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STRONGLY RECTIFIABLE AND S-HOMOGENEOUS MODULES

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Abstract

In this paper we introduce the class of strongly rectifiable and S-homogeneous modules. We study basic properties of these modules, of their pure and refined submodules, of Hill's modules and we also prove an extension of the second Prüfer's theorem.

Keywords: strongly rectifiable module, S-homogeneous module, pure submodule, refined submodule, pure composite series, Hill's module.

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1 Introduction

There are many important results in the structure theory of torsion abelian groups that are known to be true in certain classes of modules as well. The class of I-primary modules (studied by Bican [4], [5]) and the class of rectifiable modules (studied by Benabdallah and Hattab [2], [3]) are known examples of such classes of modules. In the present paper, we introduce the class of strongly rectifiable and S-homogeneous modules and we show that, among others, the extensions of Prüfer's theorems are true in this class.

The class of strongly rectifiable and S-homogeneous modules is clearly a proper subclass of the class of all rectifiable modules, but, on the other hand, making use of this restriction we can get deeper structural results, especially by an appropriate generalization of the concept of the height of an element – the maximum value of the height of a universal submodule, as defined by Benabdallah and Hattab [2], [3] is equal to ω, i.e. to the first limit ordinal, while the height of an element, defined in this paper, can be equal to any ordinal number (limit or non-limit).

2 Strongly rectifiable, S-homogeneous modules

Let R be an associative ring with identity, let I be a maximal left ideal of R. The submodule of a module M generated by all simple submodules of M isomorphic to R/I is denoted by $\mathrm{Soc}^I(M)$ and called the *I-socle of the module* M. If α is an ordinal, then $\mathrm{Soc}^I_0(M) = 0$, $\mathrm{Soc}^I_{\alpha+1}(M)/\mathrm{Soc}^I_\alpha(M) = \mathrm{Soc}^I(M/\mathrm{Soc}^I_\alpha(M))$ and $\mathrm{Soc}^I_\alpha(M) = \bigcup_{\beta<\alpha} \mathrm{Soc}^I_\beta(M)$ for α limit is called the *I-Loewy series o*f M. The smallest ordinal τ for which $\mathrm{Soc}^I_\tau(M) = \mathrm{Soc}^I_{\tau+1}(M)$ is called the *I-Loewy length of* M. We say that M is the *I-primary (*or *I-Loewy) module* if $\mathrm{Soc}^I_\tau(M) = M$. A module U is called *uniserial* if all its submodules form a finite chain $0 = U_0 \subset U_1 \subset \cdots \subset U_n = U$. The number $n = l(U)$ is called the *length of* U. A module M is said to satisfy the *rectifiability condition* if for every U and V uniserial submodules of M, $U+V$ admits a uniserial summand (i.e. $U+V$ contains a uniserial submodule W and a submodule L for which $U+V = W \oplus L$). M is said to be *locally rectifiable* if every homomorphic image of M satisfies the rectifiability condition. A module is said to be *rectifiable* if it is locally rectifiable and generated by uniserial submodules.

Throughtout this article, I denotes the maximal left ideal of R and this ideal I is supposed to be a two-sided ideal. The simple module R/I is denoted by S.

Definition. We say that a module M is *S-homogeneous* if M is rectifiable and all its uniserial submodules are I-primary.

Definition. We say that a module M is *strongly rectifiable* if M is locally rectifiable and its every cyclic submodule is uniserial.

Remark. In the theory of the I-primary modules, L. Bican studied the modules over ring satisfying two conditions (1I) and (2I) – see [4] and [5]. It is easy to see that every I-primary module over a ring satisfying conditions (1I) and (2I) from [4] and [5] is strongly rectifiable and S-homogeneous.

Theorem 1. *Let M be a strongly rectifiable and S-homogeneous module. Then every submodule and every quotient of M is strongly rectifiable and S-homogeneous.*

Proof. Since every quotient of an uniserial module is uniserial, the quotient of any strongly rectifiable module is strongly rectifiable as well.

Since M is S-homogeneous, $M = \sum_{j\in J} U_j$, where U_j are uniserial for all $j \in J$, and hence U_j are I-primary for all $j \in J$ and the smallest ordinal τ_j for which $\mathrm{Soc}^I_{\tau_j}(U_j) = U_j$ satisfies $\tau_j \le \sigma = \sup\{\tau_j \ : \ j \in J\} \le \omega$. Then $\mathrm{Soc}^I_\sigma(M) = M$, M is I-primary and hence each quotient of M is S-homogenous, due to the fact that the class of I-primary modules is closed under submodules and quotients. ∎

Definition. Let M be a module. Then we define:

(i) $H_0(M) = M$,

(ii) $H_1(M)$ is the submodule of M generated by a set $\mathcal{U}$ of all uniserial submodules U for which there is a uniserial submodule V such that $U \subset V \subseteq M$, $l(V/U) \ge 1$ (for $\mathcal{U} = \emptyset$ we put $H_1(M) = 0$),

(iii) $H_{\alpha+1}(M) = H_1(H_\alpha(M))$, for α non-limit,

(iv) $H_\alpha(M) = \bigcap_{\beta<\alpha} H_\beta(M)$, for α limit.

Let τ be the smallest ordinal for which $H_{\tau+1}(M) = H_\tau(M)$. For every element $x \in M$ we put

$$h_M(x) = \begin{cases} \alpha & \text{for } x \in H_\alpha(M) \smallsetminus H_{\alpha+1}(M) \\ \infty & \text{for } x \in H_\tau(M). \end{cases}$$

The ordinal $h_M(x)$ is called the *height of* x in M.

Lemma 2. *Let* M, N *be modules and let* $\varphi\colon M \longrightarrow N$ *be a homomorphism. Then* $h_M(x) \le h_N(\varphi(x))$ *for every* $x \in M$.

Proof. We prove that $\varphi(H_\alpha(M)) \subseteq H_\alpha(N)$. We proceed by induction on α. Let U be a uniserial submodule of $H_\alpha(M)$ for which there is a uniserial submodule V such that $U \subset V \subseteq H_\alpha(M)$, $l(V/U) = 1$. Then $\varphi(U) \subseteq \varphi(V) \subseteq \varphi(H_\alpha(M)) \subseteq H_\alpha(N)$. For $\varphi(U) \ne 0$ the map $f\colon V \xrightarrow{\varphi|_V} \varphi(V) \xrightarrow{\pi} \varphi(V)/\varphi(U)$ is an epimorphism and $\mathrm{Ker} f = U$. Therefore $\varphi(V)/\varphi(U) \cong V/\mathrm{Ker} f = V/U$ and $\varphi(U) \subseteq H_1(H_\alpha(N)) = H_{\alpha+1}(N)$. If the ordinal α is limit, then $\varphi(H_\alpha(M)) = \varphi(\bigcap_{\beta<\alpha} H_\beta(M)) \subseteq \bigcap_{\beta<\alpha} \varphi(H_\beta(M)) \subseteq \bigcap_{\beta<\alpha} H_\beta(N) = H_\alpha(N)$. ∎

We have immediately two corollaries:

Corollary 3. *Let* M *and* N *be modules and let* $\varphi\colon M \longrightarrow N$ *be an isomorphism. Then* $\varphi(H_\alpha(M)) = H_\alpha(N)$ *and hence* $H_\alpha(M) \cong H_\alpha(N)$ *for every ordinal* α. ∎

Corollary 4. *Let* $M = \bigoplus_{i \in J} M_i$ *be a direct sum of submodules* $M_i \subseteq M$ $(i \in J)$. *Then* $H_\alpha(M) = \bigoplus_{i \in J} H_\alpha(M_i)$ *for every ordinal* α. ∎

Lemma 5. *Let* M *be a rectifiable, S-homogeneous module, let* U *be a uniserial submodule of* M, *and let* $0 = U_0 \subset U_1 \subset \ldots \subset U_n = U$ *be the chain of all submodules of* U, $l(U) = n$. *Then the following statements are satisfied for all* $j = 1, \ldots, n$:

(i) $U_j / U_{j-1} \cong R/I$,

(ii) $U_{n-j} = I^j U$.

Proof. We proceed by induction on n. The result is trivially true for $n = 1$. If $l(U_{k+1}) = k+1$, $0 = U_0 \subset U_1 \subset \ldots \subset U_k \subset U_{k+1}$, then for all $j = 1, \ldots, k$ by the induction hypothesis we have $U_j/U_{j-1} \cong R/I$, $U_{k-j} = I^j U_k$. U_{k+1}/U_k is a simple submodule of M/U_k. Because M/U_k is S-homogeneous (see the proof of Theorem 1), we have $U_{k+1}/U_k \cong R/I$ and hence $IU_{k+1} \subseteq U_k$. Since $l(U_{k+1}/U_{k-1}) = 2$, we get $IU_{k+1} = U_k$. ∎

Lemma 6. *Let* M *be a rectifiable, S-homogeneous module. Then* $H_n(M) = I^n M$ *for all* $n \in \mathbb{N}$.

Proof. If U is a uniserial submodule of M such that there is a uniserial submodule V for which $U \subset V \subseteq M$, $l(V/U) = 1$, then, by Lemma 5, $IV = U$ and hence $U \subseteq IM$. Let U be a uniserial submodule of M. If $l(U) = 1$, then $U \cong R/I$ and $IU = 0 \in H_1(M)$. If $l(U) = n > 1$, then $0 = U_0 \subset U_1 \subset \ldots \subset U_n = U$ is the chain of all submodules of U and, by Lemma 5, $IU = U_{n-1} \subseteq H_1(M)$. Hence $H_1(M) = IM$. Because $H_n(M)$ is again a rectifiable and S-homogeneous module (see the proof of Theorem 1), $H_{n+1}(M) = H_1(H_n(M)) = I(H_n(M)) = I(I^n M) = I^{n+1} M$. ∎

Lemma 7. *Let* M *be a rectifiable, S-homogeneous module. Then*

$$h_M(x+y) \geq \min\{h_M(x), h_M(y)\} \text{ for every } x, y \in M .$$

If moreover $h_M(x) \neq h_M(y)$, *then* $h_M(x+y) = \min\{h_M(x), h_M(y)\}$.

Proof. If $h_M(x) \neq \infty$, $h_M(x) = \alpha$, $h_M(y) \neq \infty$, $h_M(y) = \beta$, then $x \in H_\alpha(M)$, $y \in H_\beta(M)$ and $x, y \in H_\gamma(M)$, where $\gamma = \min\{\alpha, \beta\}$. Now we have $h_M(x+y) \geq \gamma$. For $\alpha \neq \beta$, (e.g. $\alpha < \beta$), we have $\gamma = \alpha$ and if $h_M(x+y) > \gamma$, then $x \in H_{\alpha+1}(M)$, which is a contradiction. Hence $h_M(x+y) = \gamma$.

If $h_M(x) = \infty$ and $h_M(y) = \infty$, then $h_M(x+y) = \infty$.

If $h_M(x) \neq \infty$, $h_M(x) = \alpha$ and $h_M(y) = \infty$, then immediately $h_M(x+y) = \alpha$ (since otherwise $h_M(x+y) > \alpha$ implies $h_M(x) > \alpha$, a contradiction). ■

Lemma 8. *Let M be a rectifiable, S-homogeneous module and let U be a uniserial submodule of M. Then for $x, y \in U \setminus IU$ we have $h_M(x) = h_M(y)$.*

Proof. Using Lemma 5 we have $Rx = U = Ry$. ■

Let now M be a rectifiable, S-homogeneous module and let U be a uniserial submodule of M. By Lemma 8, we can denote $h_M(U) = h_M(x)$ for arbitrary $x \in U \setminus IU$. The ordinal $h_M(U)$ is called the height of U in M.

Lemma 9. *Let M be a strongly rectifiable, S-homogeneous module. Then $h_M(rx) = h_M(x)$ for all $x \in M$ and for all $r \in R \setminus I$.*

Proof. Let $U = Rx$ be a uniserial submodule of the strongly rectifiable module M and let $0 = U_0 \subset U_1 \subset \cdots \subset U_n = U$ be the chain of all submodules of U. Then $x \in U_n \setminus U_{n-1} = U \setminus IU$. Since $I + Rr = R$, there are $i \in I$, $r' \in R$ such that $x = ix + r'rx$ and therefore, by Lemma 5, $R(rx) = U$, $rx \in U_n \setminus U_{n-1} = U \setminus IU$. ■

Theorem 10. *Let M be a strongly rectifiable, S-homogeneous module, $\bigoplus_{i=1}^n U_i \subseteq M$, and let U_i, $i \in \{1, ..., n\}$, be simple submodules of M. Then there are simple submodules $V_1, V_2, ..., V_n$ of M such that $\bigoplus_{i=1}^n U_i = \bigoplus_{i=1}^n V_i$ and for every simple submodule $Z \subseteq \bigoplus_{i=1}^n U_i$ there exists $i_0 \in \{1, ..., n\}$ such that $h_M(Z) = h_M(V_{i_0})$.*

Proof. We proceed by induction on n. For $\bigoplus_{i=1}^{k+1} U_i \subseteq M$ we get by induction $\bigoplus_{i=1}^{k+1} U_i = \bigoplus_{i=1}^{k} V_i \oplus U_{k+1}$. Now we distinguish three cases.

Case 1. $h_M(U_{k+1}) = h_M(V_{i'})$ for some $i' \in \{1, ..., k\}$ and for every simple submodule $Z \subseteq U_1 \oplus ... \oplus U_{k+1}$, there is $i_0 \in \{1, ..., k\}$ such that $h_M(Z) = h_M(V_{i_0})$. In this case we can put $V_{k+1} = U_{k+1}$.

Case 2. $h_M(U_{k+1}) \neq h_M(V_i)$ for all $i \in \{1, ..., k\}$. In this case we can put $V_{k+1} = U_{k+1}$. For every simple submodule $Z \subseteq \bigoplus_{i=1}^{k+1} U_i$, we have $Z = Rz$, where $z \in Z$, $z \neq 0$ and $z = v + v_{k+1}$ for $v \in \bigoplus_{i=1}^{k} V_i$, $v_{k+1} \in V_{k+1}$. For $v = 0$, we have $h_M(Z) = h_M(V_{k+1})$. For $v_{k+1} = 0$, by the induction hypothesis, there exists $i_0 \in \{1, ..., k\}$ such that $h_M(Z) = h_M(V_{i_0})$. For $v \neq 0$, $v_{k+1} \neq 0$ the module Rv is a uniserial submodule of M and moreover, Rv is simple since $I(Rv) = 0$. By the induction hypothesis, there exists $i'_0 \in \{1, ..., k\}$ such that $h_M(Rv) = h_M(V_{i'_0})$. By Lemmas 7 and 8, we get $h_M(Z) = \min\{h_M(V_{i'_0}), h_M(V_{k+1})\}$.

Case 3. $h_M(U_{k+1}) = h_M(V_{i'})$ for some $i' \in \{1, ..., k\}$ and there exists a simple submodule $T \subseteq U_1 \oplus \cdots \oplus U_{k+1}$ such that $h_M(T) \neq h_M(V_i)$ for all $i \in \{1, ..., k\}$. In this case we can put $V_{k+1} = T$ and the proof concludes as in case 2. ∎

Lemma 11. *Let M be a strongly rectifiable, S-homogeneous module, let $N = \bigoplus_{i=1}^{n} U_i$, where U_i are simple submodules of M for all $i \in \{1, ..., n\}$. Then the set $\{h_M(x) \; : \; x \in N\}$ is finite.*

Proof. For every non-zero element $x \in N$ the module Rx is an uniserial submodule of M, $Ix = 0$, and therefore Rx is simple. By Theorem 10, $\{h_M(x) : \; x \in N\} = \{h_M(0), h_M(V_1), h_M(V_2), ..., h_M(V_n)\}$. ∎

Both Prüfer's theorems, well-known from the theory of abelian groups, hold also in the class of strongly rectifiable, S-homogeneous modules. Benabdallah and Hattab [2] showed an extension of the first Prüfer's theorem and of the Kulikov's criterion for rectifiable modules. They prove, that a rectifiable module M is a direct sum of uniserial submodules if and only if $\mathrm{Soc}(M) = \bigcup_{n=1}^{\infty} S_n$, where $S_1 \subseteq S_2 \subseteq \cdots$ and for each $n \in \mathbb{N}$ there exists $k_n \in \mathbb{N}$ such that $S_n \cap H_{k_n}(M) = 0$.

The next theorem gives an extension of the second Prüfer's theorem for strongly rectifiable and S-homogeneous modules.

Theorem 12. *Let M be a countable generated, strongly rectifiable, S-homogeneous module. Then M is a direct sum of uniserial submodules if and only if $H_\omega(M) = 0$.*

Proof. If M is a direct sum of uniserial submodules, then, by the Kulikov's criterion, $\mathrm{Soc}(M) = \bigcup_{n \in \mathbb{N}} S_n$, $S_1 \subseteq S_2 \subseteq S_3 \subseteq ...$ and for every $n \in \mathbb{N}$ there exists $k_n \in \mathbb{N}$ such that $S_n \cap H_{k_n}(M) = 0$. If $H_\omega(M) \neq 0$, then there exists a non-zero element $x \in H_\omega(M)$, and therefore, Rx is an uniserial submodule. Denote $Z = \mathrm{Soc}(Rx)$. Then there exists $n' \in \mathbb{N}$ such that $0 \neq Z \subseteq S_{n'} \cap H_{k_{n'}}(M) = 0$, which is a contradiction. Now, let $H_\omega(M) = 0$, $M = \sum_{i \in \mathbb{N}} Rm_i$. Put $S_n = \mathrm{Soc}(\sum_{i=1}^{n} Rm_i)$. By Theorem 10, there exist simple submodules $W_1, ..., W_k$ such that $S_n = \bigoplus_{i=1}^{k} W_i$ and, for every simple submodule U of S_n, there exists $i_0 \in \{1, ..., k\}$ for which $h_M(U) = h_M(W_{i_0})$. For every $i = 1, ..., k$, we have $h_M(W_i) < \omega$ (since $H_\omega(M) = 0$). Let $k_n = 1 + \max\{h_M(W_1), ..., h_M(W_k)\}$. By Lemma 8 and by Theorem 10, $S_n \cap H_{k_n}(M) = 0$ and we use the Kulikov's criterion for rectifiable modules (see [2]). ∎

3 Pure and refined submodules

The concepts of pure and refined submodules are direct analogies of the corresponding concept in the theory of I-primary modules or rectifiable modules, respectively, and we show that basic results hold also in the class of strongly rectifiable, S-homogeneous modules.

Lemma 13. *Let* M *be a strongly rectifiable, S-homogeneous module, let* $x_0 \in M$ *be a non-zero element and let* $r_0 \in R \smallsetminus I$. *Then, for every integer* $m \geq 0$, *we have* $Rx_0 = I^m x_0 + Rr_0x_0$ *and hence there exists* $d \in R \smallsetminus I$ *such that* $x_0 = dr_0x_0$.

Proof. We proceed by induction on m. Because the left ideal I is maximal, $Rx_0 = Ix_0 + Rr_0x_0$. For every $s \in R$, we have $sx_0 = u(bx_0 + t'r_0x_0) + tr_0x_0 \in I^{k+1}x_0 + Rr_0x_0 \subseteq Rx_0$, where $u \in I^k$, $b \in I$, $t, t' \in R$. ∎

Lemma 14.

1. *If* $I = I^2 + aR$ *for some* $a \in I$, *then* $I = I^m + aR$ *for every* $m \in \mathbb{N}$.
2. *If* $I = I^2 + Ra$ *for some* $a \in I$, *then* $I = I^m + Ra$ *for every* $m \in \mathbb{N}$.

Proof.

1. We proceed by induction on m. For $m = 1$ the statement is obvious. Suppose that $I = I^m + aR$ for $m = k$. Then $I = I^2 + aR = (I^k + aR)I + aR \subseteq I^{k+1} + aI + aR \subseteq I^{k+1} + aR \subseteq I$ and therefore $I = I^{k+1} + aR$.
2. The proof is similar as in the previous case. ∎

Lemma 15. *Let* M *be a strongly rectifiable, S-homogeneous module, and let* $I = I^2 + aR$ *for some* $a \in I$. *Then* $H_n(M) = a^nM = I^nM$ *for every integer* $n \geq 0$.

Proof. For an element $x \in M$, the module Rx is a uniserial submodule of M and, by Lemma 5, there is $k \geq 0$ such that $I^k x = 0$. Thus, by Lemma 14,

$Ix \subseteq I^k x + aRx \subseteq aM$. By Lemma 6, further $aM = IM = H_1(M)$. Now, by induction and by Lemma 6, $H_{n+1}(M) = H_1(H_n(M)) = IH_n(M) = aH_n(M) = I^{n+1}M = a^{n+1}M$. ∎

Lemma 16. *Let M be a strongly rectifiable, S-homogeneous module, let $I = I^2 + Ra$ for some $a \in I$. Then, for every integer $k \geq 0$, for every non-zero element $x \in M$ and for an element $b \in R$, for which $bx \in I^k x \setminus I^{k+1}x$, there exists an element $d \in R \setminus I$ such that $dbx = a^k x$.*

Proof. By induction on k. For $k = 0$ the lemma follows immediatly from Lemma 13. For $bx \in I^{k+1}x \setminus I^{k+2}x$ there is $bx \neq 0$, Rx is an uniserial submodule and, by Lemma 5, there exists an integer m such that $I^m x = 0$. By Lemma 14, $Ix = Rax$ and then $I^{k+1}x = I^k Rax = I^k ax$, consequently $bx = b'ax$, where $b' \in I^k$. By induction, there exists $d \in R \setminus I$ such that $dbx = db'ax = a^k(ax) = a^{k+1}x$. ∎

Theorem 17. *Let M be a strongly rectifiable, S-homogeneous module, let $I = I^2 + Ra$ for some $a \in I$. Then for every finitely generated submodule N of M, the set $\{h_M(x) : x \in N\}$ is finite.*

Proof. We proceed by induction on number of elements which generate the submodule N. If the submodule N is cyclic, then N is uniserial and we use Lemma 8. Let $N = \sum_{i=1}^{k} Rx_i$ be finitely generated and let the set $\{h_M(x) : x \in N\}$ be infinite. Then there exist $y_1, y_2, y_3, \ldots \in N$, such that $h_M(y_1) < h_M(y_2) < h_M(y_3) < \ldots$. For every $i \in \mathbb{N}$, $y_i = r_{i_1}x_1 + \cdots + r_{i_k}x_k$, where $r_{i_j} \in R$ for all $j = 1, \ldots, k$. Using induction we can assume $r_{i_k}x_k \neq 0$ for all $i \in \mathbb{N}$. Since Rx_k is uniserial, there exists n, $0 \leq n < l(Rx_k)$, such that the set $\{\ i \in \mathbb{N} :\ r_{i_k}x_k \in I^n x_k \setminus I^{n+1}x_k\ \}$ is infinite. We can assume that $r_{i_k}x_k \in I^n x_k \setminus I^{n+1}x_k$ for all $i \in \mathbb{N}$. By Lemma 16, for every $i \in \mathbb{N}$ there exists $d_i \in R \setminus I$ such that $d_i r_{i_k} x_k = a^n x_k$. Then $d_i y_i - d_{i+1}y_{i+1} \in \sum_{j=1}^{k-1} Rx_j$, because $d_i r_{i_k} x_k - d_{i+1} r_{(i+1)_k} x_k = 0$. By Lemmas 7 and 9, $h_M(d_1y_1 - d_2y_2) < h_M(d_2y_2 - d_3y_3) < \ldots$ which contradicts the induction hypothesis. ∎

Definition. Let M be a strongly rectifiable, S-homogeneous module. We say that a submodule N of M is *pure in* M if $H_\alpha(M/N) = (H_\alpha(M) + N)/N$ for every ordinal α.

Remark. If a module M is strongly rectifiable and S-homogeneous, then $H_\alpha(M/N) = (H_\alpha(M) + N)/N$ for every submodule N of M and for every non-limit ordinal α (by induction and by Lemma 6). We can say that the submodule N is pure in M iff $H_\alpha(M/N) = (H_\alpha(M) + N)/N$ for every limit ordinal α.

Lemma 18. *Let M be a strongly rectifiable, S-homogeneous module and let N be a submodule of M. Then N is pure in M if and only if for every element $x \in M \setminus N$ there is an element $y \in N$ such that $h_M(x+z) \leq h_M(x+y)$ for all $z \in N$.*

Proof. If N is pure in M and $x \in M \setminus N$, then $x + N$ is a non-zero element of M/N. If $h_{M/N}(x+N) = \infty$, then there is $y \in N$ for which $h_M(x+y) = \infty$. If $h_{M/N}(x+N) = \alpha$, $\alpha \neq \infty$, then $u \in H_\alpha(M)$ and there exists $y \in N$ such that $x + y = u$ and $h_M(x+y) \geq \alpha$. Since $h_{M/N}(x+N) = \alpha$, we have $h_M(x+z) \leq \alpha$ for every $z \in N$. Let the assumptions of this lemma be valid, let α be a limit ordinal. By Lemma 2, $(H_\alpha(M)+N)/N \subseteq H_\alpha(M/N)$ for all ordinals α. Let $x+N \in H_\alpha(M/N)$ be an element. Then, by the induction hypothesis, for every $\beta < \alpha$ there is $z_\beta \in H_\beta(M)$ such that $x + N = z_\beta + N$ and there is $y_\beta \in N$ for which $x + y_\beta = z_\beta$. By the assumption of this Lemma, there exists $y \in N$ such that $\beta \leq h_M(x+y_\beta) \leq h_M(x+y)$ for every $\beta < \alpha$. Hence, $x + y \in \bigcap_{\beta<\alpha} H_\beta(M) = H_\alpha(M)$. ■

Theorem 19. *If M is a strongly rectifiable, S-homogeneous module, then for every ordinal α the submodule $H_\alpha(M)$ is pure in M.*

Proof. Let us consider an element $x \in M \setminus H_\alpha(M)$. Then $h_M(x) = \sigma < \alpha$, where σ is an ordinal. For all $z \in H_\alpha(M)$, we have $h_M(z) \geq \alpha$. Therefore, by Lemma 7, $h_M(x+z) = \sigma = h_M(x)$ and we can choose $y = 0$. The rest of the proof follows from Lemma 18. ■

Lemma 20. *Let M be a strongly rectifiable, S-homogeneous module, and let K, L be submodules of M such that $K \subseteq L \subseteq M$. Then:*

1. *if L is pure in M, then L/K is pure in M/K;*
2. *if K is pure in M and L/K is pure in M/K, then L is pure in M.*

Proof.

1. Let $\overline{x} + L/K \in H_\alpha((M/K)/(L/K))$ be an element, where $\overline{x} \in M/K$ and $\overline{x} = x + K$, for an element $x \in M$. By Corollary 3, $h_{M/L}(x+L) \geq \alpha$, and then, since L is pure in M, there exist $x' \in H_\alpha(M)$ and $l \in L$ for which $x = x' + l$. Then $\overline{x} + L/K = (x'+l+K)+L/K = (x'+K)+L/K \in (H_\alpha(M/K)+L/K)/(L/K)$, due

to $(H_\alpha(M)+K)/K \subseteq H_\alpha(M/K)$ by Lemma 2. The opposite inclusion follows from Lemma 2.

2. Let $x+L \in H_\alpha(M/L)$ be an element, where $x \in M$. Then, by Corollary 3, $(x+K)+L/K \in H_\alpha((M/K)/(L/K)) = (H_\alpha(M/K)+L/K)/(L/K)$, and then there is $x' \in M$ for which $x'+K \in H_\alpha(M/K)$ and $(x+K)+L/K = (x'+K)+L/K$. Now, since K is pure in M, there exists $x'' \in H_\alpha(M)$ such that $x'+K = x''+K$. Then, since $x+K = (x'+K)+(l+K)$, we have $x' = x''+k$ and $x = x'+l+k'$ for some $k, k' \in K$, $l \in L$. Futher, we get $x+L = x''+k+l+k'+L = x''+L \in (H_\alpha(M)+L)/L$. The opposite inclusion is valid by Lemma 2. ∎

Theorem 21. *Let M be a strongly rectifiable, S-homogeneous module and suppose there is an element $a \in I$ such that $I = I^2 + Ra$. Then every finitely generated submodule N of M is pure in M.*

Proof. For an element $x \in M \setminus N$ we have $\{h_M(x+z) : z \in N\} \subseteq \{h_M(u) : u \in Rx+N\}$ and, by Theorem 17, this set is finite. Hence the set $\{h_M(x+z) : z \in N\}$ has a maximal element and the proof is finished by Lemma 18. ∎

Lemma 22. *Let $M = \bigoplus_{i\in J} M_i$ be a strongly rectifiable, S-homogeneous module, and let N_i be a submodule of M_i for every $i \in J$. Then $N = \bigoplus_{i\in J} N_i$ is pure in M if and only if N_i is pure in M_i for all $i \in J$.*

Proof. Proof follows immediately from Lemma 2 and Corollary 3. ∎

Definition. Let N be a submodule of a strongly rectifiable, S-homogeneous module M. We say that N is *refined in* M if $H_\alpha(M) \cap N = H_\alpha(N)$ for all ordinals α.

Theorem 23. *Let N be a direct summand of a strongly rectifiable, S-homogeneous module. Then N is pure and refined in M.*

Proof. Let $M = N \oplus K$ be a direct sum, where K, N are submodules of M. By Lemma 2, $H_\alpha(N) \subseteq H_\alpha(M) \cap N$, $(H_\alpha(M)+N)/N \subseteq H_\alpha(M/N)$ for all ordinals α. Applying Lemma 2 to the projection $\pi\colon M \longrightarrow N$, we get $h_N(x) \geq \alpha$ for $x \in H_\alpha(M) \cap N$. For the exact sequence $0 \longrightarrow N \longrightarrow M \xrightarrow{\psi} M/N \longrightarrow 0$, there exists a homomorphism $\varphi\colon M/N \longrightarrow M$ such that $\varphi\psi = 1_{M/N}$. Then, for an element $x+N \in H_\alpha(M/N)$, we have $\varphi(x+N) \in H_\alpha(M)$, by Lemma 2, and hence $x+N = \varphi(x+N)+N \in (H_\alpha(M)+N)/N$. ■

Lemma 24. *Let M be a strongly rectifiable, S-homogeneous module, let N be a submodule of M and let σ be an ordinal. Then the submodule N is pure in M if and only if the submodule $(N+H_\sigma(M))/H_\sigma(M)$ is pure in $M/H_\sigma(M)$ and the submodule $N \cap H_\sigma(M)$ is pure in $H_\sigma(M)$.*

Proof.

1. We first show that N is pure in M. Let $C = (M/H_\sigma(M))/((N+H_\sigma(M))/H_\sigma(M))$ and $D = ((N+H_\alpha(M))/H_\sigma(M))/((N+H_\sigma(M))/H_\sigma(M))$.

   Then $C/D \cong M/(N+H_\alpha(M)) \cong (M/N)/((N+H_\alpha(M))/N)$. By Theorem 19, the submodule $H_\sigma(M)$ is pure in M, and hence for $\alpha \leq \sigma$ we have $H_\alpha(C) = D$. Because, by Theorem 19, $H_\alpha(C/H_\alpha(C)) = 0$, we have $H_\alpha(C/D) = 0$ and thus, by Corollary 3, $H_\alpha(M/N) \subseteq (H_\alpha(M)+N)/N$.

   For $\alpha > \sigma$, there is an ordinal β such that $\alpha = \sigma+\beta$. Since $N \cap H_\sigma(M)$ is pure in $H_\sigma(M)$, we have $H_\beta(H_\sigma(M)/(N \cap H_\sigma(M))) = (H_\alpha(M)+(N \cap H_\sigma(M)))/(N \cap H_\sigma(M))$. It is easy to show that $H_\alpha(M)+(N\cap H_\sigma(M)) = H_\sigma(M) \cap (N+H_\alpha(M))$, hence $((N+H_\sigma(M))/N)/((N+H_\alpha(M))/N) \cong ((N+H_\sigma(M))/N)/(((N+H_\alpha(M))/N) \cap ((N+H_\sigma(M))/N)) \cong (N+H_\sigma(M))/(N+H_\alpha(M)) = (N+H_\alpha(M)+H_\sigma(M))/(N+H_\alpha(M)) \cong H_\sigma(M)/((N+H_\alpha(M)) \cap H_\sigma(M)) = H_\sigma(M)/(H_\alpha(M)+(N\cap H_\sigma(M))) \cong (H_\sigma(M)/(N\cap H_\sigma(M)))/((H_\alpha(M)+(N\cap H_\sigma(M)))/(N\cap H_\sigma(M))) = (H_\sigma(M)/(N\cap H_\sigma(M)))/H_\beta(H_\sigma(M)/(N\cap H_\sigma(M)))$. Because the submodule $N \cap H_\sigma(M)$ is pure in $H_\sigma(M)$, we have $H_\beta(H_\sigma(M)/(N\cap H_\sigma(M))) = (H_\beta(H_\sigma(M))+(N\cap H_\sigma(M)))/(N\cap H_\sigma(M)) = (H_\alpha(M)+(N\cap H_\sigma(M)))/(N \cap H_\sigma(M))$. By Theorem 19, we get $H_\beta(((N+H_\sigma(M))/N)/((N+H_\alpha(M))/N)) = 0$ and, therefore, $(N+H_\alpha(M))/N \supseteq H_\beta((N+H_\sigma(M))/N) = H_\beta(H_\sigma(M/N)) = H_\alpha(M/N)$.
2. Let N be pure in M and let α be an ordinal. Then $(H_{\sigma+\alpha}(M)+N)/N = H_{\sigma+\alpha}(M/N) = H_\alpha((H_\sigma(M)+N)/N)$ is pure in $(H_\sigma(M)+N)/N$ and, therefore, $H_\alpha(((H_\sigma(M)+N)/N)/((H_{\sigma+\alpha}(M)+N)/N)) = 0$.

Since $H_\sigma(M)/(N \cap H_\sigma(M)) \cong (H_\sigma(M)+N)/N$, we have $((H_\sigma(M)+N)/N)/((H_{\sigma+\alpha}(M)+N)/N) \cong (H_\sigma(M)/(N \cap H_\sigma(M)))/((H_{\sigma+\alpha}(M)+(N \cap H_\sigma(M)))/(N \cap H_\sigma(M)))$, hence $H_\alpha((H_\sigma(M)/(N \cap H_\sigma(M)))/((H_{\sigma+\alpha}(M)+(N\cap H_\sigma(M)))/(N\cap H_\sigma(M)))) = 0$ and then $H_\alpha(H_\sigma(M)/(N \cap H_\sigma(M))) \subseteq (H_{\sigma+\alpha}(M)+(N \cap H_\sigma(M)))/(N \cap H_\sigma(M))$. The other inclusion follows from Lemma 2, and thus the submodule $N \cap H_\sigma(M)$ is pure in $H_\sigma(M)$.

For an ordinal $\alpha \leq \sigma$, we have, by Theorem 19, $H_\alpha(M/(H_\alpha(M)+N)) \cong H_\alpha((M/N)/H_\alpha(M/N)) = 0$. Then $H_\alpha(C/D) \cong H_\alpha((M/H_\sigma(M))/((N + H_\alpha(M))/H_\sigma(M))) \cong H_\alpha(M/(N + H_\alpha(M))) = 0$, and $H_\alpha(C) \subseteq D$. Therefore, for all $\alpha \leq \sigma$, the condition from the definition of a pure submodule is satisfied. For an ordinal $\alpha > \sigma$, an ordinal β exists, such that $\alpha = \sigma+\beta$. Then $H_\alpha((M/H_\sigma(M))/((N+H_\sigma(M))/H_\sigma(M))) = H_\beta((H_\sigma(M/H_\sigma(M))+(N+H_\sigma(M))/H_\sigma(M))/((N+H_\sigma(M))/H_\sigma(M))) = 0$. Because $H_\alpha(M/H_\sigma(M)) = (H_\alpha(M)+H_\sigma(M))/H_\sigma(M) = 0$, the submodule $(N+H_\sigma(M))/H_\sigma(M)$ is pure in $M/H_\sigma(M)$. ∎

4 Hill's modules

Hill's modules are well-known to yield interesting results, e.g., in the theory of I-primary modules. In this paper we give an analogous definition in the class of strongly rectifiable, S-homogeneous modules, and we show that this concept has analogous properties in this class. Under an additional assumption that there is an element $a \in I$ such that $I = I^2 + Ra$, we describe the structure of Hill's modules in Theorem 30.

Definition. Let M be a strongly rectifiable, S-homogeneous module and let N be a submodule of M. We say that a well-ordered series of its submodules $N_0 \subset N_1 \subset N_2 \subset \ldots$ is a *pure composite series from* N *to* M, if:

1. $N = N_0$, $M = N_\sigma$ for an ordinal σ;
2. N_α is pure in M for all ordinals $\alpha < \sigma$;
3. $N_{\alpha+1}/N_\alpha \cong S$ for all ordinals $\alpha < \sigma$;
4. $N_\alpha = \bigcup_{\beta<\alpha} N_\beta$ for all limit ordinals α.

A pure composite series from $N = 0$ to M is said to be a *pure composite series of the module* M.

Lemma 25. *Let* M *be a strongly rectifiable, S-homogeneous module, let* $I = I^2 + Ra$ *for some* $a \in I$ *and let* N *be a countable generated submodule of* M*. Then the module* N *has a pure composite series* $0 = N_0 \subset N_1 \subset ... \subset N_\alpha = N$, *such that* $\alpha \leq \omega$ *and for every* $i < \alpha$ *the submodule* N_i *is pure in* M*.*

Proof. If $N = \sum_{i \in \mathbb{N}} Rg_i$, then we denote $N_0 = 0$, $N_n = \sum_{i=1}^{n} Rg_i$ for every $n \in \mathbb{N}$. For every integer $n \geq 0$, the module $N_{n+1}/N_n \cong Rg_{n+1}/(N_n \cap Rg_{n+1})$ is uniserial and we can denote by $0 = \overline{N}_0 \subset \overline{N}_1 \subset ... \subset \overline{N}_{k_n} = N_{n+1}/N_n$ the chain of all submodules of N_{n+1}/N_n. For each $j = 0, 1, ..., k_n$, there exists a submodule N_{n_j} such that $N_{n_j}/N_n = \overline{N}_j$, $N_n \subseteq N_{n_j} \subseteq N_{n+1}$, $N_{n_0} = N_n$, $N_{k_n} = N_{n+1}$. By Lemma 5, $N_{n_{j+1}}/N_{n_j} = \overline{N}_{j+1}/\overline{N}_j \cong S$ and all submodules N_{n_j} are finitely generated. Then, by Theorem 21, every submodule N_{n_j} is pure in M and also in N. Therefore, the series $0 = N_0 \subset N_{0_1} \subset ... \subset N_{0_{k_0}} = N_1 = N_{1_0} \subset N_{1_1} \subset ...$ is a pure composite series. ■

Definition. Let M be a strongly rectifiable, S-homogeneous module. A system $\mathcal{N}$ of submodules of M is a *Hill's system*, if the following conditions are satisfied:

1. if $N \in \mathcal{N}$, then N is pure in M;
2. $0 \in \mathcal{N}$;
3. if $\{N_i : \ i \in J\} \subseteq \mathcal{N}$, then $\sum_{i \in J} N_i \in \mathcal{N}$;
4. if X is a countable subset of M and $N \in \mathcal{N}$, then there exists $K \in \mathcal{N}$ such that $N \cup X \subseteq K$ and the module K/N is countable generated.

A reduced, strongly rectifiable, S-homogeneous module with a Hill's system is called a *Hill's module.*

Theorem 26. *Let* M *be a reduced, countable generated, strongly rectifiable, S-homogeneous module. Then* M *is a Hill's module and* $\{0, M\}$ *is its Hill's system.*

Proof. Straightforward. ■

Theorem 27. *A direct summand of a Hill's module is a Hill's module.*

Proof. Let M be a Hill's module and let $M = A \oplus B$ be a direct sum of submodules A, B, let $\mathcal{M}$ be a Hill's system of the module M. Let $\mathcal{N} = \{N : \ N$ is a submodule of A for which there exists a module $Z \in \mathcal{M}$

such that $Z = N \oplus (Z \cap B)\}$. We show that $\mathcal{N}$ is the Hill's system of the submodule A.

The first condition of the definition of Hill's system is satisfied by Lemma 22, the second and the third conditions are obvious. For the fourth condition, let X be a countable subset of A and let N be a submodule of A such that there is a module $Z \in \mathcal{M}$ for which $Z = N \oplus (Z \cap B)$. We denote $Z_0 = Z$, $A_0 = N + \sum_{x \in X} Rx$, $B_0 = Z \cap B$. Then A_0/N is countable generated and for $Z \in \mathcal{M}$ there exists a module $Z_1 \in \mathcal{M}$ such that $Z_0 + A_0 \subseteq Z_1$ and the module Z_1/Z is countable generated, too. We continue by induction. Suppose that we have modules $Z_n \in \mathcal{M}$, $A_n \subseteq A$, $B_n \subseteq B$, $Z_n \subseteq A_n \oplus B_n$, $Z + A_{n-1} \subseteq Z_n$, such that Z_n/Z and A_n/N are countable generated, $Z_{n-1} \subseteq Z_n$, $A_{n-1} \subseteq A_n$, $B_{n-1} \subseteq B_n$. Then there exists a module $Z_{n+1} \in \mathcal{M}$, $Z_n \subseteq Z_{n+1}$ such that Z_{n+1}/Z is countable generated and $Z + A_n \subseteq Z_{n+1}$. We denote $A_{n+1} = \pi_A(Z_{n+1})$, $B_{n+1} = \pi_B(Z_{n+1})$, where π_A, π_B are the projections determined by the direct sum $M = A \oplus B$. Then $Z_{n+1} \subseteq A_{n+1} \oplus B_{n+1}$ and the mapping $\varphi\colon Z_{n+1}/Z \longrightarrow A_{n+1}/N$ defined by $\varphi\colon x+Z \longmapsto \pi_A(x)+N$ for all $x \in Z_{n+1}$ is an epimorphism and, hence, the module A_{n+1}/N is countable generated. Let $L = \bigcup_{n \in \mathbb{N}_0} Z_n$ be the union of Z_n for all integers $n \geq 0$. Then $L \in \mathcal{M}$, $L = (A \cap L) \oplus (B \cap L)$ and $L \cap A \in \mathcal{N}$. The module $(L \cap A)/N$ is countable generated as an epimorphic image of the countable generated module L/Z. ■

Theorem 28. *Let M be a reduced, strongly rectifiable, S-homogeneous module, and let $M = \bigoplus_{i \in J} M_i$ be a direct sum of submodules M_i. Then M is a Hill's module if and only if M_i are Hill's modules for all $i \in J$.*

Proof. The first implication follows from Theorem 27. Let $\mathcal{N}_i$ be a Hill's system of the module M_i for every $i \in J$, and denote $\mathcal{N} = \{N : N$ is a submodule of M, $N = \bigoplus_{i \in J} N_i$, where $N_i \in \mathcal{N}_i$ for all $i \in J\}$. We show that $\mathcal{N}$ is a Hill's system of M. The first condition of the definition of a Hill's system is satisfied by Lemma 22, the second and the third conditions are obvious. Let X be a countable subset of M, $N \in \mathcal{N}$. Then $N = \bigoplus_{i \in J} N_i$, where $N_i \in \mathcal{N}_i$ for all $i \in J$. Since X is a countable set, J contains a countable subset J_0 for which $X \subseteq \bigoplus_{i \in J_0} M_i$. For every $i \in J_0$, there exists $N_i' \in \mathcal{N}_i$ such that $\pi_i(X) \cup N_i \subseteq N_i'$ and the module N_i'/N_i is countable generated (π_i are the projections generated by direct sum). If we denote

$$\overline{N}_i = \begin{cases} N_i' & \text{for } \ i \in J_0, \\ N_i & \text{for } \ i \in J \setminus J_0 \end{cases}$$

and $\overline{N} = \sum_{i\in J} \overline{N}_i = \bigoplus_{i\in J} \overline{N}_i$, then $X \cup N \subseteq \overline{N}$, $\overline{N} \in \mathcal{N}$ and the module $\overline{N}/N \cong \bigoplus_{i\in J_0} N_i'/N_i$ is countable generated. ■

Theorem 29. *Let M be a reduced, strongly rectifiable, S-homogeneous module, let σ be an ordinal for which $M/H_\sigma(M)$ is a Hill's module and let $H_{\sigma+1}(M) = 0$. Then M is a Hill's module.*

Proof. Let $\mathcal{N}$ denote the Hill's system of the module $M/H_\sigma(M)$ and let us define $\mathcal{M} = \{N : N$ is a submodule of M such that $(N + H_\sigma(M))/H_\sigma(M) \in \mathcal{N}\}$. For every $N \in \mathcal{M}$, the module $(N + H_\sigma(M))/H_\sigma(M)$ is pure in $M/H_\sigma(M)$. The module $H_\sigma(M)$ is semisimple, since $H_\sigma(M)$ is generated by uniserial submodules and $IH_\sigma(M) = H_1(H_\sigma(M)) = H_{\sigma+1}(M) = 0$. The submodule $N \cap H_\sigma(M)$ is pure in $H_\sigma(M)$ as a direct summand of $H_\sigma(M)$, therefore, by Lemma 24, the submodule N is pure in M. The second and the third conditions are obvious. Let X be a countable subset of M, and let $N \in \mathcal{M}$. Then there exists $\overline{L} \in \mathcal{N}$ such that $\{x + H_\sigma(M) : x \in X\} \cup (N + H_\sigma(M))/H_\sigma(M) \subseteq \overline{L}$ and the module $\overline{L}/((N + H_\sigma(M))/H_\sigma(M))$ is countable generated. If we put $\overline{L} = L/H_\sigma(M)$, where $H_\sigma(M) \subseteq L \subseteq M$, then $X \cup N \subseteq L$ and $L/(N + H_\sigma(M))$ is countable generated. If this module is generated by the set $\{l_i + (N + H_\sigma(M)) : i \in \mathbb{N}\}$, then we put $U = \sum_{i\in\mathbb{N}} Rl_i$ and, then, $U + N + H_\sigma(M) = L$. If the countable generated module $(\sum_{x\in X} Rx)/(\sum_{x\in X} Rx \cap N)$ is generated by the set $\{y_i + (\sum_{x\in X} Rx \cap N) : i \in \mathbb{N}\}$, we can put $V = \sum_{i\in\mathbb{N}} Ry_i$. Then $V + (\sum_{x\in X} Rx \cap N) = \sum_{x\in X} Rx$ and hence $V \subseteq L$. Now $L = L + V = U + N + H_\sigma(M) + V$ and we can put $Z = N + U + V$. It is obvious that $(Z + H_\sigma(M))/H_\sigma(M) = \overline{L} \in \mathcal{N}$. Hence, $Z \in \mathcal{M}$ and $N \cup X \subseteq Z$ and the module $Z/N \cong (U + V)/((U + V) \cap N)$ is countable generated, since U, V are countable generated. The module M is a Hill's module. ■

Theorem 30. *Let M be a reduced, strongly rectifiable, S-homogeneous module, and let $I = I^2 + Ra$ for some $a \in I$. If M is a Hill's module, then M contains a pure composite series.*

Proof. Let $\mathcal{M}$ be a Hill's system of M, let $M = \{x_\sigma : \sigma < \tau\}$ be all elements of M, where τ is an ordinal. Put $N_0 = 0$. If there is N_α for which

$N_\alpha \in \mathcal{M}$, $\{x_\sigma : \sigma < \alpha\} \subseteq N_\alpha$, then, by the definition of Hill's system, there exists $N_{\alpha+1} \in \mathcal{M}$ such that $N_\alpha \cup \{x_\alpha\} \subseteq N_{\alpha+1}$ and the module $N_{\alpha+1}/N_\alpha$ is countable generated. Therefore $\{x_\sigma : \sigma < \alpha + 1\} \subseteq N_{\alpha+1}$. If α is a limit ordinal, we put $N_\alpha = \bigcup_{\beta<\alpha} N_\beta$. By Lemma 25, for every $\alpha < \tau$, the module $N_{\alpha+1}/N_\alpha$ contains a pure composite series. If this series are $0 = N_{\alpha+1,0}/N_\alpha \subset N_{\alpha+1,1}/N_\alpha \subset ... \subset N_{\alpha+1,\varrho}/N_\alpha = N_{\alpha+1}/N_\alpha$, where $\varrho \leq \omega$, then, for all $i < \varrho$, we have $N_{\alpha+1,i+1}/N_{\alpha+1,i} \cong S$ and, by Lemmas 25 and 20, the submodules $N_{\alpha+1,i}$ are pure in M. Hence, all modules $N_{\alpha+1,i}$ create a pure composite series of M. ■

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