

## SPECTRA OF ABELIAN WEAKLY ASSOCIATIVE LATTICE GROUPS

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### Abstract

The notion of a weakly associative lattice group is a generalization of that of a lattice ordered group in which the identities of associativity of the lattice operations join and meet are replaced by the identities of weak associativity. In the paper, the spectral topologies on the sets of straightening ideals (and on some of their subsets) of abelian weakly associative lattice groups are introduced and studied.

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A *weakly associative lattice (wa-lattice)* is an algebra  $A = (A, \vee, \wedge)$  of signature  $\langle 2, 2 \rangle$  satisfying the following identities:

$$\begin{array}{ll} \text{(I)} & x \vee x = x; \qquad x \wedge x = x; \\ \text{(C)} & x \vee y = y \vee x; \qquad x \wedge y = y \wedge x; \\ \text{(Abs)} & x \vee (x \wedge y) = x; \qquad x \wedge (x \vee y) = x; \\ \text{(WA)} & ((x \wedge z) \vee (y \wedge z)) \vee z = z; \quad ((x \vee z) \wedge (y \vee z)) \wedge z = z. \end{array}$$

The *wa-lattices* have been introduced by E. Fried in [2] and by H. L. Skala in [9] and [10] as non-associative generalizations of lattices. For this, the identities of associativity of the binary operations  $\vee$  and  $\wedge$  are replaced by

weaker identities (WA) of weak associativity. Nevertheless, similarly as for lattices, the notion of a *wa*-lattice makes possible to define a binary relation  $\leq$  on  $A$  such that

$$(\forall x, y \in A) [x \leq y \iff_{df} x \wedge y = x].$$

The relation  $\leq$  is reflexive and antisymmetric (it is then called a *semi-order* on  $A$ ) and for any  $a, b \in A$  there exist  $\sup\{a, b\}$  (i. e. the join of  $a$  and  $b$ ) and  $\inf\{a, b\}$  (i. e. the meet of  $a$  and  $b$ ) in  $A$ . Hence we can equivalently view any *wa*-lattice as a set with a binary relation.

Let us recall that a *tournament* is a set  $T \neq \emptyset$  with a reflexive and antisymmetric binary relation  $\leq$  satisfying

$$(\forall x, y \in T) [x \leq y \text{ or } y \leq x].$$

Therefore any tournament is a special case of a *wa*-lattice (and also a special case of a directed graph).

Let  $G = (G; +, -, 0)$  be a group and  $(G; \vee, \wedge)$  be a *wa*-lattice. Then the system  $G = (G; +, -, 0, \vee, \wedge)$  is called a *weakly associative lattice group* (*wal-group*) if  $G$  satisfies the following mutually equivalent identities and quasi-identity:

$$\begin{aligned} (D_{\vee}) \quad & x + (y \vee z) + v = (x + y + v) \vee (x + z + v); \\ (D_{\wedge}) \quad & x + (y \wedge z) + v = (x + y + v) \wedge (x + z + v); \\ (M) \quad & y \leq z \implies x + y + u \leq x + z + u. \end{aligned}$$

(See [7] and [8]. In [10], a *wal-group* is called a *trellis group*.)

If  $G$  is a *wal-group*, then  $G^+ = \{x \in G : 0 \leq x\}$  is called the *positive cone* of  $G$  and its elements are called *positive*.

If  $G$  is a *wal-group* such that the *wa*-lattice  $(G; \leq)$  is a tournament, then  $G$  is called a *totally semi-ordered group* (*to-group*).

In contrast to lattice ordered groups (*l-groups*) and linearly ordered groups (*o-groups*) that are torsion free, there are many non-trivial finite *wal-groups* and *to-groups*. The groups that admit a total semi-order have been characterized in [3].

**Example.** Let  $T$  be a tournament (finite or infinite) and  $Aut(T)$  be the set of all automorphisms of  $T$ . Then  $Aut(T)$  is a group with respect to the composition of mappings. For  $f, g \in Aut(T)$  we put

$$f \leq g \iff_{df} (\forall t \in T) [f(t) \leq g(t)].$$

Then the group  $\text{Aut}(T)$  with  $\leq$  is a *wal*-group.

It is obvious that the class  $\mathcal{G}_{wal}$  of all *wal*-groups is a variety of algebras of type  $\langle +, -, 0, \vee, \wedge \rangle$  of signature  $\langle 2, 1, 0, 2, 2 \rangle$ . Some properties of the variety  $\mathcal{G}_{wal}$  and of the lattice **WAL** of all subvarieties of  $\mathcal{G}_{wal}$  have been investigated in [8]. (For instance, the variety  $\mathcal{G}_{wal}$  is arithmetical, the complete lattice **WAL** is distributive, and **WAL** contains the lattice **L** of all varieties of *l*-groups as a complete  $\wedge$ -subsemilattice.)

Let  $G$  be a *wal*-group. Then every its subalgebra  $A$  is called a *wal*-subgroup of  $G$ . (That means,  $A$  is a *wal*-subgroup of  $G$  if and only if it is both a subgroup and a *wa*-sublattice of  $G$ .) A normal convex *wal*-subgroup  $A$  is called a *wal*-ideal of  $G$  if it satisfies the following mutually equivalent conditions:

- (a)  $(\forall a, b \in A) (\forall x, y \in G) [x \leq a \text{ and } y \leq b \implies (\exists c \in A) [x \vee y \leq c]]$ ;
- (b)  $(\forall a, b, c \in A) (\forall x, y \in G) [x \leq a \text{ and } y \leq b \implies (x \vee y) \vee c \in A]$ .

By [7], *wal*-ideals coincide with kernels of homomorphisms of *wal*-groups. Hence the *wal*-ideals of any *wal*-group  $G$  form (with respect to ordering by set-inclusion) a complete lattice  $\mathcal{I}(G)$  which is by [8], Theorem 4, distributive. Moreover, by [8], Proposition 2,  $\mathcal{I}(G)$  is a complete sublattice of the lattice of normal subgroups of the group  $G$ , hence if  $I_\gamma \in \mathcal{I}(G)$ ,  $\gamma \in \Gamma$ , then

$$\inf\{I_\gamma : \gamma \in \Gamma\} = \bigwedge_{\gamma \in \Gamma} I_\gamma = \bigcap_{\gamma \in \Gamma} I_\gamma,$$

$$\sup\{I_\gamma : \gamma \in \Gamma\} = \bigvee_{\gamma \in \Gamma} I_\gamma = \sum_{\gamma \in \Gamma} I_\gamma.$$

If  $A$  is a *wal*-ideal of a *wal*-group  $G$ , then we can define a semi-order on the factor group  $G/A$  as follows:

$$x + A \leq y + A \iff_{df} (\exists a \in A) [x + a \leq y].$$

Then  $G/A$  with this semi-order is a *wal*-group.

A *wal*-ideal  $A$  is called *straightening* in  $G$  if it satisfies the following mutually equivalent conditions (see [7]):

- (1) If  $x, y \in G$  and  $0 \leq x \wedge y \in A$ , then  $x \in A$  or  $y \in A$ ;
- (2) If  $x, y \in G$  and  $x \wedge y = 0$ , then  $x \in A$  or  $y \in A$ ;
- (3)  $G/A$  is a *to*-group.

Further, let  $G$  be an abelian *wal*-group. Then a *wal*-ideal  $A$  of  $G$  is called a *prime ideal* of  $G$  if it is a finitely meet-irreducible element in the lattice  $\mathcal{I}(G)$  of *wal*-ideals of  $G$ , i. e. if it satisfies

$$(4) \quad (\forall I, J \in \mathcal{I}(G)) [I \cap J = A \implies I = A \text{ or } J = A].$$

By [7], Theorem 2.2, every straightening *wal*-ideal  $A$  of  $G$  satisfies the condition

$$(5) \quad \{I \in \mathcal{I}(G) : A \subseteq I\} \text{ is a linearly ordered set,}$$

and every  $A \in \mathcal{I}(G)$  that satisfies (5) is a prime ideal of  $G$ .

Note that, in contrast to  $l$ -groups where all conditions (1) – (5) are equivalent, there are prime ideals of *wal*-groups which are not straightening (see below.)

**Remark.** Let  $A$  be a prime ideal of an abelian *wal*-group  $G$  and  $I, J \in \mathcal{I}(G)$ . Let us suppose that  $I \cap J \subseteq A$ . Since by Theorem 4 of [8], the lattice  $\mathcal{I}(G)$  is distributive, we have

$$A = A \vee (I \cap J) = (A \vee I) \cap (A \vee J),$$

thus  $A = A \vee I$  or  $A = A \vee J$ , therefore  $I \subseteq A$  or  $J \subseteq A$ .

Hence every prime *wal*-ideal  $A$  satisfies the condition

$$(6) \quad (\forall I, J \in \mathcal{I}(G)) [I \cap J \subseteq A \implies I \subseteq A \text{ or } J \subseteq A].$$

It is obvious that every *wal*-ideal  $A$  which satisfies (6) is a prime ideal, and thus the conditions (4) and (6) are equivalent.

If  $G$  is an abelian *wal*-group and  $I \in \mathcal{I}(G)$ , then  $I$  is called *regular* if  $I = \bigcap_{\gamma \in \Gamma} I_\gamma$  ( $I_\gamma \in \mathcal{I}(G)$ ) implies the existence of  $\gamma_0 \in \Gamma$  such that  $I = I_{\gamma_0}$ . Obviously every regular *wal*-ideal is prime.

In this paper, *spectra* of abelian *wal*-groups, i. e. topological spaces of sets of their straightening *wal*-ideals, are investigated. (Some spectra of  $l$ -groups have been studied in [6].) For necessary results concerning  $l$ -groups and  $o$ -groups see, e.g., [1], [4], [5].

Let  $G$  be an abelian *wal*-group. Let us denote by  $\text{Spec}(G)$  the set of proper straightening *wal*-ideals of  $G$ . If  $M \subseteq G$ , put

$$\begin{aligned} S(M) &= \{P \in \text{Spec}(G) : M \not\subseteq P\}, \\ H(M) &= \{P \in \text{Spec}(G) : M \subseteq P\}. \end{aligned}$$

If  $M = \{a\}$ , then we will denote  $S(a) = S(\{a\})$  and  $H(a) = H(\{a\})$ .

Let for any  $M \subseteq G$ ,  $I(M)$  denote the *wal*-ideal of  $G$  generated by  $M$ . It is obvious that for any  $P \in \text{Spec}(G)$  we have  $M \subseteq P$  if and only if  $I(M) \subseteq P$ , hence  $S(M) = S(I(M))$  and  $H(M) = H(I(M))$ . Therefore, we will consider only  $S(I)$  and  $H(I)$  for  $I \in \mathcal{I}(G)$  and  $S(a)$  and  $H(a)$  for  $a \in G$ .

**Lemma 1.** *If  $G$  is an abelian wal-group then*

1.  $S(0) = \emptyset$ ,  $S(G) = \text{Spec}(G)$ ;
2.  $(\forall I, J \in \mathcal{I}(G)) [S(I \cap J) = S(I) \cap S(J)]$ ;
3.  $(\forall I_\gamma \in \mathcal{I}(G)) [S(\bigvee_{\gamma \in \Gamma} I_\gamma) = \bigcup_{\gamma \in \Gamma} S(I_\gamma)]$ ;
4.  $(\forall 0 \leq a, b \in G) [S(a \vee b) = S(a) \cup S(b)]$ ;
5.  $(\forall 0 \leq a, b \in G) [S(a \wedge b) = S(a) \cap S(b)]$ .

**Proof.**

1. Obvious.
2. Let  $I, J \in \mathcal{I}(G)$ ,  $P \in \text{Spec}(G)$ . Since  $P$  satisfies the condition (6), we get that  $I \cap J \not\subseteq P$  if and only if  $I \not\subseteq P$  and  $J \not\subseteq P$ . Hence  $S(I \cap J) = S(I) \cap S(J)$ .
3. Let  $I_\gamma \in \mathcal{I}(G)$ ,  $\gamma \in \Gamma$ , and  $P \in \text{Spec}(G)$ . Let  $\bigvee_{\gamma \in \Gamma} I_\gamma \not\subseteq P$ . Then there exists  $\gamma_0 \in \Gamma$  such that  $I_{\gamma_0} \not\subseteq P$ . The converse implication is valid too, so  $S(\bigvee_{\gamma \in \Gamma} I_\gamma) = \bigcup_{\gamma \in \Gamma} S(I_\gamma)$ .
4. Let  $0 \leq a, b \in G$ ,  $P \in \text{Spec}(G)$ . If  $P \in S(a) \cup S(b)$  then  $a \notin P$  or  $b \notin P$ . If  $a \vee b \in P$ , then from  $0 \leq a, b \leq a \vee b$  we get thus  $a \in P$  and  $b \in P$ , a contradiction. Therefore  $S(a) \cup S(b) \subseteq S(a \vee b)$ .  
Conversely, let  $Q \in S(a \vee b)$ . Then  $a \vee b \notin Q$ . If  $a, b \in Q$ , then  $a \vee b \in Q$ , a contradiction. Hence  $a \notin Q$  or  $b \notin Q$ , i. e.  $Q \in S(a) \cup S(b)$ , and so  $S(a \vee b) \subseteq S(a) \cup S(b)$ .
5. Let  $0 \leq a, b \in G$  and  $P \in \text{Spec}(G)$ . If  $P \in S(a) \cap S(b)$ , then  $a \notin P$  and  $b \notin P$ . But  $P$  is a straightening *wal*-ideal, hence, if  $0 \leq a \wedge b \in P$ , then, by [7],  $a \in P$  or  $b \in P$ , a contradiction. Thus  $S(a) \cap S(b) \subseteq S(a \wedge b)$ .  
Conversely, let  $Q \in S(a \wedge b)$ , i. e.  $a \wedge b \notin Q$ . If  $a \in Q$  then, because  $0 \leq a \wedge b \leq a$ , the convexity of  $Q$  implies  $a \wedge b \in Q$ , a contradiction.

Hence  $a \notin Q$ . Similarly we can prove that  $b \notin Q$ . Therefore  $S(a \wedge b) \subseteq S(a) \cap S(b)$ . ■

The following theorem is now an immediate consequence.

**Theorem 2.** *If  $G$  is an abelian  $wal$ -group and  $\text{Spec}(G)$  is the set of all proper straightening  $wal$ -ideals of  $G$ , then the sets  $S(I)$ , where  $I$  is an arbitrary  $wal$ -ideal in  $G$ , form a topology of  $\text{Spec}(G)$ .* ■

**Definition.** The topology of  $\text{Spec}(G)$  with the open sets  $S(I)$ , where  $I \in \mathcal{I}(G)$ , is called the *spectral topology* of an abelian  $wal$ -group  $G$ . The corresponding topological space is called the *spectrum* of  $G$ .

Recall that for abelian  $l$ -groups, straightening and prime ideals coincide. Now we will show that for  $wal$ -groups it is not true in general, but that there are  $wal$ -groups not being  $l$ -groups for which every prime ideal is always straightening.

**Example.** a) (See also [7]) Let  $G$  be the additive group of integers and  $G^+ = \{0, 1, 2, 4, \dots, 2n, \dots\}$ . Thus  $G$  is a  $wal$ -group which is neither an  $l$ -group nor a  $to$ -group. Let us consider the direct product  $G \times G$ . Set  $H = \{(x, 0) : x \in \mathbb{Z}\}$ . Then  $H$  is a  $wal$ -ideal of  $G \times G$  which is prime (because the only  $wal$ -ideal of  $G \times G$  that strictly contains  $H$  is  $G \times G$ ), but  $H$  is not straightening. Namely,  $(1, 4) \wedge (4, 1) = (0, 0)$ , but neither  $(1, 4)$  nor  $(4, 1)$  belongs to  $H$ .

b) Let  $G = (\mathbb{Z}, +)$  and  $G^+ = \{0, 1, -2, 3, 4, -5, \dots, 3n, 3n + 1, -(3n + 2), \dots\}$ . Then  $G$  has a unique non-trivial  $wal$ -ideal  $[3] = \text{grp}(3)$ , which is evidently prime. Obviously  $G/[3] \cong \mathbb{Z}_3$ , where  $\mathbb{Z}_3$  is the group of numbers  $0, 1, 2$  with the addition mod(3) and with  $\mathbb{Z}_3^+ = \{0, 1\}$ . Since  $\mathbb{Z}_3$  is a  $to$ -group,  $[3]$  is straightening in  $G$ . Therefore  $G$  is a  $wal$ -group (that is not an  $l$ -group) in which prime and straightening  $wal$ -ideals coincide.

**Theorem 3.** *If  $G$  is an abelian  $wal$ -group in which every its prime  $wal$ -ideal is straightening, then the mapping  $S: \mathcal{I} \rightarrow \text{Spec}(G)$  is an isomorphism of the lattice  $\mathcal{I}(G)$  onto the lattice of all open sets in  $\text{Spec}(G)$ .*

**Proof.** Let  $G \in \mathcal{Ab}_{wal}$ . Then by Lemma 1,  $S$  is a surjective lattice homomorphism. Further by [7], Corollary 2.5, every  $wal$ -ideal is an intersection of regular  $wal$ -ideals, and since every regular  $wal$ -ideal is prime, we have for any  $I \in \mathcal{I}(G)$ ,

$$I = \bigcap \{P : P \in H(I)\}.$$

Thus, if  $I, J \in \mathcal{I}(G)$  and  $S(I) = S(J)$ , then

$$I = \bigcap \{P : P \in H(I)\} = \bigcap \{Q : Q \in H(J)\} = J. \quad \blacksquare$$

Let  $G$  be any *wal*-group and  $a \in G$ . Then by the *absolute value* of  $a$  we will mean the element  $|a| = (a \vee 0) \vee (-a \vee 0)$ . It holds:

**Proposition 4.** *If  $G$  is a *wal*-group and  $a \in G$ , then  $I(a) = I(|a|)$ .*

**Proof.** Let  $I \in \mathcal{I}(G)$  and  $|a| \in I$ . Then  $0 \leq a \vee 0 \leq |a|$ , hence from the convexity of  $I$  we get  $a \vee 0, -a \vee 0 \in I$ . By [7], Proposition 1.5,  $a = (a \vee 0) - (-a \vee 0)$ , thus  $a \in I$ .

Conversely, let  $a \in I \in \mathcal{I}(G)$ . Then also  $|a| = (a \vee 0) \vee (-a \vee 0) \in I$ .  $\blacksquare$

**Example.** There exist *wal*-groups such that their positive cones are their *wa*-sublattices but also others which fail this property.

- a) It is obvious that for every *to*-group (and so also for every representable *wal*-group)  $G$ , its positive cone  $G^+$  is a *wa*-sublattice of  $G$ .
- b) Let us consider once more  $G = (\mathbb{Z}, +)$  with  $G^+ = \{0, 1, 2, 4, \dots, 2n, \dots\}$ . Then  $1, 4 \in G^+$  but  $1 \vee 4 = 5 \notin G^+$ .

**Corollary 5.** *If  $G^+$  is a *wa*-sublattice of  $G$  then every principal *wal*-ideal in  $G$  is generated by a positive element.*

**Theorem 6.** *If  $G$  is an abelian *wal*-group such that  $G^+$  is a *wa*-sublattice of  $G$ , then the sets  $S(a)$ , where  $a \in G$ , form a basis of open sets of the spectrum of the *wal*-group which is stable under finite unions and intersections.*

**Proof.** If  $I \in \mathcal{I}(G)$ , then by Lemma 1,

$$S(I) = S\left(\bigvee_{a \in I} I(a)\right) = \bigcup_{a \in I} S(a),$$

hence the sets  $S(a)$  form a basis in  $\text{Spec}(G)$ . The second assertion is a consequence of Lemma 1 and Proposition 4.  $\blacksquare$

**Theorem 7.** a) *If  $G$  is an abelian *wal*-group such that every its prime *wal*-ideal is straightening, then  $S(a)$  is compact in  $\text{Spec}(G)$  for every  $a \in G$ .*

b) *If, moreover,  $G^+$  is a *wa*-sublattice of  $G$  and  $B$  is an open compact set in  $\text{Spec}(G)$ , then  $B = S(a)$  for some  $a \in G$ .*

**Proof.** a) Let  $G \in \mathcal{Ab}_{wal}$ ,  $a \in G$ ,  $I_\gamma \in \mathcal{I}(G)$ ,  $\gamma \in \Gamma$ . Put

$$S(a) \subseteq \bigcup_{\gamma \in \Gamma} S(I_\gamma) = S\left(\bigvee_{\gamma \in \Gamma} I_\gamma\right).$$

Then by Theorem 3,  $a \in \bigvee_{\gamma \in \Gamma} I_\gamma$ . By Proposition 2 of [8],  $\bigvee_{\gamma \in \Gamma} I_\gamma = \sum_{\gamma \in \Gamma} I_\gamma$ , hence there exist  $\gamma_1, \dots, \gamma_k \in \Gamma$  such that

$$a \in \sum_{i=1}^k I_{\gamma_i} = \bigvee_{i=1}^k I_{\gamma_i}.$$

Therefore,

$$S(a) \subseteq S\left(\bigvee_{i=1}^k I_{\gamma_i}\right) = \bigcup_{i=1}^k S(I_{\gamma_i}).$$

b) Let  $B$  be an open compact set. Then  $B = \bigcup_{i=1}^n S(a_i)$ , where  $a_i \in G$ . If  $G^+$  is a  $wa$ -sublattice of  $G$ , then we can consider, by Corollary 5, that  $a_i \in G^+$ . Hence, by Lemma 1,  $\bigcup_{i=1}^n S(a_i) = S((\dots((a_1 \vee a_2) \vee a_3) \vee \dots) \vee a_n)$ . ■

**Theorem 8.** *If  $G$  is an abelian  $wa$ -group,  $P, Q \in \text{Spec}(G)$  and  $P \parallel Q$ , then  $P$  and  $Q$  have in  $\text{Spec}(G)$  disjoint neighborhoods.*

**Proof.** a) Let  $P, Q \in \text{Spec}(G)$ ,  $P \parallel Q$ . Then there exist (because every  $wa$ -ideal is generated by its positive cone)  $0 < a \in P \setminus Q$  and  $0 < b \in Q \setminus P$ . Let us denote  $u = a - (a \wedge b)$ ,  $v = b - (a \wedge b)$ . By Proposition 1.5 of [7],  $u \wedge v = 0$ . Let us suppose that  $u \in Q$ . Since  $0 \leq a \wedge b < b$ , we have  $a \wedge b \in Q$ , and hence  $a = u + (a \wedge b) \in Q$ , a contradiction. Thus  $u \notin Q$ . Similarly we can prove that  $v \notin P$ . Therefore  $P \in S(v)$  and  $Q \in S(u)$ , and because  $u \wedge v = 0$ , we get  $S(u) \cap S(v) = S(u \wedge v) = \emptyset$ . ■

The following theorem is an immediate consequence.

**Theorem 9.** *If  $G$  is an abelian  $wa$ -group and  $\mathbf{x} \subseteq \text{Spec}(G)$  a set of pairwise non-comparable straightening  $wa$ -ideals of  $G$ , then the spectral topology of  $\mathbf{x}$  is a  $T_2$ -topology.* ■

If  $\mathbf{x} \subseteq \text{Spec}(G)$ , put

$$\mathcal{D}\mathbf{x} = \bigcap \{P : P \in \mathbf{x}\}.$$

**Theorem 10.** a) *The closed sets in the spectrum of an abelian  $wa$ -group  $G$  are just all  $H(I)$ , where  $I \in \mathcal{I}(G)$ .*

b) *If  $\mathbf{x} \subseteq \text{Spec}(G)$ , then its closure is  $\bar{\mathbf{x}} = H(\mathcal{D}\mathbf{x})$ .* ■



Let us recall that a *wal*-group  $G$  is called *representable* if  $G$  is isomorphic to a subdirect sum of *to*-groups.

By Theorem 6 and Proposition 7 of [8], the class  $\mathcal{R}_{wal}$  of all representable *wal*-groups is a variety of *wal*-groups that is (in contrast to *l*-groups) non-comparable with the variety  $\mathcal{A}b_{wal}$  of all abelian *wal*-groups. It holds ([7], Theorem 2.6) that a *wal*-group  $G$  is representable if and only if the intersection of all its straightening *wal*-ideals is equal to  $\{0\}$ .

Hence we have:

**Theorem 11.** *If  $G$  is a representable abelian *wal*-group and  $\mathbf{x} \subseteq \text{Spec}(G)$ , then  $\mathbf{x}$  is dense if and only if*

$$\bigcap \{P : P \in \mathbf{x}\} = \{0\}.$$

■

Let  $G$  be an abelian *wal*-group and  $0 \neq a \in G$ . Let us denote by  $\text{val}(a)$  the set of all *wal*-ideals of  $G$  maximal with respect to not containing  $a$ . Every  $C \in \text{val}(a)$  is called a *value* of the element  $a$ . (For  $a = 0$  set  $\text{val}(a) = \emptyset$ .) By Theorem 2.4 of [7],  $\text{val}(a) \neq \emptyset$  for each  $a \neq 0$ , and by Proposition 2.3 of [7], every  $C \in \text{val}(a)$  is regular and thus also prime in  $G$ . Furthermore let us suppose that every prime *wal*-ideal of  $G$  is straightening. Then  $\text{val}(a) \subseteq \text{Spec}(G)$ . Let  $P \in S(a)$ . Then, by Theorem 2.2 of [7], the set of all *wal*-ideals of  $G$  containing  $P$  is linearly ordered, and, by Theorem 2.4 of [7], there is a *wal*-ideal in  $\text{val}(a)$  that contains  $P$ . Hence there exists exactly one *wal*-ideal  $M_P \in \text{val}(a)$  such that  $P \subseteq M_P$ .

Let us denote by  $\psi_a : S(a) \longrightarrow \text{val}(a)$  the mapping such that  $\psi_a : P \longmapsto M_P$ .

**Theorem 12.** *If  $G$  is an abelian *wal*-group such that every its prime *wal*-ideal is straightening and  $a \in G$ , then the mapping  $\psi_a$  is continuous.*

**Proof.** Let  $a \in G$ ,  $P \in S(a)$  and let  $U$  be a neighborhood of  $M_P$  in  $\text{val}(a)$ . We can suppose that  $U = S(b) \cap \text{val}(a)$  for some  $b \in G$ . If  $Q \in \text{val}(a) \setminus S(b)$  then by Theorem 8 there exist neighborhoods  $U_Q$  of  $Q$  and  $V_Q$  of  $M_P$  such that  $U_Q \cap V_Q = \emptyset$ . Let  $Q$  runs over  $\text{val}(a) \setminus S(b)$ . Then the corresponding  $U_Q$  form a covering of  $S(a) \setminus S(b)$ . By Theorem 7,  $S(a)$  is compact in  $\text{Spec}(G)$ . Moreover,  $S(a) \setminus S(b)$  is closed in  $S(a)$ , hence  $S(a) \setminus S(b)$  is also compact. Thus there exist  $n \in \mathbb{N}$  and  $Q_1, \dots, Q_n \in S(a) \setminus S(b)$  such that  $S(a) \setminus S(b) \subseteq U_{Q_1} \cup \dots \cup U_{Q_n}$ . Let us denote  $C = S(a) \setminus (U_{Q_1} \cup \dots \cup U_{Q_n})$ . We have  $V_{Q_1} \cap \dots \cap V_{Q_n} \subseteq C$  hence  $C$  is a neighborhood of  $M_P$  which is closed in  $S(a)$ , and  $C \cap \text{val}(a) \subseteq U$ . Therefore  $C \subseteq \psi_a^{-1}(C \cap \text{val}(a)) \subseteq \psi_a^{-1}(U)$ . Moreover,  $C$  is a neighborhood of  $M_P$ , thus it is also a neighborhood of  $P$ . ■

**Theorem 13.** *Let  $G$  be an abelian wal-group such that every its prime wal-ideal is straightening. If  $a \in G$ , then the set  $\text{val}(a)$  is a compact  $T_2$ -space.*

**Proof.** By Theorem 9,  $\text{val}(a)$  is a  $T_2$ -space. Further,  $\text{val}(a)$  is by Theorem 12 the image of the compact set  $S(a)$  in the continuous mapping  $\psi_a$ , hence  $\text{val}(a)$  is also compact. ■

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