

## APPLYING THE CZÉDLI-SCHMIDT SEQUENCES TO CONGRUENCE PROPERTIES OF PLANAR SEMIMODULAR LATTICES

G. GRÄTZER

Department of Mathematics  
University of Manitoba  
Winnipeg, MB R3T 2N2, Canada

e-mail: gratzer@me.com

### Abstract

Following Grätzer and Knapp, 2009, a planar semimodular lattice  $L$  is *rectangular*, if the left boundary chain has exactly one doubly-irreducible element,  $c_l$ , and the right boundary chain has exactly one doubly-irreducible element,  $c_r$ , and these elements are complementary.

The Czédli-Schmidt Sequences, introduced in 2012, construct rectangular lattices. We use them to prove some structure theorems. In particular, we prove that for a slim (no  $M_3$  sublattice) rectangular lattice  $L$ , the congruence lattice  $\text{Con } L$  has exactly  $\text{length}[c_l, 1] + \text{length}[c_r, 1]$  dual atoms and a dual atom in  $\text{Con } L$  is a congruence with exactly two classes. We also describe the prime ideals in a slim rectangular lattice.

**Keywords:** lattice, congruence, semimodular, planar, slim.

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