper finite and therefore not regular.

Suppose now that $x \uparrow$ is not a chain for some $x \in X$. Then there exist $y, z \in x \uparrow$ such that $y \not\leq z$ and $z \not\leq y$. We must have $y, z > x$ and it follows easily that



is a subsemilattice of $U(X)$. Since $x\rho$ is irreducible but $\{y, z\}\rho$ is not, $U(X)$ is not regular.

Conversely, assume that $x \uparrow$ is a finite chain for every $x \in X$. Suppose that $U(X)$ is not upper finite. Then there exists $A \in F(X)$ such that $(A\rho) \uparrow$ contains infinitely many elements, say $\{C_1\rho, C_2\rho, \dots\}$. Let $A = \{a_1, \dots, a_n\}$. Since $C_i\rho \geq A\rho$ for every i , we have

$$C_1 \cup C_2 \cup \dots \subseteq (a_1 \uparrow) \cup \dots \cup (a_n \uparrow).$$

For every $i \in \{1, \dots, n\}$, write

$$A_i = (C_1 \cup C_2 \cup \dots) \cap (a_i \uparrow).$$

Since $a_i \uparrow$ is finite, it follows that A_i is finite for every i and thus $C_1 \cup C_2 \cup \dots$ is itself finite, contradicting $\{C_1\rho, C_2\rho, \dots\}$ being an infinite set. Therefore $U(X)$ is upper finite.

Finally, let $x \in X$ and assume that $A\rho \geq x\rho$, with $A \in F(X)$. Since $A \subseteq x \uparrow$ and $x \uparrow$ is a chain, we may write $A\rho = a\rho$, where a denotes the least element of A . Thus $A\rho$ is irreducible, X is dual-ideal of $U(X)$ and $U(X)$ is regular. ■

By the preceding result, we can get an accurate picture of the irreducibles in a regular UFS. They constitute a *forest* (disjoint union) of rooted trees,

turned upside down. We adopt here the standard terminology for trees. Given a tree T and $x, y \in T$, we say that x is a *son* (respectively *father*, *descendant*, *ancestor*) of y if $x \prec y$ (respectively $x \succ y$, $x < y$, $x > y$). The elements are generally called *nodes*. Every node but one has exactly one father, the fatherless node being called the *root*. A node has *depth* n if it has exactly n ancestors.

In the case of $F(X)$, we have $|X|$ trivial trees with a single node. In the case of $I(X)$, we have a single tree described as follows: the root has $2|X|$ sons, all the other nodes have $2|X| - 1$ sons each. The reason for this is that there are $2|X|$ different ways of adjoining a letter to the empty word without cancellation, but only $2|X| - 1$ possibilities in the case of a nonempty word.

To study the Munn semigroup, we have to consider isomorphisms between principal ideals. By Lemma 13(ii), all principal ideals of a UFS are UFSs, and it should be clear from Proposition 26 that such isomorphisms must be induced by isomorphisms between the posets of irreducibles. The next step will be to devise the structure of these posets. We note that, given a forest F of the type described above, the poset obtained by adjoining a (new) greatest element 1 to F is a tree.

Lemma 42. *Let E be a regular UFS and let $e \in E$. Then Ee is a regular UFS and $\text{Irr}(Ee)$ is isomorphic to the tree obtained by identifying with the new root 1 in $(\text{Irr}(E))^1$ all the irreducibles f such that $f \geq e$.*

Proof. By Lemma 13, Ee is a UFS and $\text{Irr}(Ee) = (\text{Irr}(E))e$. By Lemma 41, we only need to prove our second assertion, since it implies, in particular, that $x \uparrow$ is a finite chain for every $x \in \text{Irr}(Ee)$. Let $f, g \in \text{Irr}(E)$ be distinct. Clearly, $f > g$ implies $fe \geq ge$, hence we need to establish when does $fe = ge$ occur.

It is immediate that $f, g \geq e$ implies $fe = ge$. Now assume that $fe = ge$, and suppose that $g \not\geq e$. We have $g \geq fe$. Since g is prime and $g \not\geq e$, we get $g \geq f$. Also $f \geq ge$ yields $f \geq g$ or $f \geq e$. Since $f \geq e$ would imply $g \geq e$, a contradiction, we must have $f \geq g$. But then we obtain $f = g$, contradicting the fact that f and g were assumed to be distinct. It follows that $g \geq e$ and, by symmetry, also $f \geq e$. Thus only the irreducibles that lie above e are identified. Since e is the maximal element of Ee , these identified irreducibles will constitute the root of the tree $\text{Irr}(Ee)$, and we may describe this by identifying them all with the new root 1 in $(\text{Irr}(E))^1$. ■

Note that the set $Y = \{f \in \text{Irr}(E) \mid f \geq e\}$ is always a dual-ideal of both

$\text{Irr}(E)$ and E (since E is regular). Moreover, the minimal elements of Y are the factors in the (unique) reduced irreducible factorization of e . In the particular case of $I(X)$, when $|X|$ is finite and greater than 1, we get a tree where the root may have an arbitrarily large number of sons (depending on the order of e) and all the other nodes have exactly $2|X| - 1$ sons each. For $|X| = 1$ (respectively X infinite) the root has 2 sons (respectively $|X|$ sons) and the other nodes 1 son (respectively $|X|$ sons) each. We can get an idea of the size of $T_{I(X)}$ by the following proposition.

Proposition 43. *Let X be a nonempty set. Then:*

- (i) *Every maximal subgroup of $T_{I(X)}$ is isomorphic to the automorphism group of a rooted tree.*
- (ii) *If $|X| = 1$ or X is infinite, all maximal subgroups of $T_{I(X)}$ are isomorphic.*
- (iii) *If $|X| > 1$, every maximal subgroup of $T_{I(X)}$ is uncountable.*

Proof. (i) It is easy to see that the maximal subgroups of T_E are the sets of automorphisms of the principal ideals Ee . In the case of a UFS, this corresponds to automorphisms of $\text{Irr}(Ee)$, in our case a rooted tree.

(ii) This follows from the description of the trees accomplished above, or alternatively from the fact that $T_{I(X)}$ is bisimple in this case ([2], Theorem 4.2).

(iii) In this case every node in the tree has at least two sons, and we can define an injective map from the set of all sequences in $\{0, 1\}$ into the group of automorphisms of the tree. We give a sketch of the argument. Let (a_n) be such a sequence. Assuming that the sons are always ordered in a certain way, we choose an automorphism that, at level n of depth, keeps or changes the relative position of the first son according to a_n being a 0 or a 1. This shows in fact that the cardinal of a maximal subgroup is at least the continuum. ■

Several other properties of $T_{I(X)}$ are discussed in [2], including the property of being E -unitary. An inverse semigroup S is said to be E -unitary if

$$ea, e \in E(S) \Rightarrow a \in E(S)$$

holds for all $a, e \in S$. The idempotents in T_E are the identity mappings on the principal ideals Ee , denoted by 1_{Ee} . Given $\varphi : Ee \rightarrow Ef$, we have $1_{Eg}\varphi = \varphi|_{Ege}$, hence T_E is E-unitary if and only if, given an isomorphism $\varphi : Ee \rightarrow Ef$ and $g \in E$,

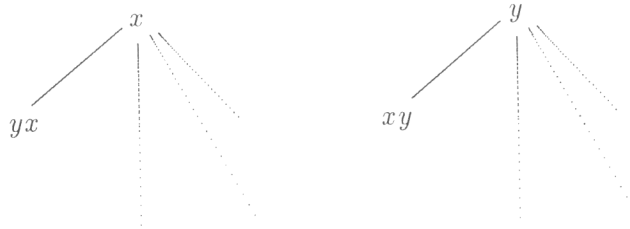
$$\varphi|_{Ege} = 1_{Ege} \Rightarrow \varphi = 1_{Ee}.$$

We can give a full account of this property in the context of regular UFSs.

Theorem 44. *Let $E = U(X)$ be a regular UFS. Then T_E is E-unitary if and only if X has at most one minimal element.*

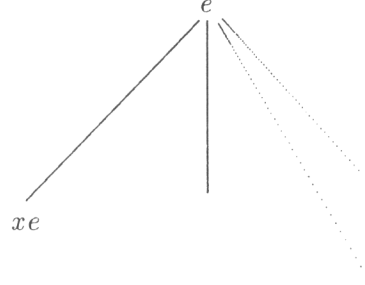
Proof. Suppose that x and y are two distinct minimal elements of X . We consider two cases.

Suppose first that x and y are both maximal elements of X . The tree $\text{Irr}(Ex)$ can be obtained from X^1 by identifying the isolated node x with the root 1 (formally, we are in fact multiplying all the irreducibles by x in E). Symmetrically, $\text{Irr}(Ey)$ can be obtained from X^1 by identifying the node y with 1, as the next figure shows.



These trees are obviously isomorphic and we may define a natural isomorphism $\varphi : Ex \rightarrow Ey$. Since $x\varphi = y$, φ is not the identity mapping. We may assume that $(yx)\varphi = xy$ and $(zx)\varphi = zy$ for every $z \in X - \{x, y\}$. It follows that $\varphi|_{Exy} = 1_{Exy}$ and so T_E is not E-unitary in this case.

Next we suppose that either x or y is not a maximal element of X . Let $e \in E$ be obtained by multiplying the fathers of x and y (one of the two may be absent). Since x and y are both minimal, neither of them can lie above the father of the other. It follows easily that $x, y \not\leq e$ and so x and y are not to be identified by the product by e . As a consequence, the root of the tree $\text{Irr}(Ee)$ will have two sons corresponding to x and y (possibly others), as the next figure shows.



We may define a nontrivial automorphism φ of Ee by permuting xe and ye and fixing all other irreducibles. However, $\varphi|_{Exye} = 1_{Exye}$ and so T_E is not E-unitary in this case too.

Conversely, assume that X has at most one minimal element. If it exists at all, we denote it by m . Suppose that $\varphi : Ee \rightarrow Ef$ is a nonidentity isomorphism and $g \in E$ is such that $\varphi|_{Ege} = 1_{Ege}$. If X is a chain, the mere fact that φ fixes the element eg implies that φ must fix all other elements and is therefore the identity mapping, hence we may assume that X is not a chain. We claim that

$$(5) \quad \exists x \in X : xe < e \text{ and } (xe)\varphi \neq xf.$$

Consider first the case $e = f$. Since $\varphi \neq 1_{Ee}$, we have $h\varphi \neq h$ for some $h \in Ee$. We may write $h = h_1 \dots h_n$ for some $h_1, \dots, h_n \in \text{Irr}(Ee) - \{e\}$, hence we must have $h_i\varphi \neq h_i$ for some $i \in \{1, \dots, n\}$. We may write $h_i = xe$ for some $x \in X$. Thus (5) holds in this case.

Assume now that $e \neq f$. Since X is not a chain and has at most one minimal element, we have $e \neq m$ and so $\text{Irr}(Ee) \neq \{e\}$. Let $\{x_1e, \dots, x_ne\}$ denote the sons of e in $\text{Irr}(Ee)$, with $x_1, \dots, x_n \in X$. It follows from Lemma 42 that x_1, \dots, x_n are the sons in X of the factors in the reduced irreducible factorization of e . Suppose that $(x_ie)\varphi = x_if$ for every $i \in \{1, \dots, n\}$. Since φ must map the sons of e in $\text{Irr}(Ee)$ onto the sons of f in $\text{Irr}(Ef)$, the same argument would imply that x_1, \dots, x_n were the sons in X of the factors in the reduced irreducible factorization of f . Since $e \neq f$, the respective reduced irreducible factorizations must be different and thus their factors cannot have the same sons, a contradiction. Therefore $(x_ie)\varphi \neq x_if$ for some $i \in \{1, \dots, n\}$ and (5) holds.

Let $x \in X$ satisfy (5). We show that

$$(6) \quad m \in X \Rightarrow m \not\leq x.$$

Suppose that $m \in X$ and $m \leq x$. Since $x \not\leq e$, we must have also $m \not\leq e$, hence me is a minimal element of $\text{Irr}(Ee)$, more precisely, the only one, since X cannot have more than one minimal element. Thus $\text{Irr}(Ef)$ must have also a minimal element to be the image of me by φ , and the unique candidate is mf . Therefore

$$(7) \quad (me)\varphi = mf.$$

We have a sequence in $\text{Irr}(Ee)$ of the form

$$(8) \quad me = y_0e \prec y_1e \prec \dots \prec y_k e = xe$$

for some $y_0, \dots, y_k \in X$. Since $x \not\leq e$, it follows from Lemma 42 that

$$(9) \quad m = y_0 \prec y_1 \prec \dots \prec y_k = x$$

in X . Suppose that $x \geq f$. By (9), the depth of mf in $\text{Irr}(Ef)$ would be at most k , but it follows from (8) and $xe < e$ that the depth of me in $\text{Irr}(Ee)$ must be at least $k + 1$. Clearly, the unique minimal element in two isomorphic trees cannot have different depths, thus $x \not\geq f$. By Lemma 42 and (9) we obtain the sequence

$$(10) \quad mf = y_0f \prec y_1f \prec \dots \prec y_k f = xf$$

in $\text{Irr}(Ef)$. Any isomorphism between trees must preserve order, in particular the father must be mapped onto the father of its son's image. Thus (7), (8) and (10) yield $(xe)\varphi = xf$, a contradiction. Thus (6) holds.

Since $m \not\leq x$, there exist descendants of x with arbitrarily large depth. Since $fge \uparrow$ is finite, there exists $x' \in X$ such that $x' \leq x$ and $x' \not\leq fge$. The argument we used to prove (6) shows that $(x'e)\varphi = x'f$ would imply $(xe)\varphi = xf$, hence x' satisfies (5) and (6) and we may assume that $x' = x$. Since φ maps $\text{Irr}(Ee)$ onto $\text{Irr}(Ef)$, we must have $(xe)\varphi = zf$ for some $z \in X$. Now

$$xge = (xge)\varphi = ((xe)(ge))\varphi = (xe)\varphi (ge)\varphi = z fge.$$

In particular, $x \geq z fge$. Since x is prime and $x \not\leq fge$, we obtain $x \geq z$. This implies that $z \not\leq fge$, hence $z \geq xge$ yields $z \geq x$ and so $z = x$, a contradiction, since $(xe)\varphi \neq xf$ by (5). Thus T_E is E -unitary. ■

Note. This work is based on a talk presented at the *Encontro Nacional da Sociedade Portuguesa de Matemática*, Braga, 9-12/2/98.

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