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THE GALOIS CORRESPONDENCE BETWEEN SUBVARIETY LATTICES AND MONOIDS OF HYPERSUBSTITUTIONS

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Abstract

Denecke and Reichel have described a method of studying the lattice of all varieties of a given type by using monoids of hypersubstitutions. In this paper we develop a Galois correspondence between monoids of hypersubstitutions of a given type and lattices of subvarieties of a given variety of that type. We then apply the results obtained to the lattice of varieties of bands (idempotent semigroups), and study the complete sublattices of this lattice obtained through the Galois correspondence.

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1 Introduction

When the collection of all varieties of a given type τ is ordered by inclusion, a complete lattice $\mathcal{L}(\tau)$ is obtained. This lattice is dually isomorphic to the lattice of all equational theories of type τ. It is of some interest to know what the lattices $\mathcal{L}(\tau)$ look like, but it has become clear that they are very complicated, even for such special cases as the lattice $\mathcal{L}^{sg}$ of all varieties of semigroups. In [10] a new method to study these lattices was proposed, using monoids of hypersubstitutions. In this paper we develop a Galois correspondence between monoids of hypersubstitutions of a given type and lattices of subvarieties of a given variety of that type. We then apply the results obtained to the lattice of varieties of bands (idempotent semigroups), and study the complete sublattices of this lattice obtained through the Galois correspondence.

In the remainder of this section we set out some notation and background information on hypersubstitutions. Section 2 sets up the Galois correspondence between sets of hypersubstitutions and collections of varieties. This correspondence is restricted in Section 3 to monoids of hypersubstitutions and subvariety lattices. Finally, Section 4 works out this correspondence in a particular example, the lattice of varieties of bands or idempotent semigroups.

We fix a type $\tau = (n_i)_{i\in I}, n_i > 0$ for all $i \in I$, and operation symbols $(f_i)_{i\in I}$ where f_i is n_i-ary. Let $W_\tau(X)$ be the set of all terms of type τ over some fixed alphabet $X = \{x_1, x_2, \ldots\}$. Terms in $W_\tau(X_n)$ with $X_n = \{x_1, x_2, \cdots, x_n\}$, $n \geq 1$, are called n-ary. An algebra of type τ is a pair $\underline{A} = (A; (f_i^{\underline{A}})_{i\in I})$, where for every $i \in I$ we denote by $f_i^{\underline{A}}$ the operation induced by the operation symbol f_i on the set A. Let $Alg(\tau)$ be the class of all algebras of type τ and let $\mathcal{L}(\tau)$ be the lattice of all varieties of algebras of type τ. Clearly, $Alg(\tau)$ is the greatest element of $\mathcal{L}(\tau)$. We denote by $\mathcal{P}(\mathcal{L}(\tau))$ the power set of $\mathcal{L}(\tau)$.

The concept of hypersubstitution will be a crucial one. A mapping $\sigma : \{f_i \mid i \in I\} \to W_\tau(X)$ which assigns to every n_i-ary operation symbol f_i an n_i-ary term of type τ will be called a *hypersubstitution of type* τ. Any hypersubstitution σ can be uniquely extended to a map $\hat{\sigma} : W_\tau(X) \to W_\tau(X)$ on terms; this is defined inductively by

(i) $\hat{\sigma}[x] := x$ for any variable x in the alphabet X, and

(ii) $\hat{\sigma}[f_i(t_1, \ldots, t_{n_i})] := \sigma(f_i)^{\underline{W_\tau(X)}}(\hat{\sigma}[t_1], \ldots, \hat{\sigma}[t_{n_i}])$.

Here $\sigma(f_i)^{\underline{W_\tau(X)}}$ denotes the term operation induced by $\sigma(f_i)$ on the term algebra $\underline{W_\tau(X)}$.

We denote by $Hyp(\tau)$ the set of all hypersubstitutions of type τ. If we define a product $\circ_h$ of hypersubstitutions by $\sigma_1 \circ_h \sigma_2 := \hat{\sigma}_1 \circ \sigma_2$, where $\circ$ is the usual composition of functions, then $(Hyp(\tau); \circ_h, \sigma_{id})$ is a monoid. Note that σ_{id} is the identity hypersubstitution, defined by $\sigma_{id}(f_i) = f_i(x_1, \cdots, x_{n_i})$ for every $i \in I$. We denote by $\mathcal{P}(Hyp(\tau))$ the power set of $Hyp(\tau)$.

Let M be any subset of the monoid $(Hyp(\tau); \circ_h, \sigma_{id})$, and let V be a variety of type τ. Then an identity $s \approx t$ of V is called an *M-hyperidentity of the variety V* if for every $\sigma \in M$ the equation $\hat{\sigma}[s] \approx \hat{\sigma}[t]$ is an identity in V. When $M = Hyp(\tau)$, an M-hyperidentity is just an ordinary *hyperidentity*. Hyperidentities and M-hyperidentities for arbitrary subsets M of $Hyp(\tau)$ are particular sentences in a second order language, and were considered first by Belousov [2], Aczél [1], and Taylor [20]. For further results and a survey on these topics see [9], [6], [7], [5], [11], [14], [21], [22], [19], and [12].

If every identity in V is an M-hyperidentity, then V is called M-solid. This generalizes the concept of solidity, introduced by Graczyńska and Schweigert in [16]: a solid variety is M-solid for $M = Hyp(\tau)$. In [10] it was shown that if M is a monoid, the collection of all M-solid varieties of type τ forms a complete sublattice $S_M(\tau)$ of $\mathcal{L}(\tau)$. If $M_1 \subseteq M_2$ then $S_{M_1}(\tau) \supseteq S_{M_2}(\tau)$, i.e. $S_{M_2}(\tau)$ is a sublattice of $S_{M_1}(\tau)$. Further, for every submonoid M of $Hyp(\tau)$, two closure operators were introduced. These are χ_M^A and χ_M^E, defined on the collections $Alg(\tau)$ and $W_\tau(X) \times W_\tau(X)$ respectively by

$$\chi_M^E[s \approx t] := \{\hat{\sigma}[s] \approx \hat{\sigma}[t] \mid \sigma \in M\} \text{ and } \chi_M^A[\underline{A}] = \{\sigma[\underline{A}] \mid \sigma \in M\},$$

where $\sigma[\underline{A}] = (A; \sigma(f_i)^{\underline{A}})_{i \in i})$ is the so-called *derived algebra*. We also define $\sigma[V] = \{\sigma[\underline{A}] \mid \underline{A} \in V\}$, for any variety V. These operators are extended to sets Σ of identities and families K of algebras of type τ by setting

$$\chi_M^E[\Sigma] := \bigcup_{s \approx t \in \Sigma} \chi_M^E[s \approx t] \text{ and } \chi_M^A[K] := \bigcup_{A \in K} \chi_M^A[\underline{A}].$$

The closure properties of χ_M^E and χ_M^A follow directly from the monoid properties of M. It follows that a variety V is M-solid iff $\chi_M^A[V] = V$, i.e. iff V is closed with respect to χ_M^A. This is equivalent to $\chi_M^A[V] \subseteq V$, since the opposite inclusion is one of the properties of the closure operator χ_M^A. Now let $\mathcal{L}(V)$ be the lattice of all subvarieties of V. The intersection of this lattice with the lattice of M-solid varieties of type τ gives a new complete sublattice of $\mathcal{L}(\tau)$, the lattice $S_M(V)$ of all M-solid subvarieties of V. Our goal is to set up a Galois correspondence between such complete sublattices $S_M(V)$ and monoids of hypersubstitutions. We will do this by considering the set of all hypersubstitutions σ of type τ for which $\sigma[V] \subseteq V$, which J. Płonka (in [17]) has called *V-proper hypersubstitutions.* Since $(\sigma_1 \circ_h \sigma_2)[\underline{A}] = \sigma_1[\sigma_2[\underline{A}]]$, the set of V-proper hypersubstitutions forms a submonoid of the monoid $Hyp(\tau)$.

We shall also need the concept of V-equivalence of hypersubstitutions, which was first defined by Denecke and Reichel in [10] (see also [17]). Two hypersubstitutions σ_1 and σ_2 of type τ are called *V-equivalent* if for every operation symbol f_i the equation $\hat{\sigma}_1[f_i(x_1, \cdots x_{n_i})] \approx \hat{\sigma}_2[f_i(x_1, \cdots x_{n_i})]$ is an identity in V. In this case we write $\sigma_1 \sim_V \sigma_2$. Denecke and Reichel proved that for arbitrary hypersubstitutions σ_1, σ_2 the following three conditions are equivalent:

(i) $\sigma_1 \sim_V \sigma_2$,

(ii) for any term t of type τ the equation $\hat{\sigma}_1[t] \approx \hat{\sigma}_2[t]$ is an identity in V,

(iii) $\sigma_1[\underline{A}] = \sigma_2[\underline{A}]$ for any algebra $\underline{A} \in V$.

2 A Galois correspondence between sets of hypersubstitutions and sets of varieties

In this section we set up the Galois correspondence between sets of hypersubstitutions and collections of varieties. We begin by outlining some basic properties of Galois correspondences which we shall need. Let A and B be any sets and let $\mathcal{P}(A)$ and $\mathcal{P}(B)$ denote their power sets. Then a pair (η, θ), with $\eta : \mathcal{P}(A) \to \mathcal{P}(B)$ and $\theta : \mathcal{P}(B) \to \mathcal{P}(A)$, is called a *Galois correspondence between A and B* if for all $T, T' \subseteq A$ and all $S, S' \subseteq B$ the following properties are satisfied:

(i) $T \subseteq T' \Longrightarrow \eta(T) \supseteq \eta(T')$, and $S \subseteq S' \Longrightarrow \theta(S) \supseteq \theta(S')$,

(ii) $T \subseteq \theta\eta(T)$ and $S \subseteq \eta\theta(S)$.

Galois correspondences between sets A and B arise in the following way. For any relation $R \subseteq A \times B$ and for every $T \subseteq A$ and $S \subseteq B$ we define

$$\eta(T) := \{b \in B \mid \forall a \in T, (a, b) \in R\} \text{ and}$$
$$\theta(S) := \{a \in A \mid \forall b \in S, (a, b) \in R\}.$$

It is well known that the pair (η, θ) defined in this way is a 0 between A and B, called the *Galois correspondence induced by the relation R.*

For any Galois correspondence (η, θ), the operators $\theta\eta$ and $\eta\theta$ are closure operators on A and B, respectively. The corresponding closure systems (families of closed sets) $\mathcal{H}_{\eta\theta}$ and $\mathcal{H}_{\theta\eta}$ form lattices (with respect to set inclusion) which are dually isomorphic. Each of them is a meet-subsemilattice of the power set lattice on the respective set. It also follows that the $\eta\theta$-closed subsets of B are exactly the sets of the form $\eta(T)$ for some $T \subseteq A$, and dually the $\theta\eta$-closed subsets of A are exactly the sets of the form $\theta(S)$ for some $S \subseteq B$. That is, $\eta\theta\eta(T) = \eta(T)$ and $\theta\eta\theta(S) = \theta(S)$.

Now we apply this general Galois theory to hypersubstitutions and subvarieties. Let V be any (fixed) variety of type τ, with $\mathcal{L}(V)$ its subvariety lattice. We define a relation $R \subseteq Hyp(\tau) \times \mathcal{L}(V)$ between $Hyp(\tau)$ and $\mathcal{L}(V)$ by setting $R := \{(\sigma, W) \mid \sigma \in Hyp(\tau) \text{ and } W \in \mathcal{L}(V) \text{ and } \sigma[W] \subseteq W\}$. That is, $(\sigma; W) \in R$ iff σ is a W-proper hypersubstitution. Clearly if $\sigma_1 \sim_W \sigma_2$, then $(\sigma_1, W) \in R$ iff $(\sigma_2, W) \in R$.

As described above, this relation R then induces a Galois correspondence (η, θ) between $Hyp(\tau)$ and $\mathcal{L}(V)$. For any $M \in \mathcal{P}(Hyp(\tau))$ and any $L \in \mathcal{P}(\mathcal{L}(V))$, we set $\bar{L} := \eta\theta(L)$ and $\bar{M} := \theta\eta(M)$. We call a subset M of $Hyp(\tau)$ closed if $M = \bar{M}$ and dually, a subset L of $\mathcal{L}(V)$ closed if $L = \bar{L}$. Then we have the following properties of the Galois correspondence.

Proposition 2.1. *Let V be a variety of type τ, let $\mathcal{L}(V)$ be its subvariety lattice, and let $Hyp(\tau)$ be the set of all hypersubstitutions of type τ. For the mappings $\eta : \mathcal{P}(Hyp(\tau)) \to \mathcal{P}(\mathcal{L}(V))$ and $\theta : \mathcal{P}(\mathcal{L}(V)) \to \mathcal{P}(Hyp(\tau))$, the following properties hold:*

(i) *if $M \subseteq M' \subseteq Hyp(\tau)$ then $\eta(M) \supseteq \eta(M')$, and if $L \subseteq L' \subseteq \mathcal{L}(V)$ then $\theta(L) \supseteq \theta(L')$;*

(ii) *for any $M \in \mathcal{P}(Hyp(\tau))$ and for any $L \in \mathcal{P}(\mathcal{L}(V))$, we have $M \subseteq \theta\eta(M)$ and $L \subseteq \eta\theta(L)$;*

(iii) *the mappings defined on $\mathcal{P}(Hyp(\tau)$ and $\mathcal{L}(V)$ by $M \to \bar{M}$ and $L \to \bar{L}$ are closure operators;*

(iv) *for any $L \subseteq \mathcal{L}(V)$, the set L is closed iff there is a set $M \subseteq Hyp(\tau)$ such that $L = \eta(M)$, and for any $M \subseteq Hyp(\tau)$, the set M is closed iff there is a set $L \subseteq \mathcal{L}(V)$ such that $M = \theta(L)$; in particular, we have $\theta\eta\theta(L) = \theta(L)$ and $\eta\theta\eta(M) = \eta(M)$;*

(v) *for any $M, M' \in \mathcal{P}(Hyp(\tau))$, $\eta(M \cup M') = \eta(M) \cap \eta(M')$, and for any $L, L' \in \mathcal{P}(\mathcal{L}(V))$, $\theta(L \cup L') = \theta(L) \cap \theta(L')$;*

(vi) *if* $L, L' \subseteq \mathcal{L}(V)$ *then* $(L, L') \in ker(\theta)$ *iff* $\bar{L} = \bar{L}'$, *and if* $M, M' \subseteq Hyp(\tau)$ *then* $(M, M') \in ker(\eta)$ *iff* $\bar{M} = \bar{M}'$. ■

Part (vi) of the Proposition means that all members of each $ker(\theta)$-class have the same closure. Thus we can define a map $\bar{\theta}$ on $\mathcal{P}(\mathcal{L}(V))/ker(\theta)$ by $\bar{\theta}([L]_{ker(\theta)}) := [\theta(L)]_{ker(\eta)}$. Dually, we define a map $\bar{\eta}$ on $\mathcal{P}(Hyp(\tau))/ker(\eta)$ by $\bar{\eta}([M]_{ker(\theta)}) := [\eta(M)]_{ker(\theta)}$. It follows from the properties of a Galois connection that these two maps are bijections.

Corollary 2.2. *The maps* $\bar{\theta}$ and $\bar{\eta}$ *are bijections between* $\mathcal{P}(\mathcal{L}(V))/ker(\theta)$ and $\mathcal{P}(Hyp(\tau))/ker(\eta)$. ■

3 Subvariety lattices and monoids of hypersubstitutions

In the previous section, we described a Galois correspondence between any sets of hypersubstitutions from $Hyp(\tau)$ and any subcollections of varieties from the lattice $\mathcal{L}(V)$ of all subvarieties of a given variety V. In this section, we consider the restriction of the correspondence to certain special kinds of sets. This is motivated by a result of Denecke and Reichel [10] that any submonoid M of $Hyp(\tau)$ determines a complete sublattice of the lattice $\mathcal{L}(V)$, the sublattice $S_M(V)$ of all M-solid subvarieties of the variety V. So it is very natural to restrict our Galois mappings θ and η to submonoids M of $Hyp(\tau)$ and to sublattices L of $\mathcal{L}(V)$, respectively.

Lemma 3.1. *For any subset L of $\mathcal{L}(V)$, the image $\theta(L)$ is a submonoid of $Hyp(\tau)$; and for any subset M of $Hyp(\tau)$, the image $\eta(M)$ is a sublattice of $\mathcal{L}(V)$.*

Proof. Let L be a subset of $\mathcal{L}(V)$. For any variety W, the set of all W-proper hypersubstitutions forms a monoid [17], a submonoid of $Hyp(\tau)$. The image $\theta(L)$ is the intersection of these monoids for every $W \in L$, and thus a monoid.

Now let M be a subset of $Hyp(\tau)$. By definition, $\eta(M)$ consists of those W in $\mathcal{L}(V)$ for which $\sigma[W] \subseteq W$, for all $\sigma \in M$. In [10] it was shown that when M is a submonoid, this is equivalent to $\chi_M^A[W] = W$, and also that χ_M^A is a closure operator on classes of algebras, with the set of all subvarieties W of V with $\chi_M^A[W] = W$ forming a (complete) sublattice $S_M(V)$ of the lattice $\mathcal{L}(V)$ consisting of all M-solid subvarieties of V.

Thus we have $\eta(M) = S_M(V)$, a sublattice of $\mathcal{L}(V)$, when M is a submonoid of $Hyp(\tau)$. But then for any arbitrary subset M of $Hyp(\tau)$, we have $\eta(M) = \eta\theta\eta(M)$ with $\theta\eta(M)$ a submonoid from the first part of this proof, so again $\eta(M)$ is a sublattice. ■

Let $S(Hyp(\tau))$ be the submonoid lattice of the monoid $Hyp(\tau)$, and let $L(\mathcal{L}(V))$ be the lattice of all sublattices of $\mathcal{L}(V)$. We now define two mappings β and α as the restrictions of θ to $L(\mathcal{L}(V))$ and η to $S(Hyp(\tau))$ respectively. Then Lemma 3.1 shows that β is a mapping from $L(\mathcal{L}(V))$ to $S(Hyp(\tau))$ and α is a mapping from $S(Hyp(\tau))$ to $L(\mathcal{L}(V))$.

Lemma 3.2. *For any $L, K \in L(\mathcal{L}(V))$ and for any $M, N \in S(Hyp(\tau))$, we have $\beta(L) \wedge \beta(K) = \beta(L \vee K)$ and $\alpha(M \vee N) = \alpha(M) \wedge \alpha(N)$.*

Proof. This follows from Proposition 2.1(v) and Lemma 3.1. ■

In the same way as $\bar{\theta}$ and $\bar{\eta}$ we define mappings $\bar{\beta}, \bar{\alpha}$ with

$$\bar{\alpha} : S(Hyp(\tau))/ker(\alpha) \rightarrow L(\mathcal{L}(V))/ker(\beta)$$

$$\text{and } \bar{\beta} : L(\mathcal{L}(V))/ker(\beta) \rightarrow S(Hyp(\tau))/ker(\alpha).$$

Corollary 3.3. *$\bar{\alpha}$ and $\bar{\beta}$ are bijections.* ■

It is clear that the maps α and β do not preserve joins, and hence are not lattice homomorphisms. Also, from Lemma 3.2 we have $\beta(L) \wedge \beta(K) = \beta(L \vee K)$, for any L and K. It is also always true that $\beta(L) \vee \beta(K)$ is contained in $\beta(L \wedge K)$, but in the next section we will give an example to show that this inclusion can be strict. Thus β is not a lattice dual-homomorphism.

Corollary 3.4. *The intersection of closed submonoids from $S(Hyp(\tau))$ is closed, and the intersection of closed sublattices from $L(\mathcal{L}(V))$ is also closed. Thus the closed objects in $S(Hyp(\tau))$, and dually of $L(\mathcal{L}(V))$, form a lattice under inclusion, with meet equal to intersection.* ■

Every submonoid M of $Hyp(\tau)$ determines a complete sublattice $\alpha(M) = S_M(V)$ of the lattice $\mathcal{L}(V)$ of all subvarieties of the variety V. Considering a set $\mathcal{M}$ of submonoids of $Hyp(\tau)$ we define the set $\mathcal{L}_{\mathcal{M}} = \{\alpha(M) \mid M \in \mathcal{M}\}$ of complete sublattices of $\mathcal{L}(V)$ and ask under which condition $\mathcal{L}_{\mathcal{M}}$ is a sublattice of the lattice $L(\mathcal{L}(V))$ of all sublattices of $\mathcal{L}(V)$. Since for submonoids M_1, M_2 of $Hyp(\tau)$ with $M_1 \subseteq M_2$ we have $\alpha(M_1) = S_{M_1}(V) \supseteq S_{M_2}(V) = \alpha(M_2)$, one conclusion is that if $\mathcal{M}$ is a chain then $\mathcal{L}_{\mathcal{M}}$ is also a chain and thus a sublattice of $L(\mathcal{L}(V))$.

From our earlier results we get the following Proposition:

Proposition 3.5. *Let $\mathcal{M}$ be a sublattice of $S(Hyp(\tau))$. Then the following conditions are equivalent:*

(i) *$\mathcal{L}_{\mathcal{M}}$ forms a sublattice of $L(\mathcal{L}(V))$,*

(ii) *for any two monoids M_1, M_2 from $\mathcal{M}$, we have $\alpha(M_1) \vee \alpha(M_2) = S_{M_1}(V) \vee S_{M_2}(V) \in \mathcal{L}_{\mathcal{M}}$.*

Proof. (i) $\Longrightarrow$ (ii) is clear.

(ii) $\Longrightarrow$ (i): Since by assumption the join of elements of $\mathcal{L}_{\mathcal{M}}$ is in $\mathcal{L}_{\mathcal{M}}$, we need only check meets. That is, we need to check that $\alpha(M_1) \wedge \alpha(M_2) \in \mathcal{L}_{\mathcal{M}}$ for any two monoids $M_1, M_2 \in \mathcal{M}$. This is immediate from Lemma 3.3 and the fact that $\mathcal{M}$ is a sublattice. ∎

Dually, for a set $\mathcal{L}$ of sublattices of $\mathcal{L}(V)$ we can consider monoids which are β-images of the lattices in $\mathcal{L}$: $\mathcal{M}_{\mathcal{L}} = \{\beta(L) \mid L \in \mathcal{L}\}$. We could state and prove a dual theorem to 3.7 which characterizes when $\mathcal{M}_{\mathcal{L}}$ is a sublattice of the lattice of all submonoids of $Hyp(\tau)$.

4 M-Solid varieties of bands

The Galois correspondence between monoids of hypersubstitutions and lattices of subvarieties gives us a tool to examine the lattice of all varieties of a given type in terms of its closed sublattices. Within type (2), there has been particular interest in hyperidentities and M-hyperidentities for varieties of semigroups, where the structure is simple enough to be accessible but rich enough to provide interesting examples. For ordinary hyperidentities, that is M- hyperidentities when $M = Hyp(\tau)$, much has been done for semigroups; see for instance [5], [18], [11], [21], and [22]. The more general M-hyperidentity approach promises to tell us more about the lattice of all semigroup varieties, but may be difficult to use since the monoid of all semigroup hypersubstitutions is infinite. One recent contribution in this direction by Denecke and Koppitz [8] describes all the finite submonoids of this monoid, and the corresponding M-solid semigroup varieties.

In this section we pursue a different approach, and consider a subvariety of the variety of all semigroups whose monoid of hypersubstitutions is finite. For this variety, the variety B of bands, we illustrate our Galois correspondence by working out the lattices of closed submonoids and of closed sublattices.

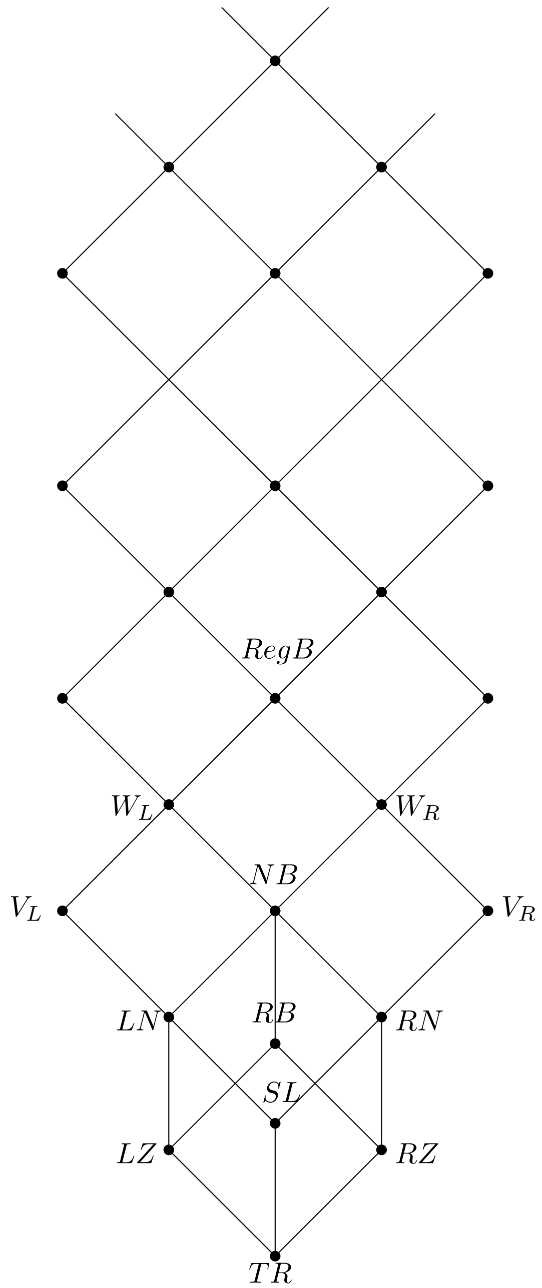

Figure 1. The Lattice of Proper Varieties of Bands

We begin with some background on bands; for more information the reader is referred to [15]. *Bands* are idempotent semigroups; that is, algebras of type (2) satisfying associativity and the idempotent law $x^2 \approx x$. (The single binary operation is usually denoted by juxtaposition.) The lattice $\mathcal{L}^B$ of all varieties of bands was completely described by Birjukov [3], Fennemore [13] and Gerhard [14]. The picture of the lattice shown in Figure 1 is due to Gerhard and Petrich [15].

There are a countably infinite number of varieties of bands, each equationally defined by associativity, idempotence, and one additional identity. In this section, we will use the notation $V(u \approx v)$ for the variety of bands determined by the additional identity $u \approx v$. An important feature of the lattice is its symmetry about a center column of self-dual varieties. Each variety $V = V(u \approx v)$ not on the center column has a dual, $V^d = V(u^d \approx v^d) \neq V$ (where u^d is just the right-to-left dual of the word u); a variety V on the center column has $V = V^d$.

For reference, we list below some of the varieties and identities to be used in this section:

$I = V(x \approx y)$, the trivial variety,

$SL = V(xy \approx yx)$, the variety of semilattices,

$RB = V(x \approx xyx)$, the variety of rectangular bands,

$NB = V(xyzw \approx xzyw)$, the variety of normal bands,

$RegB = V(xyxzx \approx xyzx)$, the variety of regular bands,

$LZ = V(xy \approx x)$, the variety of left zero semigroups,

$RZ = V(xy \approx y)$, the variety of right zero bands,

$LN = V(xyz \approx xzy)$, the variety of left normal semigroups,

$RN = V(xyz \approx yxz)$, the variety of right normal semigroups,

$V_L = V(xy \approx xyx)$,

$V_R = V(xy \approx yxy)$,

$W_L = V(xyz \approx xyxz)$,

$W_R = V(xyz \approx xzyz)$,

$C = V(xy \approx uv)$, the variety of zero semigroups.

Within the variety of bands, there are only six binary terms: x, y, xy, yx, xyx and yxy. Thus using the relation $\sim_V$ between hypersubstitutions the monoid Hyp of hypersubstitutions can be denoted as

$$\{\sigma_{xy}, \sigma_x, \sigma_y, \sigma_{yx}, \sigma_{xyx}, \sigma_{yxy}\}.$$

Then it is straightforward to work out the sixteen submonoids of Hyp, as listed here:

$M_0 = \{\sigma_{xy}\}$;	$M_8 = \{\sigma_{xy}, \sigma_y, \sigma_{yxy}\}$;
$M_1 = \{\sigma_{xy}, \sigma_x\}$;	$M_9 = \{\sigma_{xy}, \sigma_x, \sigma_y, \sigma_{xyx}\}$;
$M_2 = \{\sigma_{xy}, \sigma_y\}$;	$M_{10} = \{\sigma_{xy}, \sigma_x, \sigma_y, \sigma_{yxy}\}$;
$M_3 = \{\sigma_{xy}, \sigma_{yx}\}$;	$M_{11} = \{\sigma_{xy}, \sigma_x, \sigma_y, \sigma_{yx}\}$;
$M_4 = \{\sigma_{xy}, \sigma_{xyx}\}$;	$M_{12} = \{\sigma_{xy}, \sigma_{yx}, \sigma_{xyx}, \sigma_{yxy}\}$;
$M_5 = \{\sigma_{xy}, \sigma_{yxy}\}$;	$M_{13} = \{\sigma_{xy}, \sigma_{xyx}, \sigma_{yxy}\}$;
$M_6 = \{\sigma_{xy}, \sigma_x, \sigma_y\}$;	$M_{14} = \{\sigma_{xy}, \sigma_x, \sigma_y, \sigma_{xyx}, \sigma_{yxy}\}$;
$M_7 = \{\sigma_{xy}, \sigma_x, \sigma_{xyx}\}$;	$M_{15} = Hyp$.

These submonoids then form a lattice, shown in Figure 2 below. Using previous results about hyperidentities for band varieties [21], we see that all the submonoids except two are closed; $\overline{M}_{13} = M_{12}$ and $\overline{M}_{14} = Hyp$. Dually, we get the lattice of closed (complete) sublattices of $\mathcal{L}(B)$, sublattices of the form $S_M(B)$ for M a submonoid of Hyp.

Now we characterize the corresponding lattices of M_i-solid subvarieties of B. We deal first with the six submonoids M_0 to M_5 which have only one or two elements.

Proposition 4.1. *Let W be a non-trivial variety of bands. Then the following hold:*

(i) *W is M_0-solid iff W is any variety of bands*;

(ii) *W is M_1-solid iff $LZ \subseteq W$*;

(iii) *W is M_2-solid iff $RZ \subseteq W$*;

(iv) *W is M_3-solid iff W is self-dual*, (*that is, $W^d = W$, and the dual of any identity of W is also an identity of W*).

Proof. In [8], the corresponding lattices $S_M(Sg)$ of solid semigroup varieties were characterized, for M equal to any of M_0, M_1, M_2, M_3, M_6 or M_{11}. The same proofs and results hold for the band case. ∎

Theorem 4.2. *Let W be a nontrivial variety of bands. Then W is M_4-solid iff W is one of the varieties LZ, SL, LN, RB, V_L, NB, W_L or $RegB$.* (See Figure 1 above.)

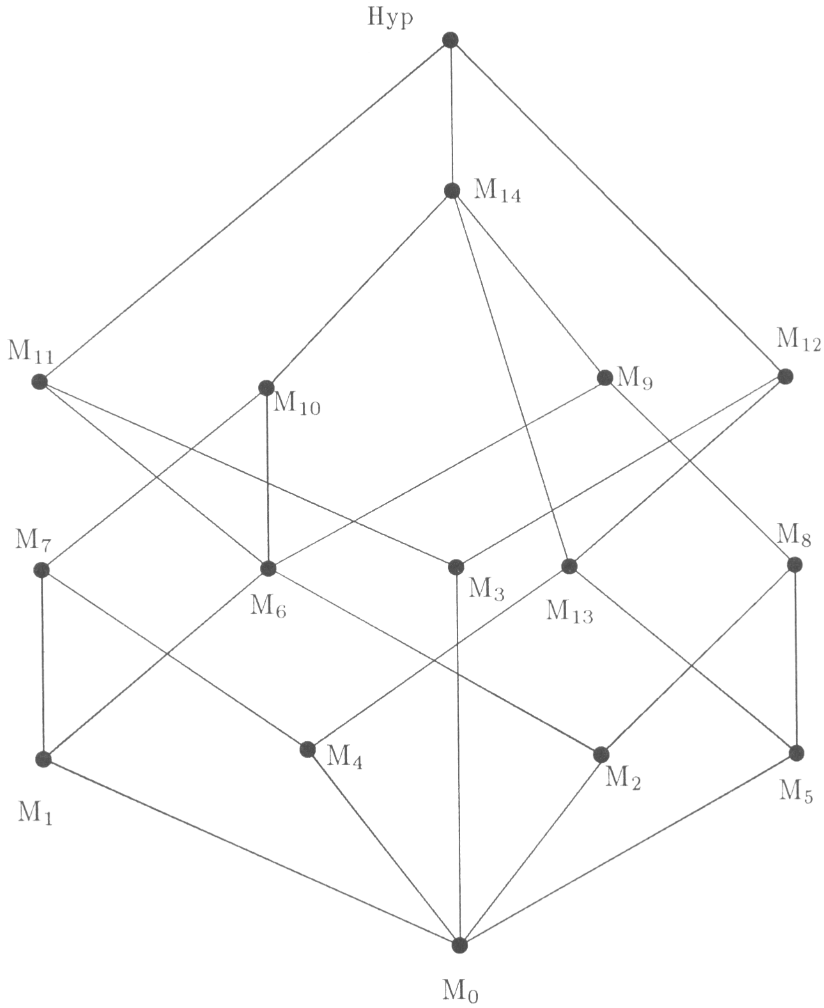

Figure 2. The Lattice of Submonoids of Hyp

Proof. For a nontrivial variety W of bands to be M_4-solid, it must be closed under application of the hypersubstitution σ_{xyx}. In particular, W must satisfy the identity $xyxzxyx \approx xyzyx$ obtained by applying $\hat{\sigma}_{xyx}$ to the associative identity $x(yz) \approx (xy)z$. This is known to require that W be a subvariety of the variety $RegB$ of regular bands. Thus we need only examine the twelve nontrivial subvarieties of $RegB$, as shown in Figure 1 above.

For W equal to any of the varieties RZ, RN, V_R and W_R, we see that W is characterized by an identity $u \approx v$ with the property that the words u and v end with the same last letter – and in fact all identities satisfied by these varieties must have this property – but also that u and v start with different letters. When we apply $\hat{\sigma}_{xyx}$ to such an identity $u \approx v$ it is easy to see that the result is the identity $uu^d \approx vv^d$. But this identity no longer has both words ending with the same last letter, and so does not hold in the variety W. Thus we have excluded these four varieties from being M_4-solid.

We now check that the remaining eight subvarieties of $RegB$ are M_4-solid. First, it is well known that the three self-dual varieties RB, NB and $RegB$ are solid, which means that they are certainly M_4-solid. For SL, it is well known that SL satisfies an identity $u \approx v$ iff the words u and v contain the same letters; applying $\hat{\sigma}_{xyx}$ does not change which letters are used in the words, to that SL still satisfies $\hat{\sigma}_{xyx}[u] \approx \hat{\sigma}_{xyx}[v]$.

We want to use a similar argument for the remaining varieties W on our list, for each one giving an equivalent condition for W to satisfy an identity $u \approx v$ which is still met by $\hat{\sigma}_{xyx}[u] \approx \hat{\sigma}_{xyx}[v]$. Let us note first that we may exclude identities of the form $x^a \approx x^b$ from this consideration: such identities result in identities of this same form when $\hat{\sigma}_{xyx}$ is applied, and always hold in any variety of bands.

The variety LZ satisfies an identity $u \approx v$ iff the words u and v start with the same first letter, a property which is preserved under application of $\hat{\sigma}_{xyx}$. The variety LN satisfies an identity $u \approx v$ iff the words u and v both contain the same letters and start with the same first letter; again this is preserved by $\hat{\sigma}_{xyx}$. Similar characterizations hold for the remaining two varieties in our list: V_L satisfies $u \approx v$ iff words u and v start with the same first two letters, while W_L does iff u and v start with the same first two letters and end with the same last letter, and both have length at least three. ■

By a dual argument we get:

Theorem 4.3. *Let W be a nontrivial variety of bands. Then W is M_5-solid iff W is one of the varieties RZ, SL, RN, RB, V_R, NB, W_R or $RegB$.* ∎

From Lemma 3.2 we know that $\alpha(M \vee N) = \alpha(M) \wedge \alpha(N)$ for any submonoids M and N of Hyp. Thus we can extend results 4.1, 4.2 and 4.3 above to characterize the M-solid varieties for any submonoid M of Hyp.

Proposition 4.4. $\alpha(M_3) = \alpha(M_{12})$, i.e. $(M_3, M_{12}) \in ker(\alpha)$.

Proof. Since $M_3 \subseteq M_{12}$ and α preserves inclusions, $\alpha(M_{12}) = S_{M_{12}}(B) \subseteq S_{M_3}(B) = \alpha(M_3)$. To show that $S_{M_3}(B) \subseteq S_{M_{12}}(B)$, we will show that any M_3-solid variety W is also closed under application of σ_{xyx} and σ_{yxy}. Let $u \approx v$ be an arbitrary identity in W. The fact that W is M_3-solid means that $u^d \approx v^d$ also holds in W. But $\hat{\sigma}_{xyx}[u] = uu^d \approx vv^d = \hat{\sigma}_{xyx}[v]$ is an immediate consequence of $u \approx v$ and $u^d \approx v^d$, so W is closed under application of σ_{xyx}. A similar argument works for σ_{yxy}. ∎

We will now show that the set $\{S_M(B) \mid M \in S(M_{15})\}$ does not form a lattice. We form $S_{M_4}(B) \vee S_{M_6}(B)$ and show that $S_{M_4}(B) \vee S_{M_6}(B) \notin \{S_M(B) \mid M \in S(M_{15})\}$. Consider $L^* := \{W \mid RB \subseteq W \subseteq B\} \cup \{W \mid W \subseteq B \text{ and } W \subseteq Mod\{xyx \approx xy\}\} = S_{M_6}(B) \cup \{W \mid W \subseteq B \text{ and } W \subseteq Mod\{xyx \approx xy\}\}$. Firstly note that $S_{M_4}(B) \subseteq L^*$ and $S_{M_6}(B) \subseteq L^*$, so $S_{M_6}(B) \cup S_{M_4}(B) \subseteq L^*$. On the other hand we have $L^* \subseteq S_{M_6}(B) \cup S_{M_4}(B) \subseteq S_{M_6}(B) \vee S_{M_4}(B)$. We want to show that $L^* = S_{M_6}(B) \vee S_{M_4}(B)$. Thus we must show that if $W \in S_{M_6}(B) \vee S_{M_4}(B)$ then W belongs to L^*. $W \in S_{M_6}(B) \vee S_{M_4}(B)$ means $W = V_1 \vee V_2$ or $W = V_1 \wedge V_2$ with $V_1 \in S_{M_6}(B), V_2 \in S_{M_4}(B)$. If $V_1 = I$ then $V_1 \vee V_2 = V_2$ and $V_1 \wedge V_2 = I = V_1$ and all is clear. Assume that $V_1 \neq I$. Then $RB \subseteq V_1 \subseteq V_1 \vee V_2$, i.e. $V_1 \vee V_2 \in L^*$. If $RB \subseteq V_2$ then $RB \subseteq V_1 \wedge V_2$, i.e. $V_1 \wedge V_2 \in L^*$ and $RB \subseteq V_2 \subseteq V_1 \vee V_2$ means $V_1 \vee V_2 \in L^*$. Now let $V_2 \subseteq Mod\{xyx \approx xy\}$. Then $V_1 \wedge V_2 \subseteq V_2 \subseteq Mod\{xyx \approx xy\}$, i.e. $V_1 \wedge V_2 \in L^*$. It is easy to check that $S_M(B) \neq L^*$ for any $M \in S(M_{15})$.

The band information worked out above can be used to give an example that $\beta(L) \vee \beta(K)$ can be properly contained in $\beta(L \wedge K)$. Both $L = \{I, LZ\}$ and $K = \{I, RZ\}$ are sublattices of $\mathcal{L}^B$. Then $\beta(L) = M_7$ and $\beta(K) = M_8$, so $\beta(L) \vee \beta(K) = M_{14}$. But $L \wedge K = \{I\}$, and $\beta(L) \wedge \beta(K)$ is a proper subset of $\beta(L \wedge K)$.

As we have pointed out, it may be difficult to study all complete sublattices of the lattice of all semigroup varieties in this way. Since the monoid of hypersubstitutions of the variety of bands is finite, we are able to obtain a full picture of the complete sublattices determined by all possible submonoids. Our results in this case suggest some interesting general questions. For example, from the fact that the operator α reverses inclusions, we know that

$$M \subseteq N \Longrightarrow S_N(\tau) = \alpha(N) \subseteq \alpha(M) = S_M(\tau).$$

One could ask what relationship exists between lattices $S_M(\tau)$ and $S_N(\tau)$ if the monoids M and N are isomorphic. In our band example, for instance, the isomorphic submonoids M_4 and M_5 lead to isomorphic complete sublattices of $\mathcal{L}^B$, but we do not know if this is true in general.

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