# MODIFICATIONS OF CSÁKÁNY'S THEOREM 

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#### Abstract

Varieties whose algebras have no idempotent element were characterized by B. Csákány by the property that no proper subalgebra of an algebra of such a variety is a congruence class. We simplify this result for permutable varieties and we give a local version of the theorem for varieties with nullary operations.


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## 1 Introduction

B. Csákány [2] proved the following statement:

Proposition. For a variety $\mathcal{V}$, the following conditions are equivalent:
(i) None of algebras in $\mathcal{V}$ having at least two elements have idempotent elements;
(ii) No algebra $\mathcal{A} \in \mathcal{V}$ has a proper subalgebra whose carrier is a class of some $\theta \in \operatorname{Con} \mathcal{A}$;
(iii) There exist $n \in N$, ternary terms $p_{1}, \ldots, p_{n}$, and unary terms $f_{1}, \ldots, f_{n}, g_{1}, \ldots, g_{n}$ such that the identities

$$
\begin{gathered}
x=p\left(f_{1}(x), x, y\right), \\
p_{i}\left(g_{i}(x), x, y\right)=p_{i+1}\left(f_{i+1}(x), x, y\right), \quad \text { for } \quad i=1, \ldots, n-1, \\
y=p_{n}\left(g_{n}(x), x, y\right)
\end{gathered}
$$

hold in $\mathcal{V}$.

It was recognized by J. Kollár [3] that each of the equivalent conditions of the Proposition is equivalent to
(iv) For all $\mathcal{A} \in \mathcal{V}$, the greatest congruence $\iota_{A}$ on $\mathcal{A}$ is a compact element of $\operatorname{Con} \mathcal{A}$.
Analyzing the proof of Proposition, we find out that these conditions are also equivalent to
(v) $\quad \theta\left(F_{\mathcal{V}}(x) \times F_{\mathcal{V}}(x)\right)=F_{\mathcal{V}}(x, y) \times F_{\mathcal{V}}(x, y)$ in $\quad C o n F_{\mathcal{V}}(x, y)$,
where $F_{\mathcal{V}}\left(x_{1}, \ldots, x_{n}\right)$ denotes the free algebra of $\mathcal{V}$ generated by the set $\left\{x_{1}, \ldots, x_{n}\right\}$ of free generators, and $\theta(M \times M)$ denotes the least congruence containing the set $M \times M$.

From this point of view, $(v)$ can be modified by several ways. We can consider a variety $\mathcal{V}$ with constants (i.e. nullary operations) and we can omit a free variable on the left-hand side of $(v)$ to obtain
(vi) $\quad \theta\left(F_{\mathcal{V}}(\emptyset) \times F_{\mathcal{V}}(\emptyset)\right)=F_{\mathcal{V}}(x, y) \times F_{\mathcal{V}}(x, y)$ in $\operatorname{Con} F_{\mathcal{V}}(x, y)$.

This condition used in [1] it is equivalent to the property that $\iota_{A}$ is generated by the set of nullary operations for each $\mathcal{A} \in \mathcal{V}$.
Further, we can also
(a) simplify Csákány's original result for permutable varieties;
(b) omit one free variable in both sides of $(v)$ to obtain
(vii) $\theta\left(F_{\mathcal{V}}(\emptyset) \times F_{\mathcal{V}}(\emptyset)\right)=F_{\mathcal{V}}(x) \times F_{\mathcal{V}}(x)$ in Con $F_{\mathcal{V}}(x)$.

In the second case we obtain a local version of Csákány's theorem. These modifications are treated in this paper.

## 2 Results

Theorem 1. Let $\mathcal{V}$ be a permutable variety. The following conditions are equivalent:
(i) None of algebras in $\mathcal{V}$ having at least two elements have idempotent elements;
(ii) No algebra $\mathcal{A} \in \mathcal{V}$ has a proper subalgebra whose carrier is a class of some $\theta \in \operatorname{Con} \mathcal{A}$;
(iii) There exist $n \in N$ and $a(2+n)$-ary term $p$ and unary terms $v_{1}, \ldots, v_{n}, w_{1}, \ldots, w_{n}$ such that the identities

$$
x=p\left(x, y, v_{1}(x), \ldots, v_{n}(x)\right)
$$

$$
y=p\left(x, y, w_{1}(x), \ldots, w_{n}(x)\right)
$$

hold in $\mathcal{V}$.
Proof. The equivalence of (i) and (ii) is proven by the Proposition. Prove (ii) $\Rightarrow$ (iii): Set $\mathcal{A}=F_{\mathcal{V}}(x, y)$ and $\mathcal{B}=F_{\mathcal{V}}(x)$. Let $\theta=\theta(B \times B) \in C o n \mathcal{A}$ (where $B$ is the carrier of $\mathcal{B}$ ). Since $\mathcal{B}$ is a subalgebra of $\mathcal{A}$, we have $\theta(B \times B)=\theta=\iota_{A}$ by (ii). However, $\mathcal{V}$ is permutable; thus $\theta(B \times B)=$ $R(B \times B)$, the least reflexive and compatible relation on $\mathcal{A}$ containing the set $B \times B$. It follows $\iota_{A}=R(B \times B)$ which yields $\langle x, y\rangle \in R(B \times B)$. Hence, there exists a $(2+n)$-ary term $p$ and elements $b_{1}, \ldots, b_{n}, b_{1}^{\prime}, \ldots, b_{n}^{\prime} \in B$ such that

$$
x=p\left(x, y, b_{1}, \ldots, b_{n}\right) \text { and } y=p\left(x, y, b_{1}^{\prime}, \ldots, b_{n}^{\prime}\right)
$$

Since $b_{i}, b_{i}^{\prime} \in F_{\mathcal{V}}(x)$, there are unary terms $v_{i}, w_{i}$ with $b_{i}=v_{i}(x)$ and $b_{i}^{\prime}=$ $w_{i}(x), i=1, \ldots, n$.

For (iii) $\Rightarrow$ (i) let $\mathcal{A} \in \mathcal{V}$ with $|A|>1$ and suppose that $a \in A$ is an idempotent element. Let $b \in A \backslash\{a\}$. Then $v_{i}(a)=a=w_{i}(a)$ and, by (iii), $a=p\left(a, b, v_{i}(a), \ldots, v_{n}(a)\right)=p(a, b, a, \ldots, a)=p\left(a, b, w_{1}(a), \ldots, w_{n}(a)\right)=b$, a contradiction.

Example 1. For a variety $\mathcal{R}$ of rings with 1 , we can set $n=2, v_{1}(x)=1=$ $w_{2}(x), v_{2}(x)=0=w_{1}(x)$ and $p(x, y, a, b)=a x+b y$. Hence, it follows that the reduct of a ring with 1 , determined by the terms $0,1, x-y+z$, and $x y$, generates a permutable variety with no idempotent elements.

In this section we consider only varieties $\mathcal{V}$ having a nullary operation which will be denoted by 0 ; it is usually called a constant. For $\mathcal{A} \in \mathcal{V}$, this constant will be denoted by $0_{A}$. We do not restrict the number of nullary operations of $\mathcal{V}$ but this 0 will be considered to be fixed.

Theorem 2. Let $\mathcal{V}$ be a variety with 0 . The following conditions are equivalent:
(i) No $\mathcal{A} \in \mathcal{V}$ having at least two elements has $0_{A}$ as an idempotent element;
(ii) For each $\mathcal{A} \in \mathcal{V}$ and each $\theta \in \operatorname{Con} \mathcal{A},\left[0_{a}\right]_{\theta}$ is not a proper subalgebra of $\mathcal{A}$;
(iii) There exist $n \in N$, binary terms $q_{1}, \ldots, q_{n}$, and unary terms $v_{1}, \ldots, v_{n}, w_{1}, \ldots, w_{n}$ such that the identities

$$
x=q_{1}\left(x, v_{1}(0)\right),
$$

$$
\begin{gathered}
q_{1}\left(x, w_{i}(0)\right)=q_{i+1}\left(x, v_{i+1}(0)\right), \quad i=1, \ldots n-1 \\
0=q_{n}\left(x, w_{n}(0)\right)
\end{gathered}
$$

hold in $\mathcal{V}$.

Proof. $(\mathrm{i}) \Rightarrow(\mathrm{ii}):$ Let $\mathcal{A} \in \mathcal{V},|A|>1, \theta \in \operatorname{Con} \mathcal{A}$ and suppose $\left[0_{A}\right]_{\theta} \neq A$. Then $\mathcal{A} / \theta$ has at least two elements, and, of course, $0_{A / \theta}=\left[0_{A}\right]_{\theta}$. Since $0_{A / \theta}$ is not an idempotent of $\mathcal{A} / \theta \in \mathcal{V},\left[0_{A}\right]_{\theta}$ cannot be a subalgebra of $\mathcal{A}$.
$($ ii $) \Rightarrow($ iii $): ~ S e t ~ \mathcal{A}=\mathcal{F}_{\mathcal{V}}(x)$ and $\mathcal{B}=\mathcal{F}_{\mathcal{V}}(\emptyset)$. Let $\theta=\theta(B \times B)$ in Con $\mathcal{A}$. Since $0_{A}=0_{B} \in B$, the class $\left[0_{A}\right]_{\theta}$ contains $B$. Hence, for every $n$-ary operation $f$ of the type of $\mathcal{V}$, for every $c_{1}, \ldots, c_{n} \in\left[0_{A}\right]_{\theta}$, and every $b_{1}, \ldots, b_{n} \in B$ we have $\left\langle c_{i}, b_{i}\right\rangle \in \theta \quad(i=1, \ldots, n)$; thus, also $\left\langle f\left(c_{1}, \ldots, c_{n}\right), f\left(b_{1}, \ldots, b_{n}\right)\right\rangle \in \theta$. But $f\left(b_{1}, \ldots, b_{n}\right) \in B \subseteq\left[0_{A}\right]_{\theta}$, whence $f\left(c_{1}, \ldots, c_{n}\right) \in\left[0_{A}\right]_{\theta}$. Thus, $\left[0_{A}\right]_{\theta}$ is a subalgebra of $\mathcal{A}$. In account of (ii), $\left[0_{A}\right]_{\theta}=A$; thus, $\langle x, 0\rangle \in \theta(B \times B)$. Hence, there exist $d_{0}, d_{1}, \ldots, d_{n} \in A$ such that $d_{0}=x, d_{n}=0$ and $\left\langle d_{i-1}, d_{i}\right\rangle=\left\langle\varphi_{i}\left(b_{i}\right), \varphi_{i}\left(b_{i}^{\prime}\right)\right\rangle \quad(i=1, \ldots, n)$ for some $b_{i}, b_{i}^{\prime} \in B$ and unary polynomials $\varphi_{i}$ over $\mathcal{A}$. Thus $b_{i}=v_{i}(0), b_{i}^{\prime}=$ $w_{i}(0)$ for some unary terms $v_{i}, w_{i}$. Of course, $\varphi_{i}(z)=q_{i}(x, z)$ for some binary terms $q_{1}, \ldots, q_{n}$. The condition (iii) is evident.
(iii) $\Rightarrow$ (i): Let $\mathcal{A} \in \mathcal{V},|A|>1,0_{A} \neq a \in A$. Suppose that $0_{A}$ is an idempotent of $\mathcal{A}$. Then $v_{i}\left(0_{A}\right)=0_{A}=w_{i}\left(0_{A}\right)$ and $a=q_{1}\left(a, v_{1}\left(0_{A}\right)\right)=q_{1}\left(a, 0_{A}\right)=q_{1}\left(a, w_{1}\left(0_{A}\right)\right)=$ $q_{2}\left(a, v_{2}\left(0_{A}\right)\right)=q_{2}\left(a, 0_{A}\right)=q_{2}\left(a, w_{2}\left(0_{A}\right)\right)=\cdots=0_{A}$, a contradiction.

Example 2. For a variety $\mathcal{P}$ of pseudocomplemented semilattices, we set $n=1, v_{1}(x)=x^{*}, w_{1}(x)=x$ and $q_{1}(x, y)=x \wedge y$. Then

$$
\begin{aligned}
& q_{1}\left(x, v_{1}(0)\right)=x \wedge 0^{*}=x \\
& q_{1}\left(x, w_{1}(0)\right)=x \wedge 0=0
\end{aligned}
$$

Analogously as previously, we can simplify Theorem 2 for permutable varieties.

Theorem 3. Let $\mathcal{V}$ be a permutable variety with 0 . The following conditions are equivalent:
(i) No $\mathcal{A} \in \mathcal{V}$ consisting of at least two elements has $0_{A}$ as an idempotent element;
(ii) For each $\mathcal{A} \in \mathcal{V}$ and each $\theta \in \operatorname{Con} \mathcal{A},\left[0_{A}\right]_{\theta}$ is not a proper subalgebra of $\mathcal{A}$;
(iii) There exist $n \in N$ and $a(1+n)$-ary term $q$ and unary terms $v_{1}, \ldots, v_{n}, w_{1}, \ldots, w_{n}$ such that the identities

$$
\begin{aligned}
& x=q\left(x, v_{1}(0), \ldots, v_{n}(0)\right), \\
& 0=q\left(x, w_{1}(0), \ldots, w_{n}(0)\right)
\end{aligned}
$$

hold in $\mathcal{V}$.
Example 3. For the variety of Boolean algebras, we can set $n=1$, $v_{1}(x)=x^{\prime}, w_{1}(x)=x$ and $q(x, y)=x \wedge y$. Then $q\left(x, v_{1}(0)\right)=x \wedge 0^{\prime}=x$, and $q\left(x, w_{1}(0)\right)=x \wedge 0=0$.

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