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ON FUZZY TOPOLOGICAL BCC-ALGEBRAS¹

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Abstract

We describe properties of subalgebras and BCC-ideals in BCCalgebras with a topology induced by a family of fuzzy sets. **Keywords:** BCC-algebra, fuzzy subalgebra, fuzzy topological subalgebra.

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1. Introduction

In 1966, Y. Imai and K. Iséki (cf. [6]) defined a class of algebras of type (2,0) called *BCK-algebras* which generalize the notion of algebra of sets with the set subtraction as the only fundamental non-nullary operation, on the other hand the notion of implication algebra (cf. [7]). The class of all BCK-algebras is a quasivariety. K. Iséki posed an interesting problem whether the class of BCK-algebras is a variety. That problem was solved by A. Wroński [11] who proved that BCK-algebras do not form a variety. In

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connection with this problem, Y. Komori [8] introduced the notion of BCCalgebras, and W.A. Dudek (cf. [1], [2]) redefined the notion of BCC-algebras by using a dual form of the ordinary definition in the sense of Y. Komori. In [4], W.A. Dudek and X.H. Zhang introduced a new notion of ideals in BCC-algebras and described connections between such ideals and congruences. W.A. Dudek and Y.B. Jun (cf. [3]) considered the fuzzification of ideals in BCC-algebras. They proved that every fuzzy BCC-ideal of a BCCalgebra is a fuzzy BCK-ideal, and showed that the converse is not true by providing a counterexample. They also proved that in a BCC-algebra every fuzzy BCK-ideal is a fuzzy BCC-subalgebra; and that in a BCK-algebra the notion of a fuzzy BCK-ideal and a fuzzy BCC-ideal coincide. The concept of a fuzzy set, which was introduced in [12], provides a natural framework for generalizing many of the concepts of general topology to what might be called fuzzy topological spaces. D.H. Foster [5] combined the structure of a fuzzy topological spaces with that of a fuzzy group, introduced by A. Rosenfeld [10], to formulate the elements of a theory of fuzzy topological groups. In the present paper, we introduce the concept of fuzzy topological subalgebras of BCC-algebras and apply some of Foster's results on homomorphic images and inverse images to fuzzy topological subalgebras.

2. Preliminaries

In the present paper we will use the definition of BCC-algebras in the sense of [2] and [4].

A nonempty set X with a constant 0 and a binary operation denoted by juxtaposition is called a *BCC-algebra* if for all $x, y, z \in X$ the following axioms hold:

- (I) ((xy)(zy))(xz) = 0,
- (II) xx = 0,
- (III) 0x = 0,
- (IV) x0 = x,
- (V) xy = 0 and yx = 0 imply x = y.

In a BCC-algebra, the following holds:

$$(1) (xy)x = 0.$$

Any BCK-algebra is a BCC-algebra, but there are BCC-algebras which are not BCK-algebras (cf. [2]). Note that a BCC-algebra is a BCK-algebra iff it satisfies:

$$(2) (xy)z = (xz)y$$

A nonempty subset S of a BCC-algebra X is called *a subalgebra* of X if it is closed under the BCC-operation. Such subalgebra contains obviously the constant 0 and is a BCC-algebra, but some subalgebras may be also BCK-algebras. Moreover, there are BCC-algebras in which all subalgebras are BCK-algebras (cf. [1]).

We now review some fuzzy logic concepts. Let X be a set. A fuzzy set A in X is characterized by a membership function $\mu_A : X \to [0, 1]$. A fuzzy set is *empty* iff its membership function is identically zero on X. If A and B are two fuzzy sets on X with respective membership functions μ_A and μ_B , then

$$A \subseteq B \iff (\forall x \in X) \ [\mu_A(x) \le \mu_B(x)]$$
$$A = B \iff (\forall x \in X) \ [\mu_A(x) = \mu_B(x)].$$

In the case $A \subset B$ we say that a fuzzy set A is *smaller* than B (cf. [12]).

The union of two fuzzy sets A and B is a fuzzy set C, written as $C = A \cup B$, whose membership function is related to those A and B by

$$(\forall x \in X) \ [\mu_C(x) = \max\{\mu_A(x), \, \mu_B(x)\}]$$

The union of A and B is the smallest fuzzy set containing both A and B.

The *intersection* of two fuzzy sets A and B is a fuzzy set D, written as $D = A \cap B$, whose membership function is related to those of A and B by

$$(\forall x \in X) \ [\mu_D(x) = \min\{\mu_A(x), \mu_B(x)\}].$$

The intersection of A and B is the largest fuzzy set which is contained in both A and B.

As in the case of ordinary sets, A and B are *disjoint* if $A \cap B$ is empty. Note that fuzzy sets in X constitute a distributive lattice with 0 and 1.

Let α be a mapping from the set X to a set Y. Let B be a fuzzy set in Y with membership function μ_B . The *inverse image* of B, denoted $\alpha^{-1}(B)$, is the fuzzy set in X with membership function $\mu_{\alpha^{-1}(B)}(x) = \mu_B(\alpha(x))$ for all $x \in X$. Conversely, let A be a fuzzy set in X with membership function μ_A . Then the *image* of A, denoted $\alpha(A)$, is the fuzzy set in Y such that

$$\mu_{\alpha(A)}(y) = \begin{cases} \sup_{z \in \alpha^{-1}(y)} \mu_A(z), & \text{if } \alpha^{-1}(y) = \{x : \alpha(x) = y\} \text{ is nonempty,} \\ 0, & \text{otherwise.} \end{cases}$$

A fuzzy topology on a set X is a family \mathcal{T} of fuzzy subsets in X which satisfies the following conditions:

- (i) For all $c \in [0, 1]$, $k_c \in \mathcal{T}$, where k_c have constant membership functions with the value c,
- (ii) If $A, B \in \mathcal{T}$, then $A \cap B \in \mathcal{T}$,
- (iii) If $A_j \in \mathcal{T}$ for all $j \in J$, then $\bigcup_{j \in J} A_j \in \mathcal{T}$.

The pair (X, \mathcal{T}) is called a *fuzzy topological space* and members of \mathcal{T} are *open fuzzy subsets*.

Let A be a fuzzy subset in X and \mathcal{T} a fuzzy topology on X. Then the *induced fuzzy topology* on A is the family of fuzzy subsets of A which are the intersection with A of \mathcal{T} -open fuzzy subsets in X. The induced fuzzy topology is denoted by \mathcal{T}_A , and the pair (A, \mathcal{T}_A) is called a *fuzzy subspace* of (X, \mathcal{T}) .

Let (X, \mathcal{T}) and (Y, \mathcal{U}) be two fuzzy topological spaces. A mapping α of (X, \mathcal{T}) into (Y, \mathcal{U}) is *fuzzy continuous* if for each open fuzzy set U in \mathcal{U} , the inverse image $\alpha^{-1}(U)$ is in \mathcal{T} . Conversely, α is *fuzzy open* if for each open fuzzy set V in \mathcal{T} , the image $\alpha(V)$ is in \mathcal{U} .

Let (A, \mathcal{T}_A) and (B, \mathcal{U}_B) be fuzzy subspaces of fuzzy topological spaces (X, \mathcal{T}) and (Y, \mathcal{U}) respectively, and let α be a mapping $(X, \mathcal{T}) \to (Y, \mathcal{U})$. Then α is a mapping of (A, \mathcal{T}_A) into (B, \mathcal{U}_B) if $\alpha(A) \subset B$. Furthermore α is relatively fuzzy continuous if for each open fuzzy set V' in \mathcal{U}_B , the intersection $\alpha^{-1}(V') \cap A$ is in \mathcal{T}_A . Conversely, α is relatively fuzzy open if for each open fuzzy set U' in \mathcal{T}_A , the image $\alpha(U')$ is in \mathcal{U}_B .

Lemma 2.1 [5]. Let (A, T_A) , (B, U_B) be fuzzy subspaces of fuzzy topological spaces (X, T), (Y, U) respectively, and let α be a fuzzy continuous mapping of (X, T) into (Y, U) such that $\alpha(A) \subset B$. Then α is a relatively fuzzy continuous mapping of (A, T_A) into (B, U_B) .

3. Fuzzy topological subalgebras

Definition 3.1 [3]. A fuzzy subset F in a BCC-algebra X with membership function μ_F is called a *fuzzy subalgebra* of X if

$$(\forall x, y \in X) \ [\mu_F(xy) \ge \min\{\mu_F(x), \mu_F(y)\}].$$

Example 3.2 [3]. Let $X = \{0, a, b, c, d\}$ be a set with Cayley table of the binary operation as follows:

•	0	a	b	c	d
0	0	0	0	0	0
a	a	0	a	0	0
b	b	b	0	0	0
c	c	c	a	0	0
d	d	с	d	c	0
Table 1					

Then X is a BCC-algebra ([4]) which is not a BCK-algebra since $(da)b \neq (db)a$. By routine calculations we know that a fuzzy subset F in X with membership function μ_F defined by $\mu_F(d) = 0.4$ and $\mu_F(x) = 0.8$ for all $x \neq d$ is a fuzzy subalgebra of X.

Proposition 3.3. Let α be a homomorphism of a BCC-algebra X into a BCC-algebra Y and G a fuzzy subalgebra of Y with membership function μ_G . Then the inverse image $\alpha^{-1}(G)$ of G is a fuzzy subalgebra of X.

Proof. Let $x, y \in X$. Then

$$\mu_{\alpha^{-1}(G)}(xy) = \mu_{G}(\alpha(xy)) = \mu_{G}(\alpha(x)\alpha(y)) \\
\geq \min\{\mu_{G}(\alpha(x)), \, \mu_{G}(\alpha(y))\} \\
= \min\{\mu_{\alpha^{-1}(G)}(x), \, \mu_{\alpha^{-1}(G)}(y)\}.$$

This completes the proof.

For images, we need the following definition [10].

Definition 3.4. A fuzzy subset F in a BCC-algebra X with membership function μ_F is said to have the *sup property* if, for any subset $T \subset X$, there exists $t_0 \in T$ such that

$$\mu_F(t_0) = \sup_{t \in T} \mu_F(t) \,.$$

Proposition 3.5. Let α be a homomorphism of a BCC-algebra X onto a BCC-algebra Y and let F be a fuzzy subalgebra of X that has the sup property. Then the image $\alpha(F)$ of F is a fuzzy subalgebra of Y.

Proof. For $u, v \in Y$, let $x_0 \in \alpha^{-1}(u), y_0 \in \alpha^{-1}(v)$ such that

$$\mu_F(x_0) = \sup_{t \in \alpha^{-1}(u)} \mu_F(t), \quad \mu_F(y_0) = \sup_{t \in \alpha^{-1}(v)} \mu_F(t)$$

Then, by the definition of $\mu_{\alpha(F)}$, we have

$$\mu_{\alpha(F)}(uv) = \sup_{t \in \alpha^{-1}(uv)} \mu_F(t) \ge \mu_F(x_0y_0)$$

$$\ge \min\{\mu_F(x_0), \ \mu_F(y_0)\}$$

$$= \min\{\sup_{t \in \alpha^{-1}(u)} \mu_F(t), \ \sup_{t \in \alpha^{-1}(v)} \mu_F(t)\}$$

$$= \min\{\mu_{\alpha(F)}(u), \ \mu_{\alpha(F)}(v)\},$$

ending the proof.

For any BCC-algebra X and any element $a \in X$ we use a_r to denote the selfmap of X defined by $a_r(x) = xa$ for all $x \in X$.

Definition 3.6. Let X be a BCC-algebra and \mathcal{T} a fuzzy topology on X. Let F be a fuzzy subalgebra of X with induced topology \mathcal{T}_F . Then F is called a *fuzzy topological subalgebra* of X if for each $a \in X$ the mapping $a_r: x \mapsto xa$ of $(F, \mathcal{T}_F) \to (F, \mathcal{T}_F)$ is relatively fuzzy continuous.

Theorem 3.7. Given BCC-algebras X, Y and a homomorphism $\alpha: X \to Y$, let \mathcal{U} and \mathcal{T} be fuzzy topologies on Y and X respectively, such that $\mathcal{T} = \alpha^{-1}(\mathcal{U})$. Let G be a fuzzy topological subalgebra of Y with membership function μ_G . Then $\alpha^{-1}(G)$ is a fuzzy topological subalgebra of X with membership function $\mu_{\alpha^{-1}(G)}$.

Proof. We have to show that, for each $a \in X$, the mapping

 $a_r: x \mapsto xa$ of $(\alpha^{-1}(\mathbf{G}), \mathcal{T}_{\alpha^{-1}(\mathbf{G})}) \to (\alpha^{-1}(\mathbf{G}), \mathcal{T}_{\alpha^{-1}(\mathbf{G})})$

is relatively fuzzy continuous. Let U be an open fuzzy set in $\mathcal{T}_{\alpha^{-1}(G)}$ on $\alpha^{-1}(G)$. Since α is a fuzzy continuous mapping of (X, \mathcal{T}) into (Y, \mathcal{U}) , it follows from Lemma 2.1 that α is a relatively fuzzy continuous mapping of $(\alpha^{-1}(G), \mathcal{T}_{\alpha^{-1}(G)})$ into (G, \mathcal{U}_G) . Note that there exists an open fuzzy set $V \in \mathcal{U}_G$ such that $\alpha^{-1}(V) = U$. The membership function of $a_r^{-1}(U)$ is given by

$$\mu_{a_r^{-1}(U)}(x) = \mu_U(a_r(x)) = \mu_U(xa) = \mu_{\alpha^{-1}(V)}(xa) = \mu_V(\alpha(xa)) = \mu_V(\alpha(x)\alpha(a)).$$

As G is a fuzzy topological subalgebra of Y, the mapping

 $b_r: y \mapsto yb$ of $(G, \mathcal{U}_G) \to (G, \mathcal{U}_G)$

is relatively fuzzy continuous for each $b \in Y$. Hence

$$\begin{split} \mu_{a_r^{-1}(U)}(x) &= \mu_V(\alpha(x)\alpha(a)) = \mu_V(\alpha(a)_r(\alpha(x))) \\ &= \mu_{\alpha(a)_r^{-1}(V)}(\alpha(x)) = \mu_{\alpha^{-1}(\alpha(a)_r^{-1}(V))}(x) \,, \end{split}$$

which implies that $a_r^{-1}(U) = \alpha^{-1}(\alpha(a)_r^{-1}(V))$ so that

$$a_r^{-1}(U) \cap \alpha^{-1}(G) = \alpha^{-1}(\alpha(a)_r^{-1}(V)) \cap \alpha^{-1}(G)$$

is open in the induced fuzzy topology on $\alpha^{-1}(G)$. This completes the proof.

We say that the membership function μ_G of a fuzzy subalgebra G of a BCC-algebra X is α -invariant [10] if, for all $x, y \in X$, $\alpha(x) = \alpha(y)$ implies $\mu_G(x) = \mu_G(y)$.

Clearly, a homomorphic image $\alpha(G)$ of G is then a fuzzy subalgebra.

Theorem 3.8. Given BCC-algebras X, Y and a homomorphism α of X onto Y, let \mathcal{T} be a fuzzy topology on X and \mathcal{U} be the fuzzy topology on Y such that $\alpha(\mathcal{T}) = \mathcal{U}$. Let F be a fuzzy topological subalgebra of X. If the membership function μ_F of F is α -invariant, then $\alpha(F)$ is a fuzzy topological subalgebra of Y.

Proof. It is sufficient to show that the mapping

 $b_r: y \mapsto yb$ of $(\alpha(F), \mathcal{U}_{\alpha(F)}) \to (\alpha(F), \mathcal{U}_{\alpha(F)})$

is relatively fuzzy continuous for each $b \in Y$. Note that α is relatively fuzzy open; for if $U' \in \mathcal{T}_F$, there exists $U \in \mathcal{T}$ such that $U' = U \cap F$ and by the α -invariance of μ_F ,

$$\alpha(U') = \alpha(U) \cap \alpha(F) \in \mathcal{U}_{\alpha(F)}$$

Let V' be an open fuzzy set in $\mathcal{U}_{\alpha(F)}$. Since α is onto, for each $b \in Y$ there exists $a \in X$ such that $b = \alpha(a)$. Hence

$$\begin{split} \mu_{\alpha^{-1}(b_r^{-1}(V'))}(x) &= \mu_{\alpha^{-1}(\alpha(a)_r^{-1}(V'))}(x) = \mu_{\alpha(a)_r^{-1}(V')}(\alpha(x)) \\ &= \mu_{V'}(\alpha(a)_r(\alpha(x))) = \mu_{V'}(\alpha(x)\alpha(a)) \\ &= \mu_{V'}(\alpha(xa)) = \mu_{\alpha^{-1}(V')}(xa) \\ &= \mu_{\alpha^{-1}(V')}(a_r(x)) = \mu_{a_r^{-1}(\alpha^{-1}(V'))}(x) \,, \end{split}$$

which implies that $\alpha^{-1}(b_r^{-1}(V')) = a_r^{-1}(\alpha^{-1}(V')).$

By hypothesis, $a_r : x \mapsto xa$ is a relatively fuzzy continuous mapping: $(F, \mathcal{T}_F) \to (F, \mathcal{T}_F)$ and α is a relatively fuzzy continuous mapping: $(F, \mathcal{T}_F) \to (\alpha(F), \mathcal{U}_{\alpha(F)})$. Hence

$$\alpha^{-1}(b_r^{-1}(V')) \cap G = a_r^{-1}(\alpha^{-1}(V')) \cap F$$

is open in \mathcal{T}_F . Since α is relatively fuzzy open,

$$\alpha(\alpha^{-1}(b_r^{-1}(V'))\cap F)=b_r^{-1}(V')\cap\alpha(F)$$

is open in $\mathcal{U}_{\alpha(F)}$. This completes the proof.

4. Fuzzy topological ideals

First we briefly review the concepts of fuzzy ideals of BCC-algebras (cf. [3]).

Definition 4.1. A fuzzy subset A in X with membership function μ_A is called a *fuzzy BCK-ideal* of X if

- (a) $(\forall x \in X) \ [\mu_A(0) \ge \mu_A(x)],$
- (b) $(\forall x, y \in X) \ [\mu_A(x) \ge \min\{\mu_A(xy), (\mu_A(y))\}].$

Definition 4.2. A fuzzy subset A in X with membership function μ_A is called a *fuzzy BCC-ideal* of X if

- (a) $(\forall x \in X) \ [\mu_A(0) \ge \mu_A(x)],$
- (c) $(\forall x, y, z \in X) \ [\mu_A(xz) \ge \min\{\mu_A((xy)z), \mu_A(y)\}].$

Putting z = 0 in (c) we see that a fuzzy BCC-ideal is a fuzzy BCK-ideal. The converse is not true, in general (cf. [3]).

Proposition 4.3. Let α be a homomorphism of a BCC-algebra X into a BCC-algebra Y and B a fuzzy BCC-ideal of Y with membership function μ_B . Then the inverse image $\alpha^{-1}(B)$ of B is a fuzzy BCC-ideal of X.

Proof. Since α is a homomorphism of $(X, \cdot, 0)$ into (Y, \cdot, θ) , then $\alpha(0) = \theta$ and, by the assumption, $\mu_B(\alpha(0)) = \mu_B(\theta) \ge \mu_B(y)$ for every $y \in Y$. In particular, $\mu_B(\alpha(0)) \ge \mu_B(\alpha(x))$ for $x \in X$. Thus $\mu_{\alpha^{-1}(B)}(0) \ge \mu_{\alpha^{-1}(B)}(x)$, which proves (a).

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Now, let $x, y, z \in X$. Then, by the assumption on μ_B , we have

$$\mu_{\alpha^{-1}(B)}(xz) = \mu_B(\alpha(xz)) = \mu_B(\alpha(x)\alpha(z))$$

$$\geq \min\{\mu_B((\alpha(x)u)\alpha(z)), \ \mu_B(u)\}$$

for all $\alpha(x), u, \alpha(z) \in Y$. In particular, for $u = \alpha(y)$, this gives

$$\begin{aligned} \mu_{\alpha^{-1}(B)}(xz) &\geq \min\{\mu_B((\alpha(x)\alpha(y))\alpha(z)), \ \mu_B(\alpha(y))\} \\ &= \min\{\mu_B(\alpha((xy)z)), \ \mu_B(\alpha(y))\} \\ &= \min\{\mu_{\alpha^{-1}(B)}((xy)z), \ \mu_{\alpha^{-1}(B)}(y)\}, \end{aligned}$$

which proves (c). The proof is complete.

Putting in the above proof z = 0, we obtain

Corollary 4.4. Let α be a homomorphism of a BCC-algebra X into a BCC-algebra Y and B a fuzzy BCK-ideal of Y with membership function μ_B . Then the inverse image $\alpha^{-1}(B)$ of B is a fuzzy BCK-ideal of X.

Since any fuzzy BCC-ideal (BCK-ideal) is a fuzzy subalgebra (cf. [3]), then a fuzzy topological BCC-ideal (BCK-ideal) is a fuzzy topological subalgebra and as a consequence of the above results and Theorem 3.7, we obtain

Corollary 4.5. Given BCC-algebras X, Y and a homomorphism $\alpha: X \to Y$, let \mathcal{U} and \mathcal{T} be fuzzy topologies on Y and X respectively, such that $\mathcal{T} = \alpha^{-1}(\mathcal{U})$. Let G be a fuzzy topological BCC-ideal (BCK-ideal) of Y with membership function μ_G . Then $\alpha^{-1}(G)$ is a fuzzy topological BCC-ideal (BCK- ideal) of X with membership function $\mu_{\alpha^{-1}(G)}$.

It is not difficult to see that if the membership function μ_G of a fuzzy BCC-ideal (BCK-ideal) G of a BCC-algebra X is α -invariant, then a homomorphic image $\alpha(G)$ of G is a fuzzy BCC-ideal (BCK-ideal). Thus from Theorem 3.8 it follows

Corollary 4.6. Given BCC-algebras X, Y and a homomorphism α of X onto Y, let \mathcal{T} be a fuzzy topology on X and \mathcal{U} be the fuzzy topology on Y such that $\alpha(\mathcal{T}) = \mathcal{U}$. Let F be a fuzzy topological BCC-ideal (BCK-ideal) of X. If the membership function μ_F of F is α -invariant, then $\alpha(F)$ is a fuzzy topological BCC-ideal (BCK-ideal) of Y.

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