# RING-LIKE OPERATIONS IN PSEUDOCOMPLEMENTED SEMILATTICES 

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#### Abstract

Ring-like operations are introduced in pseudocomplemented semilattices in such a way that in the case of Boolean pseudocomplemented semilattices one obtains the corresponding Boolean ring operations. Properties of these ring-like operations are derived and a characterization of Boolean pseudocomplemented semilattices in terms of these operations is given. Finally, ideals in the ring-like structures are defined and characterized.


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## 1. Introduction

First we will briefly report on some results (obtained in [2] and [5]) concerning ring-like operations in orthomodular lattices since ring-like operations will be introduced in pseudocomplemented semilattices in a similar way and we will obtain also similar results.

It is well-known that there is a natural bijection between Boolean algebras and Boolean rings. This correspondence between certain lattice structures and certain term-equivalent ring structures is very useful. For instance, congruence permutability of Boolean algebras follows immediately from that of rings (resp. groups). So it is natural to ask how a ring-like structure can be introduced in generalizations of Boolean algebras. In [1] and [3] a ring-like structure was introduced in orthomodular lattices and also in more general structures. Since pseudocomplemented semilattices can also be viewed as generalizations of Boolean algebras one may try to introduce a ring-like structure in pseudocomplemented semilattices. This is the aim of the present paper.

## 2. Ring-Like operations in orthomodular lattices

An orthomodular lattice is an algebra $\left(L, \vee, \wedge,{ }^{\prime}, 0,1\right)$ of type $(2,2,1,0,0)$ such that $(L, \vee, \wedge, 0,1)$ is a bounded lattice and additionally we have:
(i) $\left(x^{\prime}\right)^{\prime}=x$,
(ii) $(x \vee y)^{\prime}=x^{\prime} \wedge y^{\prime}$,
(iii) $x \vee x^{\prime}=1$,
(iv) $\quad x \leq y \Rightarrow y=x \vee\left(y \wedge x^{\prime}\right)$,
for all $x, y \in L$.
In the following let $L$ be an arbitrary, fixed orthomodular lattice. On every Boolean subalgebra of $L$ one can define ring operations + and $\cdot$ by

$$
\begin{aligned}
& x+y:=\left(x \wedge y^{\prime}\right) \vee\left(x^{\prime} \wedge y\right)=(x \vee y) \wedge\left(x^{\prime} \vee y^{\prime}\right) \\
& x y:=x \wedge y=(x \vee y) \wedge\left(x \vee y^{\prime}\right) \wedge\left(x^{\prime} \vee y\right)
\end{aligned}
$$

We now extend these operations from the Boolean subalgebras of $L$ to $L$ by defining

$$
\begin{aligned}
& x+{ }_{1} y:=\left(x \wedge y^{\prime}\right) \vee\left(x^{\prime} \wedge y\right) \\
& x+{ }_{2} y:=(x \vee y) \wedge\left(x^{\prime} \vee y^{\prime}\right) \\
& x \cdot{ }_{1} y:=x \wedge y \\
& x \cdot{ }_{2} y:=(x \vee y) \wedge\left(x \vee y^{\prime}\right) \wedge\left(x^{\prime} \vee y\right),
\end{aligned}
$$

for all $x, y \in L$. Then the following theorem holds:
Theorem 1. For arbitrary, fixed $i, j \in\{1,2\}$ the following properties are equivalent:
(i) $L$ is a Boolean algebra;
(ii) $+_{1}=+_{2}$;
(iii) $\cdot{ }_{1}={ }_{2}$;
(iv) $+_{i}$ is associative;
(v) $\cdot 2$ is associative;
(vi) ${ }_{\cdot j}$ is distributive with respect to $+_{i}$;
(vii) For every $(a, b) \in L^{2}$ the equation $a+_{i} x=b$ has at most one solution;
(viii) For every $(a, b) \in L^{2}$ the equation $a+{ }_{i} x=b$ has exactly one solution;
(ix) For every $(a, b) \in L^{2}$ the equation $a+{ }_{i} x=b$ has at least one solution.

Proof. See [2] and [5].

## 3. Ring-Like operations in pseudocomplemented semilattices

A pseudocomplemented meet-semilattice (with zero) is an algebra $S=$ $\left(S, \wedge,^{*}, 0\right)$ of type $(2,1,0)$, where $(S, \wedge, 0)$ is a meet-semilattice with smallest element 0 and where each $x \in S$ has a so-called pseudocomplement $x^{*}$, that is a greatest element $y \in S$ with the property $x \wedge y=0$, i. e. for $x, y \in S$ it holds $x \wedge y=0$ iff $y \leq x^{*}$. (The concept of a pseudocomplemented join-semilattice (with one) can be defined dually.)

In this section let $S$ denote an arbitrary, fixed pseudocomplemented meet-semilattice and let $a, b, c$ be arbitrary, fixed elements of $S$.

Put $a \sqcup b:=\left(a^{*} \wedge b^{*}\right)^{*}$ and $1:=0^{*}$.
The following facts are well-known (cf. [4]):
(i) $\quad\left({ }^{*},{ }^{*}\right)$ is a Galois correspondence between $(S, \leq)$ and $(S, \leq)$,
(ii) $a \leq b \Rightarrow a^{*} \geq b^{*}$,
(iii) $a \leq a^{* *}$,
(iv) $a^{* * *}=a^{*}$,
(v) $0^{* *}=0$,
(vi) ${ }^{* *} \in$ End $S$;
(vii) $\theta_{0}:=\operatorname{ker}^{* *} \in \operatorname{Con} S$,
(viii) $B A(S):=\left(S^{*}, \sqcup, \wedge,{ }^{*}, 0,1\right)$ is the greatest Boolean subalgebra of $\left(S, \sqcup, \wedge,{ }^{*}, 0,1\right)$.
We call $B A(S)$ the Boolean algebra induced by $S$.
Remark. (i) $\theta_{0} \in \operatorname{Con}\left(S, \sqcup, \wedge,{ }^{*}, 0,1\right)$ and from the homomorphism theorem it follows that $\left(S, \sqcup, \wedge,{ }^{*}, 0,1\right) / \theta_{0} \cong B A(S)$.
(ii) One can show that the class of all pseudocomplemented meetsemilattices forms a variety which can be defined by the following laws:
(1) $(x \wedge y) \wedge z=x \wedge(y \wedge z)$,
(2) $x \wedge y=y \wedge x$,
(3) $x \wedge x=x$,
(4) $x \wedge 0=0$,
(5) $(x \wedge y)^{*} \wedge\left(x \wedge y^{*}\right)^{*}=x^{*}$,
(6) $0^{* *}=0$,
(7) $x \wedge x^{* *}=x$.

Let $B R(S):=\left(S^{*},+, 0, \cdot, 1\right)$ denote the Boolean ring corresponding to the Boolean algebra $B A(S)$. We call $B R(S)$ the Boolean ring induced by $S$. Then

$$
\begin{aligned}
& a+b=\left(a \wedge b^{*}\right) \sqcup\left(a^{*} \wedge b\right)=(a \sqcup b) \wedge\left(a^{*} \sqcup b^{*}\right), \\
& a b=a \wedge b=(a \sqcup b) \wedge\left(a \sqcup b^{*}\right) \wedge\left(a^{*} \sqcup b\right),
\end{aligned}
$$

if $a, b \in S^{*}$. We now extend these operations from $S^{*}$ to $S$ in several "natural" ways by defining

$$
\begin{aligned}
& a+b:=a^{* *}+b^{* *}, \\
& a+1 b:=\left(a \wedge b^{*}\right) \sqcup\left(a^{*} \wedge b\right), \\
& a+_{2} b:=(a \sqcup b) \wedge\left(a^{*} \sqcup b^{*}\right), \\
& a b:=a^{* *} b^{* *}, \\
& a \cdot \cdot_{1} b:=a \wedge b, \\
& a \cdot 2 b:=(a \sqcup b) \wedge\left(a \sqcup b^{*}\right) \wedge\left(a^{*} \sqcup b\right) .
\end{aligned}
$$

Since $S^{*}$ is a subalgebra of $S$ and $S \sqcup S \subseteq S^{*}$, we have $S+S, S+{ }_{1} S, S+{ }_{2} S, S S, S \cdot{ }_{2} S \subseteq S^{*}$.

Lemma 2. $+=+_{1}=+_{2}$ and $\cdot=\cdot_{2}$
Proof. For $i \in\{1,2\}$ it holds

$$
\begin{aligned}
& a+_{i} b=\left(a+{ }_{i} b\right)^{* *}=a^{* *}++_{i} b^{* *}=a^{* *}+b^{* *}=a+b, \\
& a \cdot{ }_{2} b=\left(a \cdot{ }_{2} b\right)^{* *}=a^{* *} \cdot{ }_{2} b^{* *}=a^{* *} b^{* *}=a b .
\end{aligned}
$$

Obviously,,$+ \cdot$ and $\cdot 1$ are commutative and associative and $\cdot$ is distributive with respect to + .

To $S$, we assign the following ring-like structures: $R(S):=(S,+, 0, \cdot, 1)$ and $P(S):=\left(S,+, 0, \cdot{ }_{1}, 1\right)$.

From the homomorphism theorem, it follows that

$$
\theta_{0} \in \operatorname{Con} R(S) \cap \operatorname{Con} P(S) \text { and } R(S) / \theta_{0} \cong P(S) / \theta_{0} \cong B R(S) \text {. }
$$

Lemma 3. The following identities hold:

$$
\begin{array}{lll}
a+0=a^{* *}, & a 0=0, & a \cdot{ }_{1} 0=0, \\
a+a=0, & a a=a^{* *}, & a \cdot{ }_{1} a=a, \\
a+a^{*}=1, & a a^{*}=0, & a \cdot{ }_{1} a^{*}=0, \\
a+a^{* *}=0, & a a^{* *}=a^{* *}, & a \cdot{ }_{1} a^{* *}=a, \\
a+1=a^{*}, & a 1=a^{* *}, & a \cdot{ }_{1} 1=a .
\end{array}
$$

Proof. Straightforward.
Lemma 4. $(a+b)^{*}=a^{*}+b=a+b^{*}$.
Proof. Indeed, we have

$$
\begin{aligned}
& (a+b)^{*}=(a+b)+1=(a+1)+b=a^{*}+b, \\
& (a+b)^{*}=(a+b)+1=a+(b+1)=a+b^{*} .
\end{aligned}
$$

Corollary. $a+b=(a+b)^{* *}=\left(a^{*}+b\right)^{*}=a^{*}+b^{*}$.
The following lemma characterizes vanishing of the symmetric difference $a+b$ :

Lemma 5. The following properties are equivalent:
(i) $a+b=0$;
(ii) $a \theta_{0} b$;
(iii) There exists a $c \in S$ with $a+c=b+c$;
(iv) $a+x=b+x$ for all $x \in S$.

Proof. (i) $\Rightarrow$ (ii): $a^{* *}=a+0=a+(a+b)=(a+a)+b=0+b=b^{* *}$.
(ii) $\Rightarrow$ (iv): $a+x=a^{* *}+x=b^{* *}+x=b+x$ for all $x \in S$.
(iv) $\Rightarrow$ (iii): Straightforward.
(iii) $\Rightarrow(\mathrm{i}): a+b=(a+b)+0=(a+b)+(c+c)=(a+c)+(b+c)=$ $(a+c)+(a+c)=0$.

Lemma 6. The following properties are equivalent:
(i) $a+b=1$;
(ii) $a \theta_{0} b^{*}$;
(iii) $a^{*} \theta_{0} b$.

Proof. Indeed, we have

$$
\begin{gathered}
a+b=1 \Leftrightarrow(a+b)^{*}=1^{*} \Leftrightarrow a+b^{*}=0 \Leftrightarrow a \theta_{0} b^{*}, \\
a+b=1 \Leftrightarrow(a+b)^{*}=1^{*} \Leftrightarrow a^{*}+b=0 \Leftrightarrow a^{*} \theta_{0} b,
\end{gathered}
$$

according to Lemma 5.
Now we want to characterize Boolean pseudocomplemented semilattices. For this purpose we first need two lemmas. The first of these describes the set of all solutions of a linear equation:

Lemma 7. We have:

$$
\{x \in S \mid a+x=b\}= \begin{cases}\emptyset & \text { if } b \notin S^{*} \\ {[a+b] \theta_{0}} & \text { if } b \in S^{*}\end{cases}
$$

Proof. The first part follows from $S+S \subseteq S^{*}$. In order to prove the second part, assume $b \in S^{*}$ and let $x \in S$. Then

$$
\begin{aligned}
& a+x=b \Leftrightarrow a^{* *}+x^{* *}=b \Leftrightarrow x^{* *}=a^{* *}+b \Leftrightarrow x^{* *}= \\
& =(a+b)^{* *} \Leftrightarrow x \theta_{0} a+b \Leftrightarrow x \in[a+b] \theta_{0}
\end{aligned}
$$

Proposition 8. The following properties hold:
(i) $a b=a \cdot{ }_{1} b \Leftrightarrow a^{* *} \wedge b^{* *} \leq a \wedge b$;
(ii) $(a+b) \cdot 1 c=(a \cdot 1 c)+(b \cdot 1 c) \Leftrightarrow(a+b) \wedge c^{* *} \leq c$;
(iii) $|\{x \in S \mid a+x=b\}| \leq 1 \Leftrightarrow\left(b \notin S^{*}\right.$ or $\left.\left|[a+b] \theta_{0}\right|=1\right)$;
(iv) $|\{x \in S \mid a+x=b\}|=1 \Leftrightarrow\left(b \in S^{*}\right.$ and $\left.\left|[a+b] \theta_{0}\right|=1\right)$;
(v) $|\{x \in S \mid a+x=b\}| \geq 1 \Leftrightarrow b \in S^{*}$.

Proof. (i) is evident.
$\operatorname{Ad}$ (ii): $(a+b) \cdot 1 c=(a+b) \wedge c$ and

$$
\begin{aligned}
\left(a \cdot{ }_{1} c\right)+(b \cdot 1 c) & =(a \wedge c)+(b \wedge c)=((a \wedge c)+(b \wedge c))^{* *}= \\
& =\left(a^{* *} \wedge c^{* *}\right)+\left(b^{* *} \wedge c^{* *}\right)= \\
& =\left(a^{* *}+b^{* *}\right) \wedge c^{* *}=(a+b) \wedge c^{* *} .
\end{aligned}
$$

Hence,

$$
\begin{aligned}
(a+b) \cdot 1 c=(a \cdot 1 c)+(b \cdot 1 c) & \Leftrightarrow(a+b) \wedge c=(a+b) \wedge c^{* *} \Leftrightarrow \\
& \Leftrightarrow(a+b) \wedge c^{* *} \leq(a+b) \wedge c \Leftrightarrow \\
& \Leftrightarrow\left((a+b) \wedge c^{* *} \leq a+b\right. \\
\text { and }(a+b) \wedge c^{* *} \leq c & \Leftrightarrow(a+b) \wedge c^{* *} \leq c .
\end{aligned}
$$

The properties (iii) - (v) follow from Lemma 7 .
Now we are ready to prove the result concerning the characterization of Boolean pseudocomplemented semilattices in terms of ring-like operations.

Theorem 9. The following are equivalent:
(i) $S$ is a Boolean algebra;
(ii) $\cdot={ }_{1}$;
(iii) $\cdot{ }_{1}$ is distributive with respect to +;
(iv) The equation $a+x=b$ has at most one solution;
(v) The equation $a+x=b$ has exactly one solution;
(vi) The equation $a+x=b$ has at least one solution.

Proof. Obviously, (i) $\Rightarrow$ (ii) - (vi) and (v) $\Rightarrow$ (iv), (vi).
(ii) $\Rightarrow$ (i): According to (i) of Proposition 8, $a^{* *} \wedge b^{* *} \leq a \wedge b$. Hence $a \leq a^{* *}=a^{* *} \wedge 1^{* *} \leq a \wedge 1=a$ which implies $a=a^{* *}$.
(iii) $\Rightarrow$ (i): According to (ii) of Proposition $8,(a+b) \wedge c^{* *} \leq c$. Hence, $c \leq c^{* *}=(0+1) \wedge c^{* *} \leq c$ and, therefore, $c=c^{* *}$.
(iv) $\Rightarrow$ (i): According to (iii) of Proposition $8,\left|[a+b] \theta_{0}\right|=1$ if $b \in S^{*}$. Since $c, c^{* *} \in\left[0+c^{* *}\right] \theta_{0}$, we have $c=c^{* *}$.
(vi) $\Rightarrow$ (i) follows from (v) of Proposition 8.

## 4. Ideals in generalizations of pseudocomplemented SEMILATTICES

In the following let $\mathcal{A}$ be an algebra such that there exists a binary operation - on $A$, a unary operation * on $A$ as well as an element 0 of $A$ such that $0 x=x x^{*}=0$ and $x 0^{*}=x$ for all $x \in A$.

Every pseudocomplemented meet-semilattice $S=\left(S, \wedge,{ }^{*}, 0\right)$ can be considered as such an algebra. (Take $A:=S$ and $\cdot:=\wedge$.)

We now define the notions of an ideal of $\mathcal{A}$ and of a congruence kernel of $A$, respectively. But first let us recall the notion of a unary polynomial function on $A$ : A function $p$ from $A$ to $A$ is called a unary polynomial function on $A$ if there exists a positive integer $n$, an $n$-ary term function $t$ on $A$ and $a_{2}, \ldots, a_{n} \in A$ with $p(x)=t\left(x, a_{2}, \ldots, a_{n}\right)$ for all $x \in A$.

Definition 10. Let $B$ be a non-empty subset of the algebra $\mathcal{A}$. $B$ is called an ideal of $\mathcal{A}$ - in signs $B \triangleleft A$ - if for all $a, b, c \in B$ and for every unary polynomial function $p$ on $A, p(a)\left(p(b) c^{*}\right)^{*} \in B . B$ is called a congruence kernel of $\mathcal{A}$ if $B=[0] \theta$ for some $\theta \in \operatorname{Con} A$.

The following theorem holds for arbitrary universal algebras:
Theorem 11. A non-empty subset $B$ of $A$ is a class of some congruence on $\mathcal{A}$ iff for every unary polynomial function $p$ on $A, a, b, p(a) \in B$ implies $p(b) \in B$.
Proof. See [6].
Now, we are able to prove our final result concerning the fact that both notions defined in Definition 10 coincide:

Theorem 12. The ideals of $A$ coincide with the congruence kernels of $\mathcal{A}$.
Proof. Let $I \subseteq A$. First assume $I \triangleleft A$. Let $a, b \in I$, let $p$ be a unary polynomial function on $A$ and assume $p(a) \in I$. Then $p(b)=p(b)\left(p(a)(p(a))^{*}\right)^{*} \in I$. Hence, by Theorem 11, there exists some $c \in A$ and some $\theta \in \operatorname{Con}(\mathcal{A})$ such that $I=[c] \theta$. Let $q$ denote the unary zero polynomial function on $A$. Then $0=q(c)\left(q(c) c^{*}\right)^{*} \in I$. Hence $I=[0] \theta$ which shows that $I$ is a congruence kernel of $\mathcal{A}$.

Conversely, assume $I$ to be a congruence kernel of $\mathcal{A}$. Then there exists some $\alpha \in \operatorname{Con}(\mathcal{A})$ with $I=[0] \alpha$. Let $d, e, f \in I$ and let $r$ be a unary polynomial function on $A$. Then $r(d)\left(r(e) f^{*}\right)^{*} \alpha r(0)\left(r(0) 0^{*}\right)^{*}=0$ which shows $r(d)\left(r(e) f^{*}\right)^{*} \in[0] \alpha=I$. Hence $I \triangleleft A$.

Remark. The ring-like structures $P(S)$ (where $S$ is a pseudocomplemented meet-semilattice), Boolean quasirings (introduced in [3]) and orthopseudorings (introduced in [1]) are also special cases of the algebras considered in this section.

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