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RING-LIKE OPERATIONS IN PSEUDOCOMPLEMENTED SEMILATTICES

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Abstract

Ring-like operations are introduced in pseudocomplemented semilattices in such a way that in the case of Boolean pseudocomplemented semilattices one obtains the corresponding Boolean ring operations. Properties of these ring-like operations are derived and a characterization of Boolean pseudocomplemented semilattices in terms of these operations is given. Finally, ideals in the ring-like structures are defined and characterized.

Keywords: pseudocomplemented semilattice, Boolean algebra, Boolean ring, distributivity, linear equation, ideal, congruence kernel.

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1. Introduction

First we will briefly report on some results (obtained in [2] and [5]) concerning ring-like operations in orthomodular lattices since ring-like operations will be introduced in pseudocomplemented semilattices in a similar way and we will obtain also similar results.

It is well-known that there is a natural bijection between Boolean algebras and Boolean rings. This correspondence between certain lattice structures and certain term-equivalent ring structures is very useful. For instance, congruence permutability of Boolean algebras follows immediately from that of rings (resp. groups). So it is natural to ask how a ring-like structure can be introduced in generalizations of Boolean algebras. In [1] and [3] a ring-like structure was introduced in orthomodular lattices and also in more general structures. Since pseudocomplemented semilattices can also be viewed as generalizations of Boolean algebras one may try to introduce a ring-like structure in pseudocomplemented semilattices. This is the aim of the present paper.

2. Ring-like operations in orthomodular lattices

An *orthomodular lattice* is an algebra $(L, \vee, \wedge, ', 0, 1)$ of type $(2, 2, 1, 0, 0)$ such that $(L, \vee, \wedge, 0, 1)$ is a bounded lattice and additionally we have:

(i) $(x')' = x$,

(ii) $(x \vee y)' = x' \wedge y'$,

(iii) $x \vee x' = 1$,

(iv) $x \leq y \Rightarrow y = x \vee (y \wedge x')$,

for all $x, y \in L$.

In the following let L be an arbitrary, fixed orthomodular lattice. On every Boolean subalgebra of L one can define ring operations $+$ and $\cdot$ by

$$x + y := (x \wedge y') \vee (x' \wedge y) = (x \vee y) \wedge (x' \vee y'),$$
$$xy := x \wedge y = (x \vee y) \wedge (x \vee y') \wedge (x' \vee y).$$

We now extend these operations from the Boolean subalgebras of L to L by defining

$$x +_1 y := (x \wedge y') \vee (x' \wedge y),$$
$$x +_2 y := (x \vee y) \wedge (x' \vee y'),$$
$$x \cdot_1 y := x \wedge y,$$
$$x \cdot_2 y := (x \vee y) \wedge (x \vee y') \wedge (x' \vee y),$$

for all $x, y \in L$. Then the following theorem holds:

Theorem 1. *For arbitrary, fixed $i, j \in \{1, 2\}$ the following properties are equivalent:*

(i) *L is a Boolean algebra;*

(ii) $+_1 = +_2$;

(iii) $\cdot_1 = \cdot_2$;

(iv) *$+_i$ is associative;*

(v) *$\cdot_2$ is associative;*

(vi) *$\cdot_j$ is distributive with respect to $+_i$;*

(vii) *For every $(a, b) \in L^2$ the equation $a +_i x = b$ has at most one solution;*

(viii) *For every $(a, b) \in L^2$ the equation $a +_i x = b$ has exactly one solution;*

(ix) *For every $(a, b) \in L^2$ the equation $a +_i x = b$ has at least one solution.*

Proof. See [2] and [5]. ∎

3. RING-LIKE OPERATIONS IN PSEUDOCOMPLEMENTED SEMILATTICES

A *pseudocomplemented meet-semilattice* (with zero) is an algebra $S = (S, \wedge, ^*, 0)$ of type $(2, 1, 0)$, where $(S, \wedge, 0)$ is a meet-semilattice with smallest element 0 and where each $x \in S$ has a so-called pseudocomplement x^*, that is a greatest element $y \in S$ with the property $x \wedge y = 0$, i. e. for $x, y \in S$ it holds $x \wedge y = 0$ iff $y \le x^*$. (The concept of a *pseudocomplemented join-semilattice* (with one) can be defined dually.)

In this section let S denote an arbitrary, fixed pseudocomplemented meet-semilattice and let a, b, c be arbitrary, fixed elements of S.

Put $a \sqcup b := (a^* \wedge b^*)^*$ and $1 := 0^*$.

The following facts are well-known (cf. [4]):

(i) $(^*, ^*)$ is a Galois correspondence between $(S, \le)$ and $(S, \le)$,

(ii) $a \le b \Rightarrow a^* \ge b^*$,

(iii) $a \le a^{**}$,

(iv) $a^{***} = a^*$,

(v) $0^{**} = 0$,

(vi) $^{**} \in$ End S;

(vii) $\theta_0 := \ker^{**} \in \mathrm{Con}S$,

(viii) $BA(S) := (S^*, \sqcup, \wedge, ^*, 0, 1)$ is the greatest Boolean subalgebra of $(S, \sqcup, \wedge, ^*, 0, 1)$.

We call $BA(S)$ the *Boolean algebra induced by S*.

Remark. (i) $\theta_0 \in \mathrm{Con}(S, \sqcup, \wedge, ^*, 0, 1)$ and from the homomorphism theorem it follows that $(S, \sqcup, \wedge, ^*, 0, 1)/\theta_0 \cong BA(S)$.

(ii) One can show that the class of all pseudocomplemented meet-semilattices forms a variety which can be defined by the following laws:

(1) $(x \wedge y) \wedge z = x \wedge (y \wedge z)$,

(2) $x \wedge y = y \wedge x$,

(3) $x \wedge x = x$,

(4) $x \wedge 0 = 0$,

(5) $(x \wedge y)^* \wedge (x \wedge y^*)^* = x^*$,

(6) $0^{**} = 0$,

(7) $x \wedge x^{**} = x$.

Let $BR(S) := (S^*, +, 0, \cdot, 1)$ denote the Boolean ring corresponding to the Boolean algebra $BA(S)$. We call $BR(S)$ the *Boolean ring induced by S*. Then

$$a + b = (a \wedge b^*) \sqcup (a^* \wedge b) = (a \sqcup b) \wedge (a^* \sqcup b^*),$$
$$ab = a \wedge b = (a \sqcup b) \wedge (a \sqcup b^*) \wedge (a^* \sqcup b),$$

if $a, b \in S^*$. We now extend these operations from S^* to S in several "natural" ways by defining

$$a + b := a^{**} + b^{**},$$
$$a +_1 b := (a \wedge b^*) \sqcup (a^* \wedge b),$$
$$a +_2 b := (a \sqcup b) \wedge (a^* \sqcup b^*),$$
$$ab := a^{**}b^{**},$$
$$a \cdot_1 b := a \wedge b,$$
$$a \cdot_2 b := (a \sqcup b) \wedge (a \sqcup b^*) \wedge (a^* \sqcup b).$$

Since S^* is a subalgebra of S and $S \sqcup S \subseteq S^*$, we have
$S + S,\ S +_1 S,\ S +_2 S,\ SS,\ S \cdot_2 S \subseteq S^*$.

Lemma 2. $+ = +_1 = +_2$ *and* $\cdot = \cdot_2$

Proof. For $i \in \{1, 2\}$ it holds

$$a +_i b = (a +_i b)^{**} = a^{**} +_i b^{**} = a^{**} + b^{**} = a + b,$$
$$a \cdot_2 b = (a \cdot_2 b)^{**} = a^{**} \cdot_2 b^{**} = a^{**} b^{**} = ab. \qquad \blacksquare$$

Obviously, $+, \cdot$ and $\cdot_1$ are commutative and associative and $\cdot$ is distributive with respect to $+$.

To S, we assign the following ring-like structures: $R(S) := (S, +, 0, \cdot, 1)$ and $P(S) := (S, +, 0, \cdot_1, 1)$.

From the homomorphism theorem, it follows that

$$\theta_0 \in \mathrm{Con}R(S) \cap \mathrm{Con}P(S) \ \text{ and } \ R(S)/\theta_0 \cong P(S)/\theta_0 \cong BR(S).$$

Lemma 3. *The following identities hold*:

$$\begin{array}{lll} a + 0 = a^{**}, & a0 = 0, & a \cdot_1 0 = 0, \\ a + a = 0, & aa = a^{**}, & a \cdot_1 a = a, \\ a + a^* = 1, & aa^* = 0, & a \cdot_1 a^* = 0, \\ a + a^{**} = 0, & aa^{**} = a^{**}, & a \cdot_1 a^{**} = a, \\ a + 1 = a^*, & a1 = a^{**}, & a \cdot_1 1 = a. \end{array}$$

Proof. Straightforward. ■

Lemma 4. $(a + b)^* = a^* + b = a + b^*$.

Proof. Indeed, we have

$$(a + b)^* = (a + b) + 1 = (a + 1) + b = a^* + b,$$
$$(a + b)^* = (a + b) + 1 = a + (b + 1) = a + b^*. \qquad \blacksquare$$

Corollary. $a + b = (a + b)^{**} = (a^* + b)^* = a^* + b^*$. ■

The following lemma characterizes vanishing of the symmetric difference $a + b$:

Lemma 5. *The following properties are equivalent:*

(i) $a + b = 0$;

(ii) $a\theta_0 b$;

(iii) *There exists a* $c \in S$ *with* $a + c = b + c$;

(iv) $a + x = b + x$ *for all* $x \in S$.

Proof. (i) $\Rightarrow$ (ii): $a^{**} = a + 0 = a + (a + b) = (a + a) + b = 0 + b = b^{**}$.
(ii) $\Rightarrow$ (iv): $a + x = a^{**} + x = b^{**} + x = b + x$ for all $x \in S$.
(iv) $\Rightarrow$ (iii): Straightforward.
(iii) $\Rightarrow$ (i): $a + b = (a + b) + 0 = (a + b) + (c + c) = (a + c) + (b + c) = (a + c) + (a + c) = 0$. ■

Lemma 6. *The following properties are equivalent:*

(i) $a + b = 1$;

(ii) $a\ \theta_0\ b^*$;

(iii) $a^*\ \theta_0\ b$.

Proof. Indeed, we have

$$a + b = 1 \Leftrightarrow (a + b)^* = 1^* \Leftrightarrow a + b^* = 0 \Leftrightarrow a\ \theta_0\ b^*,$$
$$a + b = 1 \Leftrightarrow (a + b)^* = 1^* \Leftrightarrow a^* + b = 0 \Leftrightarrow a^*\ \theta_0\ b,$$

according to Lemma 5. ■

Now we want to characterize Boolean pseudocomplemented semilattices. For this purpose we first need two lemmas. The first of these describes the set of all solutions of a linear equation:

Lemma 7. *We have*:

$$\{x \in S \mid a + x = b\} = \begin{cases} \emptyset & \text{if } \ b \notin S^*, \\ [a + b]\theta_0 & \text{if } \ b \in S^*. \end{cases}$$

Proof. The first part follows from $S + S \subseteq S^*$. In order to prove the second part, assume $b \in S^*$ and let $x \in S$. Then

$$a + x = b \Leftrightarrow a^{**} + x^{**} = b \Leftrightarrow x^{**} = a^{**} + b \Leftrightarrow x^{**} = \\ = (a + b)^{**} \Leftrightarrow x\ \theta_0\ a + b \Leftrightarrow x \in [a + b]\theta_0.$$ ■

Proposition 8. *The following properties hold*:

(i) $ab = a \cdot_1 b \Leftrightarrow a^{**} \wedge b^{**} \leq a \wedge b$;

(ii) $(a+b) \cdot_1 c = (a \cdot_1 c) + (b \cdot_1 c) \Leftrightarrow (a+b) \wedge c^{**} \leq c$;

(iii) $|\{x \in S \mid a + x = b\}| \leq 1 \Leftrightarrow (b \notin S^* \text{ or } |[a+b]\theta_0| = 1)$;

(iv) $|\{x \in S \mid a + x = b\}| = 1 \Leftrightarrow (b \in S^* \text{ and } |[a+b]\theta_0| = 1)$;

(v) $|\{x \in S \mid a + x = b\}| \geq 1 \Leftrightarrow b \in S^*$.

Proof. (i) is evident.
Ad (ii): $(a+b) \cdot_1 c = (a+b) \wedge c$ and

$$\begin{aligned}(a \cdot_1 c) + (b \cdot_1 c) &= (a \wedge c) + (b \wedge c) = ((a \wedge c) + (b \wedge c))^{**} = \\ &= (a^{**} \wedge c^{**}) + (b^{**} \wedge c^{**}) = \\ &= (a^{**} + b^{**}) \wedge c^{**} = (a+b) \wedge c^{**}.\end{aligned}$$

Hence,

$$\begin{aligned}(a+b) \cdot_1 c = (a \cdot_1 c) + (b \cdot_1 c) &\Leftrightarrow (a+b) \wedge c = (a+b) \wedge c^{**} \Leftrightarrow \\ &\Leftrightarrow (a+b) \wedge c^{**} \leq (a+b) \wedge c \Leftrightarrow \\ &\Leftrightarrow ((a+b) \wedge c^{**} \leq a+b \\ \text{and } (a+b) \wedge c^{**} \leq c &\Leftrightarrow (a+b) \wedge c^{**} \leq c.\end{aligned}$$

The properties (iii) – (v) follow from Lemma 7. ■

Now we are ready to prove the result concerning the characterization of Boolean pseudocomplemented semilattices in terms of ring-like operations.

Theorem 9. *The following are equivalent:*

(i) *S is a Boolean algebra;*

(ii) $\cdot = \cdot_1$;

(iii) *$\cdot_1$ is distributive with respect to $+$;*

(iv) *The equation $a + x = b$ has at most one solution;*

(v) *The equation $a + x = b$ has exactly one solution;*

(vi) *The equation $a + x = b$ has at least one solution.*

Proof. Obviously, (i) $\Rightarrow$ (ii) – (vi) and (v) $\Rightarrow$ (iv), (vi).
(ii) $\Rightarrow$ (i): According to (i) of Proposition 8, $a^{**} \wedge b^{**} \leq a \wedge b$. Hence $a \leq a^{**} = a^{**} \wedge 1^{**} \leq a \wedge 1 = a$ which implies $a = a^{**}$.
(iii) $\Rightarrow$ (i): According to (ii) of Proposition 8, $(a+b) \wedge c^{**} \leq c$. Hence, $c \leq c^{**} = (0+1) \wedge c^{**} \leq c$ and, therefore, $c = c^{**}$.

(iv) $\Rightarrow$ (i): According to (iii) of Proposition 8, $|[a+b]\theta_0| = 1$ if $b \in S^*$. Since $c, c^{**} \in [0 + c^{**}]\theta_0$, we have $c = c^{**}$.
(vi) $\Rightarrow$ (i) follows from (v) of Proposition 8. ■

4. Ideals in generalizations of pseudocomplemented semilattices

In the following let $\mathcal{A}$ be an algebra such that there exists a binary operation $\cdot$ on A, a unary operation * on A as well as an element 0 of A such that $0x = xx^* = 0$ and $x0^* = x$ for all $x \in A$.

Every pseudocomplemented meet-semilattice $S = (S, \wedge, ^*, 0)$ can be considered as such an algebra. (Take $A := S$ and $\cdot := \wedge$.)

We now define the notions of an ideal of $\mathcal{A}$ and of a congruence kernel of A, respectively. But first let us recall the notion of a unary polynomial function on A: A function p from A to A is called a *unary polynomial* function on A if there exists a positive integer n, an n-ary term function t on A and $a_2, \ldots, a_n \in A$ with $p(x) = t(x, a_2, \ldots, a_n)$ for all $x \in A$.

Definition 10. Let B be a non-empty subset of the algebra $\mathcal{A}$. B is called an *ideal of* $\mathcal{A}$ − in signs $B \lhd A$ − if for all $a, b, c \in B$ and for every unary polynomial function p on $A, p(a)(p(b)c^*)^* \in B$. B is called a *congruence kernel* of $\mathcal{A}$ if $B = [0]\theta$ for some $\theta \in \text{Con}A$.

The following theorem holds for arbitrary universal algebras:

Theorem 11. *A non-empty subset B of A is a class of some congruence on $\mathcal{A}$ iff for every unary polynomial function p on A, $a, b, p(a) \in B$ implies $p(b) \in B$.*

Proof. See [6]. ■

Now, we are able to prove our final result concerning the fact that both notions defined in Definition 10 coincide:

Theorem 12. *The ideals of A coincide with the congruence kernels of $\mathcal{A}$.*

Proof. Let $I \subseteq A$. First assume $I \lhd A$. Let $a, b \in I$, let p be a unary polynomial function on A and assume $p(a) \in I$. Then $p(b) = p(b)(p(a)(p(a))^*)^* \in I$. Hence, by Theorem 11, there exists some $c \in A$ and some $\theta \in \text{Con}(\mathcal{A})$ such that $I = [c]\theta$. Let q denote the unary zero polynomial function on A. Then $0 = q(c)(q(c)c^*)^* \in I$. Hence $I = [0]\theta$ which shows that I is a congruence kernel of $\mathcal{A}$.

Conversely, assume I to be a congruence kernel of $\mathcal{A}$. Then there exists some $\alpha \in \mathrm{Con}(\mathcal{A})$ with $I = [0]\alpha$. Let $d, e, f \in I$ and let r be a unary polynomial function on A. Then $r(d)(r(e)f^*)^* \;\alpha\; r(0)(r(0)0^*)^* = 0$ which shows $r(d)(r(e)f^*)^* \in [0]\alpha = I$. Hence $I \lhd A$. ■

Remark. The ring-like structures $P(S)$ (where S is a pseudocomplemented meet-semilattice), Boolean quasirings (introduced in [3]) and orthopseudorings (introduced in [1]) are also special cases of the algebras considered in this section.

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